

HOCHSCHILD COHOMOLOGY AND GROUP ACTIONS

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ABSTRACT. Given a finite group action on a stable ∞ -category linear over a field, we establish a formula for the Hochschild cohomology of the category of invariants, assuming the order of the group is coprime to the characteristic of the base field. The formula shows that this cohomology group splits canonically with one summand given by the invariant subspace of the Hochschild cohomology of the original category. We also prove that Serre functors act trivially on Hochschild cohomology, and combine this with our formula to give a useful mechanism for computing the Hochschild cohomology of certain fractional Calabi–Yau categories.

1. INTRODUCTION

Let \mathbf{k} be an algebraically closed field. The Hochschild cohomology of a scheme X over \mathbf{k} can be defined as the graded \mathbf{k} -algebra

$$\mathrm{HH}^\bullet(X) = \mathrm{Ext}_{X \times X}^\bullet(\mathcal{O}_\Delta, \mathcal{O}_\Delta),$$

where $\Delta \subset X \times X$ denotes the diagonal. This invariant can in fact be defined directly in terms of the derived category of perfect complexes $\mathrm{Perf}(X)$, regarded as a \mathbf{k} -linear stable ∞ -category (or DG category). Indeed, \mathcal{O}_Δ is the Fourier–Mukai kernel for the identity functor of $\mathrm{Perf}(X)$, and $\mathrm{HH}^\bullet(X)$ is identified with the space of “derived endomorphisms” of this functor. This recipe makes sense for any \mathbf{k} -linear stable ∞ -category \mathcal{C} , and defines a graded \mathbf{k} -algebra $\mathrm{HH}^\bullet(\mathcal{C})$ known as its Hochschild cohomology. This is a fundamental and well-studied invariant of the category \mathcal{C} , which in particular controls its infinitesimal deformation theory.

The purpose of this paper is to study the Hochschild cohomology of \mathcal{C} in the presence of the action of a finite group G . In this situation, we can form a category \mathcal{C}^G of invariants. To give a feeling for this construction: if $\mathcal{C} = \mathrm{Perf}(X)$ for a smooth variety X , the G -action on \mathcal{C} is induced by one on X , and the order of G is invertible in \mathbf{k} , then $\mathcal{C}^G \simeq \mathrm{Perf}([X/G])$ where $[X/G]$ denotes the quotient stack (see [3]). We address the question: What is the relation between $\mathrm{HH}^\bullet(\mathcal{C}^G)$ and the space $\mathrm{HH}^\bullet(\mathcal{C})^G$ of invariants for the induced G -action on $\mathrm{HH}^\bullet(\mathcal{C})$?

Theorem 1.1. *Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with an action by a finite group G . Assume the order of G is coprime to the characteristic of \mathbf{k} . Then there is a canonical splitting of $\mathrm{HH}^\bullet(\mathcal{C}^G)$ with $\mathrm{HH}^\bullet(\mathcal{C})^G$ as a summand.*

In fact, we explicitly identify the complementary summand to $\mathrm{HH}^\bullet(\mathcal{C})^G$; for the precise statement, see Theorem 4.3.

Our original motivation for proving Theorem 1.1 was that it can be used to study the Hochschild cohomology of certain fractional Calabi–Yau categories. Recall that \mathcal{C} is called *fractional Calabi–Yau* if it is proper and has a Serre functor $S_{\mathcal{C}}$ satisfying $S_{\mathcal{C}}^q \cong [p]$ for some integers p and $q \neq 0$, and *Calabi–Yau* if we can take $q = 1$. The primordial example of a fractional Calabi–Yau category is $\mathcal{C} = \mathrm{Perf}(X)$, where X is a smooth proper \mathbf{k} -scheme with

torsion canonical bundle ω_X . By [8] there is also a large, interesting supply of fractional Calabi–Yau categories arising from semiorthogonal decompositions of Fano varieties. For instance, by [8, Corollary 4.1], for every smooth hypersurface $X \subset \mathbf{P}^n$ of degree $d \leq n$ there is a semiorthogonal decomposition

$$\mathrm{Perf}(X) = \langle \mathcal{C}_X, \mathcal{O}_X, \mathcal{O}_X(1), \dots, \mathcal{O}_X(n-d) \rangle, \quad (1.1)$$

where \mathcal{C}_X is fractional Calabi–Yau.

In general, it is a difficult problem to compute the Hochschild cohomology of a category \mathcal{C} . If \mathcal{C} is Calabi–Yau, however, the situation simplifies. In this case, $\mathrm{HH}^\bullet(\mathcal{C})$ is isomorphic as a graded vector space, up to a shift, to the *Hochschild homology* $\mathrm{HH}_\bullet(\mathcal{C})$. The problem of computing $\mathrm{HH}_\bullet(\mathcal{C})$ is often much more tractable than computing $\mathrm{HH}^\bullet(\mathcal{C})$, because Hochschild homology is additive under semiorthogonal decompositions (see [6]).

If \mathcal{C} is fractional Calabi–Yau, one would also like an effective mechanism for analyzing Hochschild cohomology in terms of Hochschild homology. This is what we achieve for a large class of fractional Calabi–Yau categories. Namely, suppose \mathcal{C} admits an autoequivalence σ generating a \mathbf{Z}/q -action, such that $S_{\mathcal{C}} \cong \sigma \circ [n]$. Again, there are many such categories, e.g. for any fixed q , there are infinitely many (d, n) such that \mathcal{C}_X in (1.1) satisfies this condition. If q is coprime to the characteristic of \mathbf{k} , then the invariant category $\mathcal{C}^{\mathbf{Z}/q}$ is Calabi–Yau (Lemma 6.4), hence $\mathrm{HH}^\bullet(\mathcal{C}^{\mathbf{Z}/q})$ is controlled by $\mathrm{HH}_\bullet(\mathcal{C}^{\mathbf{Z}/q})$, and Theorem 1.1 expresses $\mathrm{HH}^\bullet(\mathcal{C})^{\mathbf{Z}/q}$ as a summand of $\mathrm{HH}^\bullet(\mathcal{C}^{\mathbf{Z}/q})$. In fact, we prove the following result of independent interest, which implies $\mathrm{HH}^\bullet(\mathcal{C})^{\mathbf{Z}/q} = \mathrm{HH}^\bullet(\mathcal{C})$.

Theorem 1.2. *Let \mathcal{C} be a proper \mathbf{k} -linear stable ∞ -category which admits a Serre functor $S_{\mathcal{C}}$. Then the induced map $S_{\mathcal{C}*}: \mathrm{HH}^\bullet(\mathcal{C}) \rightarrow \mathrm{HH}^\bullet(\mathcal{C})$ is the identity.*

Thus, in the situation above we obtain a splitting of $\mathrm{HH}^\bullet(\mathcal{C}^{\mathbf{Z}/q})$ with $\mathrm{HH}^\bullet(\mathcal{C})$ as a summand. When $q = 2$, the complementary summand is entirely controlled by the Hochschild homology of \mathcal{C} (for $q > 2$ there are other contributions):

Corollary 1.3. *Let \mathcal{C} be a proper \mathbf{k} -linear stable ∞ -category such that $S_{\mathcal{C}} = \sigma \circ [n]$ is a Serre functor, where n is an integer and σ is the autoequivalence corresponding to the generator of a $\mathbf{Z}/2$ -action on \mathcal{C} . Assume the characteristic of \mathbf{k} is not 2. Then there is an isomorphism*

$$\mathrm{HH}^\bullet(\mathcal{C}^{\mathbf{Z}/2}) \cong \mathrm{HH}^\bullet(\mathcal{C}) \oplus (\mathrm{HH}_\bullet(\mathcal{C})^{\mathbf{Z}/2}[-n]). \quad (1.2)$$

Hence, in the situation of Corollary 1.3, the computation of $\mathrm{HH}^\bullet(\mathcal{C})$ reduces to an often more tractable question about Hochschild homology. As an illustration of this method, we prove the following result, which identifies the infinitesimal deformation spaces of a quartic fourfold $X \subset \mathbf{P}^5$ and the category \mathcal{C}_X defined by (1.1).

Proposition 1.4. *Assume the characteristic of \mathbf{k} is not 2, 3, or 5. Let $X \subset \mathbf{P}^5$ be a smooth quartic fourfold over \mathbf{k} . Then $\dim \mathrm{HH}^2(\mathcal{C}_X) = 90$ and the natural map*

$$H^1(X, T_X) \rightarrow \mathrm{HH}^2(\mathcal{C}_X)$$

is an isomorphism.

