THE CORRESPONDENCE THEOREMS FOR HOMOMORPHISMS AND SUBGROUPS

Let $G$ be a group, and $K \triangleleft G$ a normal subgroup. Let $\pi : G \to G/K$ be the homomorphism $g \mapsto [g]$. You may find it helpful to visualize this setup as in Figure 1. The goal of the correspondence theorems is the following.

**Goal.** Describe objects associated to $G/K$ (like homomorphism or subgroups) in terms of objects associated to $G$ and $K$.

![Figure 1. The setup of the correspondence theorems](image)

By the first isomorphism theorem, if we have a surjective homomorphism $\phi : G \to G'$ and we let $K = \ker \phi$, then $G' \cong G/K$. So, via the first isomorphism theorem, our setup is equivalent to the setup of a surjective homomorphism $\phi : G \to G'$.

1. CORRESPONDENCE BETWEEN HOMOMORPHISMS

Let $H$ be a group and $\psi : G/K \to H$ a homomorphism. Let $\alpha = \psi \circ \pi$. Then $\alpha : G \to H$ is a homomorphism. Notice that $\alpha$ has the property that it maps $K$ to $\{e\}$. Indeed, for any $k \in K$, we have $\alpha(k) = \psi([k]) = \psi([e]) = e$. We thus get a correspondence

$$
\begin{align*}
\{ \text{Homomorphisms from } G/K \text{ to } H \} & \to \{ \text{Homomorphisms from } G \text{ to } H \text{ that map } K \text{ to } \{e\} \} \\
\psi & \mapsto \alpha = \psi \circ \pi.
\end{align*}
$$

**Theorem 1.** Let $G$ be a group, $K \triangleleft G$ a normal subgroup and $\pi : G \to G/K$ the homomorphism $g \mapsto [g]$. Then we have a bijective correspondence

$$
\begin{align*}
\{ \text{Homomorphisms from } G/K \text{ to } H \} & \to \{ \text{Homomorphisms from } G \text{ to } H \text{ that map } K \text{ to } \{e\} \} \\
\psi & \mapsto \psi \circ \pi.
\end{align*}
$$

Defined by
Proof. First, we show that the correspondence is injective. Suppose $\psi_1: G/K \to H$ and $\psi_2: G/K \to H$ are two homomorphisms such that $\psi_1 \circ \pi = \psi_2 \circ \pi$. Then we get

$$\psi_1(\pi(g)) = \psi_1([g]) = \psi_2([g]) = \psi_2(\pi(g))$$

for all $g \in G$. Since all elements of $G/K$ are of the form $[g]$, we get $\psi_1 = \psi_2$.

Second, we show that the correspondence is surjective. For this, given a homomorphism $\alpha: G \to H$ that sends $K$ to $\{e\}$, we must construct a homomorphism $\psi: G/K \to H$ such that $\alpha = \psi \circ \pi$. To construct $\psi$, we must specify $\psi(c)$ for every element $c$ of $G/K$. Recall that elements of $G/K$ are cosets of $K$ in $G$. Define

$$(1) \quad \psi(c) = \alpha(g) \text{ for } c = [g].$$

See Figure 2 for a pictorial description of our definition. We must ensure that our definition of $\psi(c)$ does not depend on the representative $g$ chosen for $c$. To this end, suppose $[g_1] = [g_2] = c$. Then $g_1 = g_2k$ for some $k \in K$. Since $\alpha$ maps $K$ to $\{e\}$, we get

$$\alpha(g_1) = \alpha(g_2k) = \alpha(g_2)\alpha(k) = \alpha(g_2).$$

Thus, the value of $\psi(c)$ does not depend on which representative we choose. Therefore, (1) gives a well-defined function $\psi: G/K \to H$.

Having constructed $\psi$, checking that it is a homomorphism is straightforward. Indeed, by the definition of $\psi$ in (1), we have

$$\psi([g]) = \alpha(g).$$

Hence, we get

$$\psi([g_1][g_2]) = \psi([g_1g_2]) = \alpha(g_1g_2) = \alpha(g_1)\alpha(g_2) = \psi([g_1])\psi([g_2]).$$

Finally, saying $\psi([g]) = \alpha(g)$ is the same as saying $\psi(\pi(g)) = \alpha(g)$, which in turn, is the same as saying $\alpha = \psi \circ \pi$. Therefore, the correspondence is surjective as desired. □

Example. Let us understand the correspondence theorem in a specific instance, namely for $Z \to Z/2Z = Z_2$. By the theorem, we have a one-to-one correspondence

$$\{\text{Homomorphisms } Z_2 \to G\} \leftrightarrow \{\text{Homomorphisms } Z \to G \text{ that send } 2Z \text{ to } \{e\}\}.$$  

How do we go from left to right? Let a homomorphism $\psi: Z_2 \to G$ be given. Then we get $\alpha: Z \to G$ by setting $\alpha(i) = \psi([i])$. 

