NONCOMMUTATIVE HOMOLOGICAL PROJECTIVE DUALITY

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Abstract. We generalize Kuznetsov's theory of homological projective duality to the setting of noncommutative algebraic geometry. Simultaneously, we develop the theory over general base schemes and remove the usual transversality hypotheses.

Contents

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1. Introduction

The purpose of this paper is twofold. The first goal is to set up a robust framework for noncommutative algebraic geometry, using recent advances in higher category theory. The second and main goal is to generalize Kuznetsov's theory [9] of homological projective duality (HPD) to this setting. These results are applied in our work [14] with Kuznetsov on categorical joins, which was one of the main motivations for this paper.

In this introduction, we focus on the HPD part of our work.

1.1. **Kuznetsov's HPD.** The input for the theory of HPD is:

– A smooth projective variety X over an algebraically closed field of characteristic 0, with a morphism $X \to \mathbf{P}(V)$ to a projective space.
– A right Lefschetz decomposition of the category $\mathrm{Perf}(X)$ of perfect complexes.

Let $\mathcal{O}_X(H)$ denote the pullback of $\mathcal{O}_{\mathbf{P}(V)}(1)$. A right Lefschetz decomposition of $\mathrm{Perf}(X)$ is a semiorthogonal decomposition of the form

$$\mathrm{Perf}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle ,$$

where $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1}$ is a descending chain of categories and $\mathcal{A}_i(iH)$ denotes the image of $\mathcal{A}_i$ under the autoequivalence of $\mathrm{Perf}(X)$ given by tensoring with $\mathcal{O}_X(iH)$. The key property of such a decomposition — from which its name derives, by analogy with the Lefschetz hyperplane theorem — is that it induces semiorthogonal decompositions of the base changes of X along linear subspaces $L \subset V$. More precisely, let s denote the codimension of L in V. Then if $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$ has the expected dimension $\dim(X) - s$, pullback along $X \times_{\mathbf{P}(V)} \mathbf{P}(L) \to X$ embeds the categories $\mathcal{A}_i(iH)$ into $\mathrm{Perf}\big(X \times_{\mathbf{P}(V)} \mathbf{P}(L)\big)$ for $i \geq s$, and there is a semiorthogonal decomposition

$$\mathrm{Perf}(X \times_{\mathbf{P}(V)} \mathbf{P}(L)) = \langle \mathcal{K}_L(X), \mathcal{A}_s(H), \mathcal{A}_{s+1}(2H), \ldots, \mathcal{A}_{m-1}((m-s)H) \rangle . \tag{1.1}$$

The category $\mathcal{K}_L(X)$ should be thought of as the "interesting component" of the category of perfect complexes on $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$ — it is what is left after removing the pieces coming from the ambient variety X.

Given the above data, Kuznetsov constructs the *HPD category* $\mathrm{Perf}(X)^\vee$, which is a $\mathbf{P}(V^\vee)$-linear category that can be thought of as a total space for the categories $\mathcal{K}_L(X)$ as L ranges over the hyperplanes in V. The main theorem of [9] shows that if $\mathrm{Perf}(X)^\vee$ is geometric, i.e. if there exists a variety Y together with a morphism $Y \to \mathbf{P}(V^\vee)$ and a $\mathbf{P}(V^\vee)$-linear equivalence $\mathrm{Perf}(Y) \simeq \mathrm{Perf}(X)^\vee$, then $\mathrm{Perf}(Y)$ has a natural Lefschetz decomposition such that the "interesting components" of orthogonal linear sections of X and Y are equivalent. To be precise, we must also assume that the Lefschetz decomposition of X satisfies a mild nondegeneracy condition, namely that its length m satisfies $m < \dim(V)$. Then if $\mathcal{O}_Y(H')$ denotes the pullback of $\mathcal{O}_{\mathbf{P}(V^\vee)}(1)$, the theorem is as follows.