For another application of Corollary 1.3 to computing Hochschild cohomology, we refer to [10, Proposition 5.7].

Organization of the paper. In Section 2, we briefly review some facts about \mathbf{k} -linear stable ∞ -categories. In Section 3, we study finite group actions on such categories, and in particular prove that the norm functor is an equivalence when the order of the group is invertible in \mathbf{k} . Then in Section 4 we prove Theorem 1.1, in Section 5 we prove Theorem 1.2, and in Section 6 we prove Corollary 1.3 and Proposition 1.4.

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2. PRELIMINARIES ON LINEAR STABLE ∞ -CATEGORIES

In this paper, we work in the context of \mathbf{k} -linear stable ∞ -categories. We assume familiarity with this theory, but in this section we summarize some of the key points as we spell out our conventions. An uninitiated reader should be able to follow the main lines of reasoning, if they are willing to take for granted that a theory of ∞ -categories exists and behaves analogously to ordinary category theory. For background, see [11], [12], [13], or the survey [5, Chapter I.1].

2.1. Small stable categories. Let $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$ denote the ∞ -category of finite complexes of finite-dimensional \mathbf{k} -vector spaces. The category $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$ is stable and has a natural symmetric monoidal structure. We use the term *\mathbf{k} -linear stable ∞ -category* to mean a small idempotent-complete stable ∞ -category \mathcal{C} equipped with a module structure over $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$, such that the action functor

$$\begin{aligned} \mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}} \times \mathcal{C} &\rightarrow \mathcal{C} \\ (V, X) &\mapsto V \otimes X \end{aligned}$$

is exact in both variables. All of the functors between \mathbf{k} -linear stable ∞ -categories considered below will be \mathbf{k} -linear and exact, so we often omit these adjectives.

A reader unfamiliar with the above language will not lose much by pretending that \mathcal{C} is a small pretriangulated DG category over \mathbf{k} . We note that the homotopy category $\mathrm{h}\mathcal{C}$ of \mathcal{C} is indeed a \mathbf{k} -linear triangulated category.

The theory of \mathbf{k} -linear stable ∞ -categories has several technical advantages over the classical theory of DG categories. First, due to the foundations set up in [11], [12], the theory of ∞ -categories is a robust generalization of ordinary category theory. In particular, there are ∞ -categorical notions of commutativity of a diagram, and of limits and colimits, which are formally very similar to the corresponding notions for ordinary categories. Another important feature is that the collection of all \mathbf{k} -linear stable ∞ -categories (with morphisms between them the exact \mathbf{k} -linear functors) can be organized into an ∞ -category $\mathrm{Cat}_{\mathbf{k}}^{\mathrm{st}}$, which admits small limits and colimits.

2.1.1. Examples. Given a scheme X over \mathbf{k} , there is a \mathbf{k} -linear stable ∞ -category $\mathrm{Perf}(X)$, whose homotopy category is the usual derived category of perfect complexes. For this paper, the motivating example of a \mathbf{k} -linear stable ∞ -category is a category appearing as a semiorthogonal component in $\mathrm{Perf}(X)$, e.g. the category \mathcal{C}_X defined by (1.1). For a discussion of semiorthogonal decompositions in the context of stable ∞ -categories, see [13, Section 7.2]. We note that giving a semiorthogonal decomposition of a \mathbf{k} -linear stable ∞ -category \mathcal{C} is equivalent to giving a semiorthogonal decomposition of the triangulated category $\mathrm{h}\mathcal{C}$.

2.1.2. *Mapping objects.* For objects $X, Y \in \mathcal{C}$ in an ∞ -category \mathcal{C} , we denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the space of maps from X to Y . Let $\mathrm{Vect}_{\mathbf{k}}$ denote the ∞ -category of complexes of \mathbf{k} -vector spaces. If \mathcal{C} is a \mathbf{k} -linear stable ∞ -category, then there is a mapping object $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) \in \mathrm{Vect}_{\mathbf{k}}$ characterized by equivalences

$$\mathrm{Map}_{\mathrm{Vect}_{\mathbf{k}}}(V, \underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)) \simeq \mathrm{Map}_{\mathcal{C}}(V \otimes X, Y) \quad (2.1)$$

for $V \in \mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$.

2.2. **Large stable categories.** We review here the “large” version of \mathbf{k} -linear stable ∞ -categories. The only time this material will be needed is in the proof of Proposition 3.4 and its attendant lemmas.

The ∞ -category $\mathrm{Vect}_{\mathbf{k}}$ of complexes of \mathbf{k} -vector spaces has a natural symmetric monoidal structure. A *presentable \mathbf{k} -linear stable ∞ -category* is a presentable stable ∞ -category \mathcal{D} equipped with a module structure over $\mathrm{Vect}_{\mathbf{k}}$. Recall that a presentable ∞ -category is one which admits small colimits and satisfies a mild set-theoretic condition — roughly, that \mathcal{D} is generated under sufficiently filtered colimits by a small subcategory (see [11, Chapter 5] for details). As a basic example, given a scheme X over \mathbf{k} , there is a presentable \mathbf{k} -linear stable ∞ -category $\mathrm{QCoh}(X)$, whose homotopy category is the usual unbounded derived category of quasi-coherent sheaves.

There is an ∞ -category $\mathrm{PrCat}_{\mathbf{k}}^{\mathrm{st}}$ whose objects are the presentable \mathbf{k} -linear stable ∞ -categories and whose morphisms are the colimit preserving \mathbf{k} -linear functors. Just as $\mathrm{Cat}_{\mathbf{k}}^{\mathrm{st}}$, the category $\mathrm{PrCat}_{\mathbf{k}}^{\mathrm{st}}$ admits small limits and colimits. Moreover, for any $\mathcal{D} \in \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{st}}$ and objects $X, Y \in \mathcal{D}$, there is a mapping object $\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \in \mathrm{Vect}_{\mathbf{k}}$ characterized by equivalences as in (2.1), where now V is allowed to be any object in $\mathrm{Vect}_{\mathbf{k}}$.

Remark 2.1. In the literature, the term “ \mathbf{k} -linear stable ∞ -category” is often taken to mean a *presentable \mathbf{k} -linear stable ∞ -category* in the sense described above. We have instead reserved this term for the small version of these categories, because we will almost exclusively deal with categories of this type.

The small and presentable versions of \mathbf{k} -linear stable ∞ -categories are related via the operation of Ind-completion. Namely, there is a functor

$$\mathrm{Ind}: \mathrm{Cat}_{\mathbf{k}}^{\mathrm{st}} \rightarrow \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{st}}$$

which takes $\mathcal{C} \in \mathrm{Cat}_{\mathbf{k}}^{\mathrm{st}}$ to its *Ind-completion* $\mathrm{Ind}(\mathcal{C}) \in \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{st}}$. Roughly, $\mathrm{Ind}(\mathcal{C})$ is obtained from \mathcal{C} by freely adjoining all filtered colimits.

Remark 2.2. If X is a quasi-compact separated scheme over \mathbf{k} (or more generally a perfect stack in the sense of [1]), then there is an equivalence $\mathrm{Ind}(\mathrm{Perf}(X)) \simeq \mathrm{QCoh}(X)$.

For the details of Ind-completion, see [11, Chapter 5] or [5, Chapter I.1, Section 7.2]. All that we need for our purposes are the following facts.

Lemma 2.3. *Ind-completion satisfies the following properties:*

- (1) *The functor $\mathrm{Ind}: \mathrm{Cat}_{\mathbf{k}}^{\mathrm{st}} \rightarrow \mathrm{PrCat}_{\mathbf{k}}^{\mathrm{st}}$ commutes with colimits.*
- (2) *There is a natural fully faithful functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ of $\mathrm{Vect}_{\mathbf{k}}^{\mathrm{fd}}$ -module categories whose essential image is the subcategory $\mathrm{Ind}(\mathcal{C})^c \subset \mathrm{Ind}(\mathcal{C})$ of compact objects, and which therefore induces an equivalence $\mathcal{C} \simeq \mathrm{Ind}(\mathcal{C})^c$.*

3. GROUP ACTIONS ON CATEGORIES

In this section, we discuss group actions on \mathbf{k} -linear stable ∞ -categories. Besides recalling the basic definitions, we prove that the norm functor from the category of coinvariants to the category of invariants for the action of a finite group is an equivalence, provided the order of the group is prime to the characteristic of \mathbf{k} (Proposition 3.4).

3.1. Group actions. Let G be a finite group. We denote by BG the classifying space of G , regarded as an ∞ -category (i.e. BG is the nerve of the ordinary category with a single object whose endomorphisms are given by the group G).