![Figure 2. The construction of $\psi$ from $\alpha$](image-url)
How do we go from right to left? Let a homomorphism \( \alpha: \mathbb{Z} \rightarrow G \) be given. Suppose \( g = \alpha(1) \). Then \( \alpha(i) = g^i \) because \( \alpha \) is a homomorphism. See that \( \alpha \) takes \( 2\mathbb{Z} \) to \( \{e\} \) if and only if \( \alpha(2) = g^2 = e \). Then we get \( \psi: \mathbb{Z}_2 \rightarrow G \) by setting \( \psi([i]) = g^i \). The choice of the representative \( i \) does not matter: two representatives will differ by an even integer and \( g \) raised to an even integer is \( e \), so the answer will not change. Thus, we get a well-defined \( \psi: \mathbb{Z}_2 \rightarrow G \).

Thus, the homomorphisms from \( \mathbb{Z}_2 \) to a group \( G \) are in one-to-one correspondence with elements \( g \in G \) with \( g^2 = e \). For example, all homomorphisms from \( \mathbb{Z}_2 \rightarrow S_4 \) are given by \( [i] \mapsto g^i \) where \( g \) is one of the following:

- \( \text{id} \), \( (12) \), \( (13) \), \( (14) \), \( (23) \), \( (24) \), \( (34) \), \( (12)(34) \), \( (13)(24) \), \( (14)(23) \).

So there are 10 homomorphisms from \( \mathbb{Z}_2 \) to \( S_4 \).

2. CORRESPONDENCE BETWEEN SUBGROUPS

This material is covered beautifully in the book (section §2.10). So use these notes as a supplement and not a replacement to that section. The setup in the book is that of a surjective homomorphism \( G \rightarrow G' \). Our setup is that of a quotient by a normal subgroup \( K \), namely \( G \rightarrow G/K \). By the first isomorphism theorem, these are two equivalent situations.

Let \( H \subset G/K \) be a subgroup. Define \( H \subset G \) by \( H = \pi^{-1}(H) = \{ g \in G \mid [g] \in H \} \).

It is easy to check (check it!) that \( H \) is a subgroup of \( G \). Moreover, since \([k] = [e]\) for all \( k \in K \) and \([e] \in H \), the subgroup \( H \) contains \( K \). We thus get a correspondence

\[
\{\text{Subgroups of } G/K\} \rightarrow \{\text{Subgroups of } G \text{ that contain } K\}
\]

\( \Hrightarrow H = \pi^{-1}(H) \). (LR)

Let us construct a correspondence in the opposite direction. Let \( H \subset G \) be a subgroup containing \( K \). Define \( H \subset G/K \) by \( H = \pi(H) \).

Then \( H \) is a subgroup of \( G/K \). Indeed, it the image of the group \( H \) under the homomorphism \( \pi \). We thus get a correspondence

\[
\{\text{Subgroups of } G \text{ that contain } K\} \rightarrow \{\text{Subgroups of } G/K\}
\]

\( H \mapsto H = \pi(H) \). (RL)

**Theorem 2.** Let \( K \triangleleft G \) be a normal subgroup and \( \pi: G \rightarrow G/K \) the standard homomorphism \( g \mapsto [g] \). We have a bijective correspondence

\[
\{\text{Subgroups of } G/K\} \leftrightarrow \{\text{Subgroups of } G \text{ that contain } K\}
\]

where the left to right direction is given by (LR) and the right to left direction is given by (RL).

We will use the following lemma in the proof of the theorem. The proof of this lemma is a simple exercise with sets and functions. It has nothing to do with group theory. So I am leaving it to you.
Lemma 3. Let $f : A \to B$ be a function between two sets. For a subset $T \subset B$ denote by $f^{-1}(T)$ the subset of $A$ given by

$$f^{-1}(T) = \{ a \in A \mid f(a) \in T \}.$$ 

For a subset $S \subset A$, denote by $f(S)$ the subset of $B$ given by

$$f(S) = \{ f(s) \mid s \in S \}.$$ 

Then $f(f^{-1}(T)) \subset T$ and $S \subset f^{-1}(f(S))$. Moreover, if $f$ is surjective, then $f(f^{-1}(T)) = T$.

Proof of the theorem. Since we already have functions to and fro, what remains is to check that their compositions in both directions equal the identity.

Let us start with an $H$ on the left. By applying (LR) to $H$, we get $H = \pi^{-1}(H)$. We must show that applying (RL) to $H$ takes us back to $H$, that is $\pi(H) = \overline{H}$. Since $\pi$ is surjective, this follows from Lemma 3.

Now, let us start with an $H$ on the right. By applying (RL) to $H$, we get $H = \pi(H)$. We must show that applying (LR) to $\overline{H}$ takes us back to $H$, that is $\pi^{-1}(H) = H$. The inclusion $H \subset \pi^{-1}(H)$ follows from Lemma 3. For the opposite inclusion, consider an element $g \in \pi^{-1}(H)$. Then $\pi(g) = [g] \in H$. Since $\overline{H} = \pi(H)$, there is an $h \in H$ such that $[g] = [h]$. But $[g] = [h]$ means $g = hk$ for some $k \in K$. Since $H$ contains $K$, we get that $k \in H$. Therefore $g = hk \in H$. We thus get $\pi^{-1}(H) \subset H$. Together with the opposite inclusion, we get $\pi^{-1}(H) = H$. \hfill \Box

Remark 4. Under the correspondence of subgroups, normal subgroups of $G/K$ correspond to normal subgroups of $G$. This follows from Artin’s Proposition 2.10.4 applied to the surjective homomorphism $\pi : G \to G/K$. 

4