Theorem 1.1 ([9]). *The variety Y is smooth and $\mathrm{Perf}(Y)$ admits a natural left Lefschetz decomposition, i.e. a semiorthogonal decomposition*

$$\mathrm{Perf}(Y) = \big\langle \mathcal{B}_{n-1}(-(n-1)H'), \ldots, \mathcal{B}_1(-H'), \mathcal{B}_0 \big\rangle$$

where $\mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_{n-1}$, such that the following holds. Let $L \subset V$ be a linear subspace, let $L^\perp = \ker(V^\vee \to L^\vee)$ be its orthogonal, and let $r = \dim(L)$ and $s = \dim(L^\perp)$. Assume the fiber products $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$ and $Y \times_{\mathbf{P}(V^\vee)} \mathbf{P}(L^\perp)$ are of expected dimension. Let

$$\mathrm{Perf}(Y \times_{\mathbf{P}(V^\vee)} \mathbf{P}(L^\perp)) = \big\langle \mathcal{B}_{n-1}(-(n-r)H'), \ldots, \mathcal{B}_{r+1}(-2H'), \mathcal{B}_r(-H'), \mathcal{K}_{L^\perp}(Y) \big\rangle \tag{1.2}$$

be the induced semiorthogonal decomposition analogous to (1.1). *Then there is an equivalence of categories*

$$\mathcal{K}_L(X) \simeq \mathcal{K}_{L^\perp}(Y).$$

Remark 1.2. Kuznetsov also proves a version of the theorem for the bounded derived categories of coherent sheaves on $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$ and $Y \times_{\mathbf{P}(V^\vee)} \mathbf{P}(L^\perp)$. Namely, these categories have semiorthogonal decompositions analogous to (1.1) and (1.2), with equivalent "interesting components".

Theorem 1.1 is the source of many semiorthogonal decompositions and derived equivalences in algebraic geometry. For a survey of such results, we refer to [12, 24].

Finally, we note that the HPD category is closely related to classical projective duality. Namely, define the projective dual $X^\vee \subset \mathbf{P}(V^\vee)$ of $X \to \mathbf{P}(V)$ to be the locus of hyperplanes $H \in \mathbf{P}(V^\vee)$ such that the base change $X \times_{\mathbf{P}} H$ is singular; this reduces to the usual notion when $X \to \mathbf{P}(V^\vee)$ is a closed immersion. Then Kuznetsov shows [9, Theorem 7.9] that for Y as above, the locus of critical values of the morphism $Y \to \mathbf{P}(V^\vee)$ coincides with $X^\vee$.

1.2. **Results.** In practice, the HPD category $\mathrm{Perf}(X)^\vee$ is often *not* geometric in the above sense. Instead, usually the best one can hope for is that $\mathrm{Perf}(X)^\vee$ is "close to geometric", e.g. equivalent to the bounded derived category of coherent $\mathcal{R}$-modules $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y, \mathcal{R})$ on a variety Y

equipped with a sheaf of finite (noncommutative) $\mathcal{O}_Y$-algebras $\mathcal{R}$ (see [12] for examples), or to the derived category of a gauged Landau-Ginzburg model (see [2] for examples).

In general, the HPD category $\mathrm{Perf}(X)^\vee$ only has the structure of a "noncommutative scheme" over $\mathbf{P}(V^\vee)$, in the sense of Kontsevich. More precisely, using Lurie's foundational work [17], we define a "noncommutative scheme" over a scheme S to be a small idempotent-complete stable ∞-category $\mathcal{C}$, equipped with a $\mathrm{Perf}(S)$-module structure; for short, we say $\mathcal{C}$ is an *S-linear category*. To orient the reader, we note that the category of perfect complexes on an S-scheme is an S-linear category.

The first goal of this paper is to develop some foundational material about linear categories. We leave the detailed discussion of our results in this direction to the main text.

The second and main goal is to generalize HPD to the case of categories linear over a fixed (quasi-compact and separated) base scheme S. Namely, for a vector bundle V on S, we develop a theory of HPD where:

(1) The input variety X itself is replaced with a $\mathbf{P}(V)$-linear category.
(2) There are no geometricity assumptions on the $\mathbf{P}(V^\vee)$-linear HPD category.

The existence of such a version of HPD is essential for our work [14] with Kuznetsov, which was one of the main motivations for this paper.

The input for our theory is a *nondegenerate right Lefschetz category over* $\mathbf{P}(V)$, which consists of the following data:

– A $\mathbf{P}(V)$-linear category $\mathcal{C}$.
– A nondegenerate right Lefschetz decomposition of $\mathcal{C}$.