Definition 3.1. Let \mathcal{D} be an ∞ -category, and let $X \in \mathcal{D}$ be an object. An *action* of G on X is a functor $\phi: BG \rightarrow \mathcal{D}$ which carries the unique object $* \in BG$ to $X \in \mathcal{D}$. Given such an action, the G -invariants X^G and G -coinvariants X_G are defined by

$$X^G = \lim(\phi) \quad \text{and} \quad X_G = \operatorname{colim}(\phi),$$

provided the displayed limit and colimit exist. In this case, we denote by $p: X^G \rightarrow X$ and $q: X \rightarrow X_G$ the canonical morphisms.

Remark 3.2. What we have called an action of G could more precisely be called a *left action* of G . There is also a notion of a *right action* of G on an object $X \in \mathcal{D}$, namely, a functor $\psi: BG^{\operatorname{op}} \rightarrow \mathcal{D}$ from the opposite category of BG which carries the basepoint to X . Note that any left action $\phi: BG \rightarrow \mathcal{D}$ on X gives rise to a right action by composing with the equivalence $BG^{\operatorname{op}} \simeq BG$ induced by inversion in G .

To relate the above definition to the classical notion of a group action, note that ϕ specifies for each $g \in G$ (thought of as an endomorphism of the basepoint $* \in BG$) an equivalence $\phi_g: X \rightarrow X$. The data of the entire action functor $\phi: BG \rightarrow \mathcal{D}$ then specifies certain compatibilities among the ϕ_g .

We will be particularly interested in the case of Definition 3.1 where $\mathcal{D} = \operatorname{Cat}_{\mathbf{k}}^{\operatorname{st}}$, i.e. where G acts on a \mathbf{k} -linear stable ∞ -category $\mathcal{C} \in \operatorname{Cat}_{\mathbf{k}}^{\operatorname{st}}$. Since $\operatorname{Cat}_{\mathbf{k}}^{\operatorname{st}}$ admits limits and colimits, the G -invariants \mathcal{C}^G and coinvariants \mathcal{C}_G always exist in this situation.

The objects and morphisms of \mathcal{C}^G can be described in relatively concrete terms, as follows. By the universal property of the projection functor $\mathcal{C}^G \rightarrow \mathcal{C}$, an object of $\tilde{X} \in \mathcal{C}^G$ corresponds to an object $X \in \mathcal{C}$ together with a *linearization*, i.e. a family of equivalences $\ell_g: X \rightarrow \phi_g(X)$ satisfying certain compatibilities. By abuse of notation, given an object $X \in \mathcal{C}$ with a linearization, we will often denote by the same symbol X the corresponding object of \mathcal{C}^G .

Given another object $\tilde{Y} \in \mathcal{C}^G$ corresponding to $Y \in \mathcal{C}$ with linearizations $m_g: Y \rightarrow \phi_g(Y)$, there is a natural action of G on $\underline{\operatorname{Map}}_{\mathcal{C}}(X, Y)$, i.e. a functor $\rho: BG \rightarrow \operatorname{Vect}_{\mathbf{k}}$ sending the basepoint $* \in BG$ to $\underline{\operatorname{Map}}_{\mathcal{C}}(X, Y) \in \operatorname{Vect}_{\mathbf{k}}$. Concretely, the image of $f: X \rightarrow Y \in \underline{\operatorname{Map}}_{\mathcal{C}}(X, Y)$ under $\rho_g: \underline{\operatorname{Map}}_{\mathcal{C}}(X, Y) \rightarrow \underline{\operatorname{Map}}_{\mathcal{C}}(X, Y)$ is determined by the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\rho_g(f)} & Y \\ \ell_g \downarrow & & \downarrow m_g \\ \phi_g(X) & \xrightarrow{\phi_g(f)} & \phi_g(Y) \end{array}$$

The morphisms in the category \mathcal{C}^G are then described by the formula

$$\underline{\operatorname{Map}}_{\mathcal{C}^G}(\tilde{X}, \tilde{Y}) \simeq \underline{\operatorname{Map}}_{\mathcal{C}}(X, Y)^G. \quad (3.1)$$

3.2. Induction and restriction. Let G be a finite group and $H \subset G$ a subgroup. Recall that there are natural induction and restriction functors between the classical categories of linear representations of H and G , and that these functors are mutually left and right adjoint. There is an analogous relation between the categories of invariants for group actions on a \mathbf{k} -linear stable ∞ -category.

Namely, let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with a G -action. The functor $BH \rightarrow BG$ induces an H -action on \mathcal{C} . We denote by

$$\mathrm{Res}_H^G: \mathcal{C}^G \rightarrow \mathcal{C}^H$$

the resulting *restriction functor* from G -invariants to H -invariants. Further, by choosing representatives for the cosets of G/H , we obtain a functor

$$\bigoplus_{[g] \in G/H} \phi_g: \mathcal{C} \rightarrow \mathcal{C}, \quad X \mapsto \bigoplus_{[g] \in G/H} \phi_g(X).$$

The precomposition with the projection $\mathcal{C}^H \rightarrow \mathcal{C}$ lifts to the *induction functor*

$$\mathrm{Ind}_H^G: \mathcal{C}^H \rightarrow \mathcal{C}^G,$$

which fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^H & \xrightarrow{\mathrm{Ind}_H^G} & \mathcal{C}^G \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\bigoplus_{[g] \in G/H} \phi_g} & \mathcal{C} \end{array}$$

If H is the trivial group, we write $\mathrm{Av}: \mathcal{C} \rightarrow \mathcal{C}^G$ for the induction functor and call it the *averaging functor* (also known as the *inflation functor*).

In the setting of triangulated categories with a group action, it is well-known that the induction and restriction functors are mutually left and right adjoint (see [3, Lemma 3.8] for the case where H is the trivial group). In our setup, we have the following analogue.

Lemma 3.3. *Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with an action by a finite group G , and let $H \subset G$ be a subgroup. Then the functors Ind_H^G and Res_H^G are mutually left and right adjoint.*

3.3. The norm functor. Let G be a finite group and V be a (ordinary) \mathbf{k} -vector space with a G -action. Then there is a *norm map* $\mathrm{Nm}: V_G \rightarrow V^G$, induced by the map $V \rightarrow V$ given by $x \mapsto \sum_{g \in G} g(x)$. If the order of G is invertible in \mathbf{k} , it is easy to see that the norm map is an isomorphism. Under the same assumption, the analogous statement holds in the ∞ -categorical setting where $V \in \mathrm{Vect}_{\mathbf{k}}$ and G acts on V : the norm map $V_G \rightarrow V^G$ is an equivalence. Our goal below is to describe what happens when V is replaced by a \mathbf{k} -linear stable ∞ -category.

Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with a G -action. By the universal properties of \mathcal{C}_G and \mathcal{C}^G , the functor $\bigoplus_{g \in G} \phi_g: \mathcal{C} \rightarrow \mathcal{C}$ induces the *norm functor*

$$\mathrm{Nm}: \mathcal{C}_G \rightarrow \mathcal{C}^G,$$

which is characterized by the existence of a factorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\bigoplus_{g \in G} \phi_g} & \mathcal{C} \\ q \downarrow & & \uparrow p \\ \mathcal{C}_G & \xrightarrow{\mathrm{Nm}} & \mathcal{C}^G \end{array}$$

Note that the composition $\text{Nm} \circ q: \mathcal{C} \rightarrow \mathcal{C}^G$ is nothing but the averaging functor Av . We aim to prove:

Proposition 3.4. *Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with an action by a finite group G . Assume the order of G is coprime to the characteristic of \mathbf{k} . Then the norm functor $\text{Nm}: \mathcal{C}_G \rightarrow \mathcal{C}^G$ is an equivalence.*

We will prove the proposition by relating it to the situation for presentable \mathbf{k} -linear stable ∞ -categories. First note that if \mathcal{D} is such a category with a G -action, then the same construction as above provides a norm functor $\text{Nm}: \mathcal{D}_G \rightarrow \mathcal{D}^G$. In this setting, we have the following analogue of Proposition 3.4, which holds even without any assumption on the characteristic of \mathbf{k} .

Lemma 3.5. *Let \mathcal{D} be a presentable \mathbf{k} -linear stable ∞ -category with an action by a finite group G . Then the norm functor $\text{Nm}: \mathcal{D}_G \rightarrow \mathcal{D}^G$ is an equivalence.*

Proof. We use the following fundamental relation between colimits and limits in the presentable setting. Let $F: I \rightarrow \text{PrCat}_{\mathbf{k}}^{\text{st}}$ be a functor from a small ∞ -category I to $\text{PrCat}_{\mathbf{k}}^{\text{st}}$; we think of F as a diagram of categories $F(i) = \mathcal{C}_i$ for $i \in I$. There is a functor $G: I^{\text{op}} \rightarrow \text{PrCat}_{\mathbf{k}}^{\text{st}}$ obtained by “passing to right adjoints”, such that for every $i \in I$ we have $G(i) = \mathcal{C}_i$, and for every morphism $a: i \rightarrow j$ in I regarded as a morphism $j \rightarrow i$ in I^{op} , we have $G(a) = F(a)^\dagger: \mathcal{C}_j \rightarrow \mathcal{C}_i$ where $F(a)^\dagger$ is the right adjoint to $F(a)$ (see [5, Chapter I.1, Section 2.4]). Then the key fact is that there is an equivalence $\text{colim}(F) \simeq \text{lim}(G)$, induced by the left adjoints $p_i^*: \mathcal{C}_i \rightarrow \text{lim}(G)$ to the natural functors $p_i: \text{lim}(G) \rightarrow \mathcal{C}_i$ for $i \in I$ (see [5, Chapter I.1, Section 2.5.8]).

Let us apply this to the functor $\phi: BG \rightarrow \text{PrCat}_{\mathbf{k}}^{\text{st}}$ encoding the action of G on \mathcal{D} . The functor $\psi: BG^{\text{op}} \rightarrow \text{PrCat}_{\mathbf{k}}^{\text{st}}$ obtained by passing to right adjoints is nothing but the right G -action induced by ϕ , as described in Remark 3.2. In particular, since ψ differs from ϕ by composition with the equivalence $BG^{\text{op}} \simeq BG$, we see that $\text{lim}(\psi) \simeq \text{lim}(\phi) = \mathcal{D}^G$. Hence applying the key fact from above, we find that there is an equivalence $\mathcal{D}_G \simeq \mathcal{D}^G$. Moreover, this equivalence is induced by the left adjoint to $p: \mathcal{D}^G \rightarrow \mathcal{D}$, i.e. by the averaging functor $\text{Av}: \mathcal{D} \rightarrow \mathcal{D}^G$, and hence is given by the norm functor. \square

We will also need the following result. We use the notation \mathcal{D}^c for the full subcategory of compact objects of an ∞ -category \mathcal{D} .

Lemma 3.6. *Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with an action by a finite group G . Then the natural fully faithful functor $\mathcal{C} \hookrightarrow \text{Ind}(\mathcal{C})$ induces an equivalence $\mathcal{C}_G \simeq (\text{Ind}(\mathcal{C})_G)^c$. If the order of G is coprime to the characteristic of \mathbf{k} , then $\mathcal{C} \hookrightarrow \text{Ind}(\mathcal{C})$ also induces an equivalence $\mathcal{C}^G \simeq (\text{Ind}(\mathcal{C})^G)^c$.*

Proof. For the coinvariants, note that by Lemma 2.3(1) we have $\text{Ind}(\mathcal{C})_G \simeq \text{Ind}(\mathcal{C}_G)$. Hence by Lemma 2.3(2) the functor $\mathcal{C}_G \rightarrow \text{Ind}(\mathcal{C})_G$ factors through an equivalence $\mathcal{C}_G \simeq (\text{Ind}(\mathcal{C})_G)^c$. For the invariants, observe that the natural functor $\mathcal{C}^G \rightarrow \text{Ind}(\mathcal{C})^G$ is fully faithful, by fully faithfulness of $\mathcal{C} \hookrightarrow \text{Ind}(\mathcal{C})$ combined with the description (3.1) (which also holds in the presentable setting) of mapping spaces in a category of invariants. This realizes \mathcal{C}^G as the full subcategory of $\text{Ind}(\mathcal{C})^G$ consisting of objects whose image under $\text{Ind}(\mathcal{C})^G \rightarrow \text{Ind}(\mathcal{C})$ is in $\mathcal{C} \simeq \text{Ind}(\mathcal{C})^c \subset \text{Ind}(\mathcal{C})$. Now the equivalence $\mathcal{C}^G \simeq (\text{Ind}(\mathcal{C})^G)^c$ is a consequence of the following lemma. \square

Lemma 3.7. *Let \mathcal{D} be a presentable \mathbf{k} -linear stable ∞ -category with an action by a finite group G . Assume the order of G is coprime to the characteristic of \mathbf{k} . Then an object of \mathcal{D}^G is compact if and only if its image under $\mathcal{D}^G \rightarrow \mathcal{D}$ is compact.*

Proof. Let \tilde{X} be an object of \mathcal{D}^G and X its image in \mathcal{D} . Assume first that \tilde{X} is compact. Then by adjointness of induction and restriction, for $Y \in \mathcal{D}$ we have

$$\underline{\mathrm{Map}}_{\mathcal{D}}(X, Y) \simeq \underline{\mathrm{Map}}_{\mathcal{D}^G}(\tilde{X}, \mathrm{Av}(Y)).$$

The functor Av commutes with colimits because it is a left adjoint, and $\underline{\mathrm{Map}}_{\mathcal{D}^G}(\tilde{X}, -)$ commutes with filtered colimits by compactness of \tilde{X} . Hence the above equivalence shows that $\underline{\mathrm{Map}}_{\mathcal{D}}(X, -)$ commutes with filtered colimits, i.e. that X is compact.

Now assume that X is compact. Let $p: \mathcal{D}^G \rightarrow \mathcal{D}$ be the projection. Then for $\tilde{Y} \in \mathcal{D}^G$, we have

$$\underline{\mathrm{Map}}_{\mathcal{D}^G}(\tilde{X}, \tilde{Y}) \simeq \underline{\mathrm{Map}}_{\mathcal{D}}(X, p(\tilde{Y}))^G \simeq \underline{\mathrm{Map}}_{\mathcal{D}}(X, p(\tilde{Y}))_G,$$

where the first equivalence is (3.1) and the second is given by the inverse of the norm map (here we use the assumption on the characteristic of \mathbf{k}). Note that $p: \mathcal{D}^G \rightarrow \mathcal{D}$ commutes with colimits because it is a left adjoint, $\underline{\mathrm{Map}}_{\mathcal{D}}(X, -)$ commutes with filtered colimits by compactness of X , and taking G -coinvariants commutes with colimits because by definition it is given by a colimit. Hence the above equivalence shows that $\underline{\mathrm{Map}}_{\mathcal{D}^G}(\tilde{X}, -)$ commutes with filtered colimits, i.e. that \tilde{X} is compact. \square

Proof of Proposition 3.4. By Lemma 3.5 the norm functor $\mathrm{Nm}: \mathrm{Ind}(\mathcal{C})_G \rightarrow \mathrm{Ind}(\mathcal{C})^G$ is an equivalence, and hence so is its restriction to the full subcategories of compact objects. But by Lemma 3.6, this restriction is identified with $\mathrm{Nm}: \mathcal{C}_G \rightarrow \mathcal{C}^G$. \square

4. HOCHSCHILD COHOMOLOGY AND GROUP INVARIANTS

Our goal in this section is to prove Theorem 1.1, or rather the more precise statement Theorem 4.3 below. We start by recalling the definition of Hochschild cohomology in our setting.

4.1. Hochschild cohomology. If \mathcal{C} and \mathcal{D} are \mathbf{k} -linear stable ∞ -categories, then the \mathbf{k} -linear exact functors from \mathcal{C} to \mathcal{D} form the objects of a \mathbf{k} -linear stable ∞ -category $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{D})$.

Definition 4.1. Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category. The *Hochschild cochain complex* of \mathcal{C} is defined as

$$\mathrm{HC}^{\bullet}(\mathcal{C}) = \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}}) \in \mathrm{Vect}_{\mathbf{k}}.$$

The *Hochschild cohomology* $\mathrm{HH}^{\bullet}(\mathcal{C})$ of \mathcal{C} is the cohomology of this complex.

Remark 4.2. There is a natural algebra structure on the Hochschild cohomology $\mathrm{HH}^{\bullet}(\mathcal{C})$, induced by composition in $\underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \mathrm{id}_{\mathcal{C}})$.