As above, a right Lefschetz decomposition means a semiorthogonal decomposition

$$\mathcal{C} = \langle \mathcal{A}_0, \mathcal{A}_1(H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle$$

where $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1}$, and nondegeneracy means $m < \mathrm{rank}(V)$. Given any subbundle $L \subset V$, Lurie's work [19, 17] can be used to make sense of the base change category

$$\mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{P}(L)),$$

which plays the role of the linear section of $\mathcal{C}$ by $\mathbf{P}(L)$. Indeed, the results of [3] imply that if there exists a morphism of schemes $X \to \mathbf{P}(V)$ and a $\mathbf{P}(V)$-linear equivalence $\mathrm{Perf}(X) \simeq \mathcal{C}$, then the above base changed category recovers $\mathrm{Perf}(X \times_{\mathbf{P}(V)} \mathbf{P}(L))$.[1] Further, there is an induced semiorthogonal decomposition

$$\mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P}(V))} \mathrm{Perf}(\mathbf{P}(L)) = \langle \mathcal{K}_L(\mathcal{C}), \mathcal{A}_s(H), \mathcal{A}_{s+1}(2H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \tag{1.3}$$

where s denotes the corank of $L \subset V$.

We construct a $\mathbf{P}(V^\vee)$-linear HPD category $\mathcal{C}^\vee$ in the above setup, and prove the following version of Theorem 1.1.

Theorem 1.3. *Assume $\mathcal{C}$ is smooth and proper over S. Then the category $\mathcal{C}^\vee$ is smooth and proper over S, and admits a natural left Lefschetz decomposition*

$$\mathcal{C}^\vee = \left\langle \mathcal{A}^\vee_{n-1}(-(n-1)H'), \dots, \mathcal{A}^\vee_1(-H'), \mathcal{A}^\vee_0 \right\rangle$$

[1]Here, $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$ denotes the derived fiber product, which agrees with the usual scheme-theoretic fiber product if X and $\mathbf{P}(L)$ are Tor-independent over $\mathbf{P}(V)$.

such that the following holds. Let $L \subset V$ be a subbundle, let $L^\perp = \ker(V^\vee \to L^\vee)$ be its orthogonal, and let $r = \mathrm{rank}(L)$ and $s = \mathrm{rank}(L^\perp)$. Let

$$\mathcal{C}^\vee \otimes_{\mathrm{Perf}(\mathbf{P}(V^\vee))} \mathrm{Perf}(\mathbf{P}(L^\perp)) = \left\langle \mathcal{A}^\vee_{n-1}(-(n-r)H'), \dots, \mathcal{A}^\vee_{r+1}(-2H'), \mathcal{A}^\vee_r(-H'), \mathcal{K}_{L^\perp}(\mathcal{C}^\vee) \right\rangle,$$

be the induced semiorthogonal decomposition analogous to (1.3)*. Then there is an equivalence of categories*

$$\mathcal{K}_L(\mathcal{C}) \simeq \mathcal{K}_{L^\perp}(\mathcal{C}^\vee).$$

If the categories $\mathcal{C}$ and $\mathcal{C}^\vee$ are geometric, i.e. if $\mathcal{C} \simeq \mathrm{Perf}(X)$ for a $\mathbf{P}(V)$-scheme X and $\mathcal{C}^\vee \simeq \mathrm{Perf}(Y)$ for a $\mathbf{P}(V^\vee)$-scheme Y, then Theorem 1.3 recovers Theorem 1.1. In fact, using [3], Theorem 1.3 implies Theorem 1.1 holds without any expected dimension assumptions, provided the fiber products $X \times_{\mathbf{P}(V)} \mathbf{P}(L)$ and $Y \times_{\mathbf{P}(V^\vee)} \mathbf{P}(L^\perp)$ are taken in the derived sense; this answers a question left open in [9]. Further, using [4] we show that when the base scheme S is noetherian and defined over a field of characteristic 0, Theorem 1.3 also implies a similar result for bounded derived categories of coherent sheaves, recovering the result mentioned in Remark 1.2.

As in [9], we deduce Theorem 1.3 from a stronger result (Theorem 8.7), which we call the *main theorem of HPD* — roughly, it says Theorem 1.3 holds in families as L varies in $\mathrm{Gr}(r, V)$. As a consequence, we prove that HPD is a duality, i.e. $(\mathcal{C}^\vee)^\vee \simeq \mathcal{C}$.