Hochschild cohomology is not functorial with respect to arbitrary functors of \mathbf{k} -linear stable ∞ -categories. But, of course, it is functorial with respect to equivalences. Namely, if $\Phi: \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence, conjugation by Φ induces an equivalence

$$\Phi_*: \mathrm{HC}^{\bullet}(\mathcal{C}) \xrightarrow{\sim} \mathrm{HC}^{\bullet}(\mathcal{D})$$

and hence an isomorphism $\Phi_*: \mathrm{HH}^\bullet(\mathcal{C}) \xrightarrow{\sim} \mathrm{HH}^\bullet(\mathcal{D})$. Explicitly, the image $\Phi_*(a) \in \mathrm{HC}^\bullet(\mathcal{D})$ of an element $a: \mathrm{id}_{\mathcal{C}} \rightarrow \mathrm{id}_{\mathcal{C}} \in \mathrm{HC}^\bullet(\mathcal{C})$ is determined by the commutative diagram

$$\begin{array}{ccc} \mathrm{id}_{\mathcal{D}} & \xrightarrow{\Phi_*(a)} & \mathrm{id}_{\mathcal{D}} \\ \wr \downarrow & & \downarrow \wr \\ \Phi \circ \mathrm{id}_{\mathcal{C}} \circ \Phi^{-1} & \xrightarrow{\Phi a \Phi^{-1}} & \Phi \circ \mathrm{id}_{\mathcal{C}} \circ \Phi^{-1} \end{array}$$

4.2. The main theorem. Note that given a \mathbf{k} -linear stable ∞ -category \mathcal{C} with an action by a finite group G , there is an induced action of $G \times G$ on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$. Concretely, $(g_1, g_2) \in G \times G$ acts on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$ by sending $F: \mathcal{C} \rightarrow \mathcal{C} \in \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$ to $\phi_{g_2} \circ F \circ \phi_{g_1}^{-1}$. Via the diagonal embedding $G \subset G \times G$, this restricts to the conjugation action of G on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$.

Theorem 4.3. *Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with an action by a finite group G . Assume the order of G is coprime to the characteristic of \mathbf{k} . Then there is an isomorphism*

$$\mathrm{HH}^\bullet(\mathcal{C}^G) \cong \mathrm{HH}^\bullet(\mathcal{C})^G \oplus \left(\bigoplus_{g \neq 1 \in G} \mathrm{H}^\bullet(\mathrm{Map}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \phi_g)) \right)^G,$$

where $\phi_g: \mathcal{C} \rightarrow \mathcal{C}$ is the autoequivalence corresponding to $g \in G$, and the G -action on the right side is induced by the conjugation action of G on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$.

Remark 4.4. The isomorphism of Theorem 4.3 is compatible with the algebra structure on Hochschild cohomology, i.e. it realizes $\mathrm{HH}^\bullet(\mathcal{C})^G$ as a subalgebra of $\mathrm{HH}^\bullet(\mathcal{C}^G)$. This follows from the proof given below.

Remark 4.5. Theorem 4.3 holds verbatim for a presentable \mathbf{k} -linear stable ∞ -category \mathcal{D} in place of \mathcal{C} , with the same proof given below.

We will need the following lemma for the proof of the theorem.

Lemma 4.6. *Let \mathcal{C} be a \mathbf{k} -linear stable ∞ -category with an action by a finite group G . Assume the order of G is coprime to the characteristic of \mathbf{k} . Then there is an equivalence*

$$\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}^G, \mathcal{C}^G) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^{G \times G}$$

under which the identity $\mathrm{id}_{\mathcal{C}^G}$ corresponds to the functor $\bigoplus_{g \in G} \phi_g: \mathcal{C} \rightarrow \mathcal{C}$ (with the natural $G \times G$ -linearization).

Proof. The norm equivalence $\mathrm{Nm}: \mathcal{C}_G \rightarrow \mathcal{C}^G$ induces an equivalence

$$\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}^G, \mathcal{C}^G) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}_G, \mathcal{C}^G).$$

The formation of the functor category $\mathrm{Fun}_{\mathbf{k}}(-, -)$ between \mathbf{k} -linear stable ∞ -categories takes colimits in the first variable to limits, and limits in the second variable to limits. Applying this to $\mathcal{C}_G = \mathrm{colim}_{BG} \mathcal{C}$ and then $\mathcal{C}^G = \mathrm{lim}_{BG} \mathcal{C}$, we find

$$\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}_G, \mathcal{C}^G) \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C}^G)^G \simeq (\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^G)^G.$$

In the final term, the outer G -action is induced by the action of G on the first copy of \mathcal{C} , and the inner G -action by the action of G on the second copy of \mathcal{C} . Thus, the outer G -action is induced by the restriction of the $G \times G$ -action on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$ to the first factor, and the inner G -action is identified with the restriction of the $G \times G$ -action to the second factor. It follows that

$$(\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^G)^G \simeq \mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^{G \times G}.$$

All together this proves the equivalence stated in the lemma, and tracing through the intermediate equivalences above shows that $\text{id}_{\mathcal{C}^G}$ corresponds to $\bigoplus_{g \in G} \phi_g$. \square

Proof of Theorem 4.3. By the definition of the Hochschild cochain complex combined with Lemma 4.6, we have

$$\text{HC}^\bullet(\mathcal{C}^G) = \underline{\text{Map}}_{\text{Fun}_{\mathbf{k}}(\mathcal{C}^G, \mathcal{C}^G)}(\text{id}_{\mathcal{C}^G}, \text{id}_{\mathcal{C}^G}) \simeq \underline{\text{Map}}_{\text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^{G \times G}}(\bigoplus_{g \in G} \phi_g, \bigoplus_{g \in G} \phi_g).$$

Recall that G acts on $\text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$ via restriction along the diagonal embedding $G \subset G \times G$. Notice that the induction of $\text{id}_{\mathcal{C}} \in \text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^G$ along the diagonal is

$$\text{Ind}_G^{G \times G}(\text{id}_{\mathcal{C}}) = \bigoplus_{g \in G} \phi_g \in \text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^{G \times G}.$$

Hence by adjointness of induction and restriction, we can rewrite the above expression as

$$\begin{aligned} \text{HC}^\bullet(\mathcal{C}^G) &\simeq \underline{\text{Map}}_{\text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})^G}(\text{id}_{\mathcal{C}}, \bigoplus_{g \in G} \phi_g) \\ &\simeq \left(\bigoplus_{g \in G} \underline{\text{Map}}_{\text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(\text{id}_{\mathcal{C}}, \phi_g) \right)^G \\ &\simeq \left(\text{HC}^\bullet(\mathcal{C}) \oplus \bigoplus_{g \neq 1 \in G} \underline{\text{Map}}_{\text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(\text{id}_{\mathcal{C}}, \phi_g) \right)^G \end{aligned}$$

By our assumption on the characteristic of \mathbf{k} , the operation of taking group invariants commutes with taking cohomology. So by taking cohomology, the theorem follows. \square

5. THE ACTION OF A SERRE FUNCTOR ON HOCHSCHILD COHOMOLOGY

Our goal in this section is to prove Theorem 1.2. We start by recalling the definition of a Serre functor in our setting.

5.1. Serre functors. A \mathbf{k} -linear stable ∞ -category \mathcal{C} is *proper* if for all $X, Y \in \mathcal{C}$, the mapping object $\underline{\text{Map}}_{\mathcal{C}}(X, Y)$ lies in the essential image of $\text{Vect}_{\mathbf{k}}^{\text{fd}} \rightarrow \text{Vect}_{\mathbf{k}}$, i.e. if its total cohomology $\bigoplus_i H^i(\underline{\text{Map}}_{\mathcal{C}}(X, Y))$ is finite-dimensional. In this situation, we have functors

$$\begin{aligned} \underline{\text{Map}}_{\mathcal{C}}: \mathcal{C}^{\text{op}} \times \mathcal{C} &\rightarrow \text{Vect}_{\mathbf{k}}^{\text{fd}} & \underline{\text{Map}}_{\mathcal{C}}^{\text{op}}: \mathcal{C}^{\text{op}} \times \mathcal{C} &\rightarrow (\text{Vect}_{\mathbf{k}}^{\text{fd}})^{\text{op}} \\ (X, Y) &\mapsto \underline{\text{Map}}_{\mathcal{C}}(X, Y) & (X, Y) &\mapsto \underline{\text{Map}}_{\mathcal{C}}(Y, X) \end{aligned}$$

Note that there is an equivalence $(-)^{\vee}: (\text{Vect}_{\mathbf{k}}^{\text{fd}})^{\text{op}} \xrightarrow{\sim} \text{Vect}_{\mathbf{k}}^{\text{fd}}$ given by dualization of complexes. Here and below, given an ∞ -category \mathcal{C} , we denote by \mathcal{C}^{op} its opposite category.

Definition 5.1. Let \mathcal{C} be a proper \mathbf{k} -linear stable ∞ -category. A *Serre functor* for \mathcal{C} is an autoequivalence $S_{\mathcal{C}}: \mathcal{C} \xrightarrow{\sim} \mathcal{C}$, such that there is a commutative diagram

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} \times \mathcal{C} & \xrightarrow{\text{id} \times S_{\mathcal{C}}} & \mathcal{C}^{\text{op}} \times \mathcal{C} \\ \underline{\text{Map}}_{\mathcal{C}}^{\text{op}} \downarrow & & \downarrow \underline{\text{Map}}_{\mathcal{C}} \\ (\text{Vect}_{\mathbf{k}}^{\text{fd}})^{\text{op}} & \xrightarrow{(-)^{\vee}} & \text{Vect}_{\mathbf{k}}^{\text{fd}} \end{array}$$

In other words, a Serre functor for \mathcal{C} is characterized by the existence of natural equivalences

$$\underline{\text{Map}}_{\mathcal{C}}(X, S_{\mathcal{C}}(Y)) \simeq \underline{\text{Map}}_{\mathcal{C}}(Y, X)^{\vee}$$

for all objects $X, Y \in \mathcal{C}$. Note that if $S_{\mathcal{C}}$ is a Serre functor for \mathcal{C} , then it induces an autoequivalence of the homotopy category $\text{h}\mathcal{C}$, which is a Serre functor in the usual sense of triangulated categories, as defined in [2].

Remark 5.2. If X is a smooth proper variety, then $\text{Perf}(X)$ admits a Serre functor, given by the usual formula (see [4, Section 9], where Serre duality is developed in a much more general setup).

5.2. Action of the Serre functor. Now we turn to the proof of Theorem 1.2. We need the following lemma, which can be verified by unwinding the definitions.

Lemma 5.3. *Let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be functors between \mathbf{k} -linear stable ∞ -categories, and let $a: F \rightarrow G$ be a point of the mapping space $\text{Map}_{\text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{D})}(F, G)$. Assume F and G admit right adjoints $F^!$ and $G^!$. Let $\eta_F: \text{id}_{\mathcal{C}} \rightarrow F^! \circ F$ be the unit of the adjunction between F and $F^!$, and let $\epsilon_G: G \circ G^! \rightarrow \text{id}_{\mathcal{D}}$ be the counit of the adjunction between G and $G^!$. Define $a^!: G^! \rightarrow F^!$ as the composition*

$$a^!: G^! = \text{id}_{\mathcal{C}} \circ G^! \xrightarrow{\eta_F G^!} F^! \circ F \circ G^! \xrightarrow{F^! a G^!} F^! \circ G \circ G^! \xrightarrow{F^! \epsilon_G} F^! \circ \text{id}_{\mathcal{D}} = F^!.$$

Then there is functorially in $X, Y \in \mathcal{C}$ a commutative diagram

$$\begin{array}{ccc} \underline{\text{Map}}_{\mathcal{D}}(F(X), Y) & \longleftarrow & \underline{\text{Map}}_{\mathcal{D}}(G(X), Y) \\ \wr \downarrow & & \downarrow \wr \\ \underline{\text{Map}}_{\mathcal{C}}(X, F^!(Y)) & \longleftarrow & \underline{\text{Map}}_{\mathcal{C}}(X, G^!(Y)) \end{array}$$

where the top horizontal arrow is induced by $a: F \rightarrow G$, the bottom horizontal arrow is induced by $a^!: G^! \rightarrow F^!$, and the vertical arrows are given by adjunction.

It is well known that a Serre functor commutes with all autoequivalences. The following result gives a sense in which this commutation is functorial with respect to morphisms of autoequivalences.

Lemma 5.4. *Let \mathcal{C} be a proper \mathbf{k} -linear stable ∞ -category, which admits a Serre functor $S_{\mathcal{C}}$. Let $F, G: \mathcal{C} \rightarrow \mathcal{C}$ be autoequivalences of \mathcal{C} . Let $a: F \rightarrow G$ be an object of $\text{Map}_{\text{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(F, G)$, and let $(a^!)^!: F \rightarrow G$ be the morphism obtained by applying the construction of Lemma 5.3 twice (note that F and its inverse F^{-1} are mutually left and right adjoint, and similarly for G and G^{-1}). Then there is a commutative diagram of functors*

$$\begin{array}{ccc} S_{\mathcal{C}} \circ F & \xrightarrow{S_{\mathcal{C}} a} & S_{\mathcal{C}} \circ G \\ \wr \downarrow & & \downarrow \wr \\ F \circ S_{\mathcal{C}} & \xrightarrow{(a^!)^! S_{\mathcal{C}}} & G \circ S_{\mathcal{C}} \end{array}$$

where the vertical arrows are equivalences.

Proof. We have functorially in $X, Y \in \mathcal{C}$ a diagram

$$\begin{array}{ccc}
\underline{\mathrm{Map}}_{\mathcal{C}}(X, S_{\mathcal{C}} \circ F(Y)) & \longrightarrow & \underline{\mathrm{Map}}_{\mathcal{C}}(X, S_{\mathcal{C}} \circ G(Y)) \\
\wr \downarrow & & \downarrow \wr \\
\underline{\mathrm{Map}}_{\mathcal{C}}(F(Y), X)^{\vee} & \longrightarrow & \underline{\mathrm{Map}}_{\mathcal{C}}(G(Y), X)^{\vee} \\
\wr \downarrow & & \downarrow \wr \\
\underline{\mathrm{Map}}_{\mathcal{C}}(Y, F^{-1}(X))^{\vee} & \longrightarrow & \underline{\mathrm{Map}}_{\mathcal{C}}(Y, G^{-1}(X))^{\vee} \\
\wr \downarrow & & \downarrow \wr \\
\underline{\mathrm{Map}}_{\mathcal{C}}(F^{-1}(X), S(Y)) & \longrightarrow & \underline{\mathrm{Map}}_{\mathcal{C}}(G^{-1}(X), S_{\mathcal{C}}(Y)) \\
\wr \downarrow & & \downarrow \wr \\
\underline{\mathrm{Map}}_{\mathcal{C}}(X, F \circ S(Y)) & \longrightarrow & \underline{\mathrm{Map}}_{\mathcal{C}}(X, G \circ S_{\mathcal{C}}(Y)).
\end{array}$$

The first and third vertical equivalences in the diagram are given by the defining equivalences of a Serre functor, and the second and fourth are given by the adjunctions between F and F^{-1} and G and G^{-1} . The first and second horizontal morphisms are induced by $a: F \rightarrow G$, the third and fourth by $a^!: G^{-1} \rightarrow F^{-1}$, and the last by $(a^!)^!: F \rightarrow G$. The first and third squares in the diagram commute since the defining equivalences of a Serre functor are functorial, and the second and fourth squares commute by Lemma 5.3. Hence the diagram is commutative. The outer square in the diagram thus induces the desired diagram of functors. \square

Proof of Theorem 1.2. By the definition of the action of $S_{\mathcal{C}}$ on $\mathrm{HH}^{\bullet}(\mathcal{C})$, it suffices to show that for any $a: \mathrm{id}_{\mathcal{C}} \rightarrow \mathrm{id}_{\mathcal{C}} \in \mathrm{HC}^{\bullet}(\mathcal{C})$, there is a commutative diagram

$$\begin{array}{ccc}
S_{\mathcal{C}} \circ \mathrm{id}_{\mathcal{C}} & \xrightarrow{S_{\mathcal{C}}a} & S_{\mathcal{C}} \circ \mathrm{id}_{\mathcal{C}} \\
\mathrm{id} \downarrow & & \downarrow \mathrm{id} \\
\mathrm{id}_{\mathcal{C}} \circ S_{\mathcal{C}} & \xrightarrow{aS_{\mathcal{C}}} & \mathrm{id}_{\mathcal{C}} \circ S_{\mathcal{C}}
\end{array}$$

This follows from Lemma 5.4 with $F = G = \mathrm{id}_{\mathcal{C}}$. Indeed, in this case it is easy to see that $(a^!)^! = a$ and that the equivalences $S_{\mathcal{C}} \circ F \simeq F \circ S_{\mathcal{C}}$ and $S_{\mathcal{C}} \circ G \simeq G \circ S_{\mathcal{C}}$ constructed in the proof of the lemma are given by the identity. \square

6. APPLICATION TO FRACTIONAL CALABI–YAU CATEGORIES

In this section, we deduce Corollary 1.3 from the introduction (restated as Corollary 6.3 below), and apply it to the example of quartic fourfolds.