Our method of proof is closely modeled on [9], but there are several difficulties to overcome in our setting. First, it takes some work to even formulate the objects appearing in the proof. For instance, in the framework of [9] where $\mathcal{C} \simeq \mathrm{Perf}(X)$ is geometric, a key role is played by the spaces of universal linear sections of $X \to \mathbf{P}(V)$. Our basic observation here is that all of the necessary constructions can be made in the case where $\mathcal{C} = \mathrm{Perf}(\mathbf{P}(V))$, and then transported to general $\mathcal{C}$ by base change. A more serious obstacle is that Kuznetsov uses Fourier–Mukai kernels in an essential way. Namely, all of the functors arising in his proof are of the form $\Phi_{\mathcal{E}} \colon \mathrm{Perf}(Z_1) \to \mathrm{Perf}(Z_2)$, where Z_1 and Z_2 are schemes over a base scheme T and $\Phi_{\mathcal{E}}$ is the functor given by a Fourier–Mukai kernel $\mathcal{E} \in \mathrm{Perf}(Z_2 \times_T Z_1)$. There is a precise relationship between various operations on the kernel $\mathcal{E}$ (e.g. pullback along a morphism $f \times \mathrm{id} \colon Z_2' \times_T Z_1 \to Z_2 \times_T Z_1$ where $f \colon Z_2' \to Z_2$ is a morphism) and operations on the associated functor $\Phi_{\mathcal{E}}$ (e.g. composition with pullback along a morphism $f \colon Z_2' \to Z_2$). Kuznetsov uses this to deduce facts about functors $\Phi_{\mathcal{E}}$ via the geometry of the kernel spaces $Z_2 \times_T Z_1$. To import these ideas into our setting, we develop a robust formalism of Fourier–Mukai kernels for certain categories arising from base change. Throughout, we crucially use recent advances in higher category theory and derived algebraic geometry, especially [17, 3].

Finally, we show that our version of HPD is closely related to classical projective duality. Namely, for any category $\mathcal{D}$ linear over a base T, we introduce the notion of its *critical locus* $\mathrm{Crit}_T(\mathcal{D})$, which is the locus of points in T parameterizing the singular fibers of $\mathcal{D}$. For any $\mathbf{P}(V)$-linear category $\mathcal{C}$ which is smooth and proper over S, we use this notion to define the *classical projective dual* $\mathrm{CPD}(\mathcal{C}) \subset \mathbf{P}(V^\vee)$. If $\mathcal{C}$ is a right Lefschetz category, we prove $\mathrm{CPD}(\mathcal{C}) = \mathrm{Crit}_{\mathbf{P}(V^\vee)}(\mathcal{C}^\vee)$.

1.3. **Further directions.** We regard our results as part of a larger program of "homological projective geometry", whose goal is to find categorical analogs of results from classical projective geometry, and to bring them to bear on the structure of derived categories. In this theory, Lefschetz categories over $\mathbf{P}(V)$ play the role of projective varieties embedded in $\mathbf{P}(V)$. The prototype of homological projective geometry is HPD. In [14] we show the classical notion of the join of projective varieties also fits into this framework. Namely, we introduce categorical joins, and show they are related to HPD in the same way classical joins are related to classical projective duality. An interesting feature of homological projective geometry is that all of the operations which are known so far (i.e. HPD and categorical joins) preserve smoothness of the objects involved, whereas in classical projective geometry this is far from true (i.e. projective duals and joins of smooth varieties are usually singular).

In this paper, we show that HPD can be formulated over quite general base schemes, which need not be defined over a field. Working over a field, there are a number of examples where the HPD category admits a close-to-geometric model [12]; we believe many of these results should extend to more general base schemes.

Finally, we note that the theory developed here may hold over even more general bases, namely over spectral schemes in the sense of [18], e.g. over the sphere spectrum.

1.4. **Related work.** In his unpublished habilitation thesis [10], Kuznetsov developed a version of HPD which works for input categories $\mathcal{C}$ that are realizable (compatibly with their $\mathbf{P}(V)$-linear structure) as an admissible subcategory of the derived category of a variety. Due to my inadequate Russian I was not able to read [10], but its existence served as an inspiration for this paper.