6.1. Hochschild cohomology in the $\mathbf{Z}/2$ fractional Calabi–Yau case. In the statement of Corollary 1.3, the Hochschild homology $\mathrm{HH}_{\bullet}(\mathcal{C})$ of \mathcal{C} appears. In our context, this invariant can be defined as follows.

Definition 6.1. Let \mathcal{C} be a proper \mathbf{k} -linear stable ∞ -category admitting a Serre functor $S_{\mathcal{C}}$. The *Hochschild chain complex* of \mathcal{C} is defined as

$$\mathrm{HC}_{\bullet}(\mathcal{C}) = \underline{\mathrm{Map}}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(\mathrm{id}_{\mathcal{C}}, S_{\mathcal{C}}) \in \mathrm{Vect}_{\mathbf{k}}.$$

The *Hochschild homology* $\mathrm{HH}_\bullet(\mathcal{C})$ of \mathcal{C} is the cohomology of this complex.

Remark 6.2. If \mathcal{C} is a Calabi–Yau category, say $S_{\mathcal{C}} = [n]$, then it follows immediately from our definitions that there is an isomorphism of graded vector spaces $\mathrm{HH}^\bullet(\mathcal{C}) \cong \mathrm{HH}_\bullet(\mathcal{C})[-n]$.

Corollary 6.3. *Let \mathcal{C} be a proper \mathbf{k} -linear stable ∞ -category such that $S_{\mathcal{C}} = \sigma \circ [n]$ is a Serre functor, where n is an integer and σ is the autoequivalence corresponding to the generator of a $\mathbf{Z}/2$ -action on \mathcal{C} . Assume the characteristic of \mathbf{k} is not 2. Then there is an isomorphism*

$$\mathrm{HH}^\bullet(\mathcal{C}^{\mathbf{Z}/2}) \cong \mathrm{HH}^\bullet(\mathcal{C}) \oplus (\mathrm{HH}_\bullet(\mathcal{C})^{\mathbf{Z}/2}[-n]),$$

where $\mathbf{Z}/2$ acts on $\mathrm{HH}_\bullet(\mathcal{C})$ via conjugation by σ on $\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})$.

Proof. Theorem 4.3 gives

$$\mathrm{HH}^\bullet(\mathcal{C}^{\mathbf{Z}/2}) \cong \mathrm{HH}^\bullet(\mathcal{C})^{\mathbf{Z}/2} \oplus \mathrm{H}^\bullet(\mathrm{Map}_{\mathrm{Fun}_{\mathbf{k}}(\mathcal{C}, \mathcal{C})}(\mathrm{id}_{\mathcal{C}}, \sigma))^{\mathbf{Z}/2}.$$

Since $\sigma = S_{\mathcal{C}} \circ [-n]$ the second summand is as claimed, and it suffices to see that σ acts trivially on $\mathrm{HH}^\bullet(\mathcal{C})$. But the shift functor $[-n]$ clearly acts trivially on $\mathrm{HH}^\bullet(\mathcal{C})$, and so does $S_{\mathcal{C}}$ by Theorem 1.2. \square

The invariant category $\mathcal{C}^{\mathbf{Z}/2}$ appearing in Corollary 6.3 is Calabi–Yau. More generally:

Lemma 6.4. *Let \mathcal{C} be a proper \mathbf{k} -linear stable ∞ -category such that $S_{\mathcal{C}} = \sigma \circ [n]$ is a Serre functor, where n is an integer and σ is the autoequivalence corresponding to the generator of a \mathbf{Z}/q -action on \mathcal{C} . Assume q is coprime to the characteristic of \mathbf{k} . Then the shift functor $[n]: \mathcal{C}^{\mathbf{Z}/q} \rightarrow \mathcal{C}^{\mathbf{Z}/q}$ is a Serre functor for $\mathcal{C}^{\mathbf{Z}/q}$.*

Proof. Let $G = \mathbf{Z}/q$. Note that \mathcal{C}^G is proper, by the description of the mapping spaces (3.1) and our assumption on the characteristic of \mathbf{k} . By the universal property of $\mathcal{C}^G \rightarrow \mathcal{C}$, the Serre functor $S_{\mathcal{C}} = \sigma \circ [n]$ induces an autoequivalence $S_{\mathcal{C}}^G: \mathcal{C}^G \rightarrow \mathcal{C}^G$. We show below that $S_{\mathcal{C}}^G$ is indeed a Serre functor for \mathcal{C}^G ; since the autoequivalence of \mathcal{C}^G induced by σ is equivalent to the identity, this will prove the lemma. More generally, the following argument proves that if \mathcal{C} has an action by a finite group G of order coprime to the characteristic of \mathbf{k} , and \mathcal{C} admits a Serre functor $S_{\mathcal{C}}$ such that the composition

$$\mathcal{C}^G \xrightarrow{p} \mathcal{C} \xrightarrow{S_{\mathcal{C}}} \mathcal{C}$$

lifts to an autoequivalence $S_{\mathcal{C}}^G: \mathcal{C}^G \rightarrow \mathcal{C}^G$, then $S_{\mathcal{C}}^G$ is a Serre functor for \mathcal{C}^G . The condition that $S_{\mathcal{C}} \circ p$ lifts should be automatic (it is closely related to Lemma 5.4), but we do not address this point here.

We have functorially in $\tilde{X}, \tilde{Y} \in \mathcal{C}^G$ equivalences

$$\mathrm{Map}_{\mathcal{C}^G}(\tilde{X}, S_{\mathcal{C}}^G(\tilde{Y})) \simeq \mathrm{Map}_{\mathcal{C}}(X, S_{\mathcal{C}}(Y))^G \simeq (\mathrm{Map}_{\mathcal{C}}(Y, X)^\vee)^G. \quad (6.1)$$

By functoriality of the defining equivalences of a Serre functor, the action of G on $\mathrm{Map}_{\mathcal{C}}(Y, X)^\vee$ is induced by the action of G on $\mathrm{Map}_{\mathcal{C}}(Y, X)$. Moreover, by definition

$$\mathrm{Map}_{\mathcal{C}}(Y, X)^\vee = \mathrm{Map}_{\mathrm{Vect}_{\mathbf{k}}}(\mathrm{Map}_{\mathcal{C}}(Y, X), \mathbf{k}),$$

so since the formation of the mapping space $\mathrm{Map}_{\mathrm{Vect}_{\mathbf{k}}}(-, -)$ takes colimits in the first variable to limits, we find

$$(\mathrm{Map}_{\mathcal{C}}(Y, X)^\vee)^G \simeq (\mathrm{Map}_{\mathcal{C}}(Y, X)_G)^\vee. \quad (6.2)$$

Further, we have

$$\underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)_G \simeq \underline{\mathrm{Map}}_{\mathcal{C}}(Y, X)^G \simeq \underline{\mathrm{Map}}_{\mathcal{C}_G}(\tilde{Y}, \tilde{X}), \quad (6.3)$$

where the first equivalence is given by the norm map (using the assumption on the characteristic of \mathbf{k}) and the second by (3.1). Finally, combining (6.1)–(6.3) gives the required functorial equivalences

$$\underline{\mathrm{Map}}_{\mathcal{C}_G}(\tilde{X}, \mathrm{S}_{\mathcal{C}}^G(\tilde{Y})) \simeq \underline{\mathrm{Map}}_{\mathcal{C}_G}(\tilde{Y}, \tilde{X})^\vee. \quad \square$$

Remark 6.5. In the situation of Lemma 6.4, the category $\mathcal{C}^{\mathbf{Z}/q}$ should be regarded as the “canonical Calabi–Yau cover” of \mathcal{C} . Indeed, assume X is a smooth proper variety of dimension n , whose canonical bundle satisfies $\omega_X^q \cong \mathcal{O}_X$. Then $\mathcal{C} = \mathrm{Perf}(X)$ satisfies the assumptions of Lemma 6.4 with \mathbf{Z}/q -action given by tensoring with ω_X , and the invariant category $\mathcal{C}^{\mathbf{Z}/q}$ recovers the derived category of the canonical Calabi–Yau cover of X . Namely, there is a q -fold étale cover $\tilde{X} = \mathrm{Spec}_X(\mathcal{R}) \rightarrow X$, where

$$\mathcal{R} = \mathcal{O}_X \oplus \omega_X \oplus \cdots \oplus \omega_X^{q-1}$$

with algebra structure determined by $\omega_X^q \cong \mathcal{O}_X$. The variety \tilde{X} is Calabi–Yau in the sense that $\omega_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$, and there is an equivalence $\mathcal{C}^{\mathbf{Z}/q} \simeq \mathrm{Perf}(\tilde{X})$ (see [3, Theorem 1.1]).

6.2. Quartic fourfolds. We assume for this section that the characteristic of \mathbf{k} is not 2, 3, or 5; we state explicitly where this assumption is used below. Let $X \subset \mathbf{P}^5$ be a smooth quartic fourfold over \mathbf{k} , and let $Y \rightarrow \mathbf{P}^5$ be the double cover of \mathbf{P}^5 branched along X . Let $\mathcal{C}_X \subset \mathrm{Perf}(X)$ and $\mathcal{C}_Y \subset \mathrm{Perf}(Y)$ be the \mathbf{k} -linear stable ∞ -categories defined by the semiorthogonal decompositions

$$\mathrm{Perf}(X) = \langle \mathcal{C}_X, \mathcal{O}_X, \mathcal{O}_X(1) \rangle, \quad (6.4)$$

$$\mathrm{Perf}(Y) = \langle \mathcal{C}_Y, \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2), \mathcal{O}_Y(3) \rangle. \quad (6.5)$$

The following result guarantees the conditions of Corollary 6.3 hold for \mathcal{C}_X , and identifies the category of $\mathbf{Z}/2$ -invariants with \mathcal{C}_Y .

Lemma 6.6. *There is an autoequivalence σ of \mathcal{C}_X corresponding to the generator of a $\mathbf{Z}/2$ -action, such that:*

- (1) $\mathrm{S}_{\mathcal{C}_X} = \sigma \circ [3]$ is a Serre functor for \mathcal{C}_X .
- (2) There is an equivalence $\mathcal{C}_X^{\mathbf{Z}/2} \simeq \mathcal{C}_Y$.

Further, $\mathrm{S}_{\mathcal{C}_Y} = [3]$ is a Serre functor for \mathcal{C}_Y .

Proof. Part (1) is a special case of [8, Corollary 4.1]. More precisely, the proof of [8, Theorem 3.5] shows we can take $\sigma = \mathcal{O}_X^2[-1]$, where $\mathcal{O}_X: \mathcal{C}_X \rightarrow \mathcal{C}_X$ is the “rotation functor”. Now part (2) follows from [9, Proposition 7.10] (we use $\mathrm{char}(\mathbf{k}) \neq 2$ here). Strictly speaking, these results are stated in the references at the level of homotopy categories of \mathcal{C}_X and \mathcal{C}_Y , but they also hold at the level of stable ∞ -categories. Finally, the statement about the Serre functor of \mathcal{C}_Y follows from Lemma 6.4; alternatively, it holds by [8, Corollary 4.6]. \square

To study $\mathrm{HH}^\bullet(\mathcal{C}_X)$ using Corollary 6.3, we need the following computations of $\mathrm{HH}_\bullet(\mathcal{C}_X)$ and $\mathrm{HH}^\bullet(\mathcal{C}_Y)$.

Lemma 6.7. *There are isomorphisms of graded vector spaces*

$$\mathrm{HH}_\bullet(\mathcal{C}_X) \cong \mathbf{k}^{21}[2] \oplus \mathbf{k}^{144} \oplus \mathbf{k}^{21}[-2], \quad (6.6)$$

$$\mathrm{HH}^\bullet(\mathcal{C}_Y) \cong \mathbf{k} \oplus \mathbf{k}^{90}[-2] \oplus \mathbf{k}^2[-3] \oplus \mathbf{k}^{90}[-4] \oplus \mathbf{k}[-6]. \quad (6.7)$$

Proof. It is straightforward to check that the Hodge diamonds of X and Y are, respectively, as follows:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ & 0 & 21 & 142 & 21 & 0 & \\ & 0 & 0 & 0 & 0 & 0 & \\ & & 0 & 1 & 0 & & \\ & & & 0 & 1 & & \end{array} \quad \text{and} \quad \begin{array}{ccccccc} & & & & 1 & & \\ & & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 90 & 1 & 90 & 1 & 0 \\ & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & 0 & 1 & 0 & & \\ & & & & 0 & 1 & & \end{array}$$

The Hochschild–Kostant–Rosenberg theorem allows us to compute from this the Hochschild homology of X and Y . (Note that this theorem holds for a variety over a field of positive characteristic provided the dimension is less than the characteristic, and hence applies to X and Y by our assumption on $\mathrm{char}(\mathbf{k})$.) The result is:

$$\mathrm{HH}_\bullet(X) \cong \mathbf{k}^{21}[2] \oplus \mathbf{k}^{146} \oplus \mathbf{k}^{21}[-2],$$

$$\mathrm{HH}_\bullet(Y) \cong \mathbf{k}[3] \oplus \mathbf{k}^{90}[1] \oplus \mathbf{k}^6 \oplus \mathbf{k}^{90}[-1] \oplus \mathbf{k}[-3].$$

By the additivity of Hochschild homology under semiorthogonal decompositions (see [6]), the defining semiorthogonal decompositions (6.4) and (6.5), and the isomorphism of graded vector spaces $\mathrm{HH}^\bullet(\mathcal{C}_Y) \cong \mathrm{HH}_\bullet(\mathcal{C}_Y)[-3]$ (see Remark 6.2), we obtain the desired isomorphisms (6.6) and (6.7). \square

Putting the above together with Corollary 6.3, we obtain:

Lemma 6.8. *The canonical map $\mathrm{HH}^\bullet(\mathcal{C}_X) \rightarrow \mathrm{HH}^\bullet(\mathcal{C}_Y)$ given by Corollary 6.3 is an isomorphism in degrees not equal to 3. Explicitly, we have*

$$\mathrm{HH}^\bullet(\mathcal{C}_X) \cong \mathbf{k} \oplus \mathbf{k}^{90}[-2] \oplus \mathbf{k}^d[-3] \oplus \mathbf{k}^{90}[-4] \oplus \mathbf{k}[-6]$$

for some $0 \leq d \leq 2$.

Proof. By Corollary 6.3 combined with Lemma 6.6, there is a splitting

$$\mathrm{HH}^\bullet(\mathcal{C}_Y) \cong \mathrm{HH}^\bullet(\mathcal{C}_X) \oplus (\mathrm{HH}_\bullet(\mathcal{C}_X)^{\mathbf{Z}/2}[-3]).$$

Now the result follows for degree reasons from the computation of Lemma 6.7. \square

Remark 6.9. The value of d in Lemma 6.8 would follow from a better understanding of the action of $\mathbf{Z}/2$ on $\mathrm{HH}_0(\mathcal{C}_X)$. Namely, the above proof shows that $\mathrm{HH}_\bullet(\mathcal{C}_X)^{\mathbf{Z}/2} \cong \mathbf{k}^{2-d}$.

Finally, we explain how Lemma 6.8 implies Proposition 1.4 from the introduction, using the results of [7]. First, we note that there is a canonical restriction morphism

$$\mathrm{HH}^\bullet(X) \rightarrow \mathrm{HH}^\bullet(\mathcal{C}_X),$$

see [7, Section 3.1]. By the Hochschild–Kostant–Rosenberg theorem, there is a canonical decomposition

$$\mathrm{HH}^2(X) \cong \mathrm{H}^0(X, \wedge^2 T_X) \oplus \mathrm{H}^1(X, T_X) \oplus \mathrm{H}^2(X, \mathcal{O}_X).$$

We aim to show that the composition

$$\mathrm{H}^1(X, T_X) \rightarrow \mathrm{HH}^2(X) \rightarrow \mathrm{HH}^2(\mathcal{C}_X)$$

is an isomorphism. In fact, we show slightly more:

Proposition 6.10. *Both maps $H^1(X, T_X) \rightarrow HH^2(X)$ and $HH^2(X) \rightarrow HH^2(\mathbb{C}_X)$ are isomorphisms.*

Proof. We have

$$\dim H^1(X, T_X) = 90 = \dim HH^2(\mathbb{C}_X).$$

The first equality holds by an easy computation, and the second by Lemma 6.8. Hence the claim amounts to the injectivity of $HH^2(X) \rightarrow HH^2(\mathbb{C}_X)$. An easy computation shows that the pseudoheight of the exceptional collection $\mathcal{O}_X, \mathcal{O}_X(1) \in \text{Perf}(X)$, as defined in [7, Definition 4.4], is equal to 3. Thus [7, Corollary 4.6] gives the injectivity of $HH^2(X) \rightarrow HH^2(\mathbb{C}_X)$. \square

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