More recently, HPD has been revisited from several points of view. In [2], Ballard–Deliu–Favero–Isik–Katzarkov use variation of GIT to realize in certain cases the HPD category as the derived category of a gauged Landau–Ginzburg model, and to give a new proof of Theorem 1.1 in these cases.

In [24], Thomas reproved Theorem 1.1 using a reinterpretation of Kuznetsov's original proof. He handles the case where $\mathrm{Perf}(X)^\vee$ need not be geometric, but X is required to be a genuine variety (not "noncommutative") and he works with a special class of Lefschetz decompositions (the "rectangular" ones).

During the preparation of this paper, two other works on HPD appeared. In [8], Jiang–Leung–Xie build on the argument from [24] to prove a generalization of Theorem 1.1, where the the pair $(\mathbf{P}(L), \mathbf{P}(L^\perp))$ is replaced by any HPD pair. We independently discovered (a more general form of) this result, which will appear as an application in [14].

Finally, in [22] Rennemo builds on [2] to develop a version of HPD for categories $\mathcal{C}$ that are admissible subcategories in the derived category of a smooth quotient stack.

Our results are stronger than the above in several respects: we handle linear categories without any geometricity assumptions; we work at the enhanced level of stable ∞-categories (which allows us to give natural derived algebraic geometry interpretations in non-transverse situations); and we work over general base schemes. Another novel feature of our work is our formalism of Fourier–Mukai kernels, which will be needed in [14].

1.5. **Organization of the paper.** Part I of this paper is dedicated to foundational material on noncommutative algebraic geometry. We begin in §2 by introducing our framework for linear categories. In §3 we discuss semiorthogonal decompositions of such categories. In §4 we define the notions of smoothness and properness of a linear category, and discuss their behavior under base change and semiorthogonal decompositions. We also introduce the notion of the critical locus of a linear category. In §5 we develop our formalism of Fourier–Mukai kernels.

In Part II of the paper, we use the material from Part I to formulate and prove our results on homological projective duality. First, in §6 we define Lefschetz categories and describe their behavior under passage to linear sections. In §7 we define the HPD category, explain its relation to classical projective duality, and modulo a generation statement show that it has a natural Lefschetz decomposition. Finally, in §8 we prove the main theorem of HPD (Theorem 8.7), and use it to deduce Theorem 1.3 and the other results from above.

1.6. **Conventions.** All schemes are assumed quasi-compact and separated, and S denotes a base scheme which is fixed throughout the paper. A vector bundle on a scheme X is a finite locally free $\mathcal{O}_X$-module. Given a vector bundle V on X, we denote by

$$\mathrm{Gr}(r, V) \to X$$

the Grassmannian parameterizing rank r *subbundles* of V, and we set $\mathbf{P}(V) = \mathrm{Gr}(1, V)$. In particular, with our conventions the pushforward of $\mathcal{O}_{\mathbf{P}(V)}(1)$ to X is $V^\vee$.

1.7. **Acknowledgements.** I thank Sasha Kuznetsov for introducing me to the theory of homological projective duality, and for many useful conversations surrounding this topic. This paper owes a great intellectual debt to his work [9], the ideas of which permeate Part II of this work. I am also grateful to Daniel Halpern-Leistner for helpful discussions about ∞-categories and derived algebraic geometry.

Part I. Noncommutative algebraic geometry

2. Linear categories

In this section, we introduce the notion of an S-linear category. These categories are discussed in §2.3, after some remarks on derived algebraic geometry in §2.1 and a review of some relevant aspects of the theory of stable ∞-categories in §2.2. We also discuss "large" (i.e. presentable) versions of S-linear categories, which are useful for technical reasons. Finally, in §2.4 we define the bounded coherent category of an S-linear category; under reasonable assumptions, this recovers the bounded derived category of coherent sheaves on an S-scheme from its category of perfect complexes.

2.1. **Derived algebraic geometry.** We work in the setting of derived algebraic geometry, see [18, 7]. This means we regard schemes as objects of the ∞-category of derived schemes. In particular given morphisms of schemes $X \to S$ and $Y \to S$, the symbol

$$X \times_S Y$$

denotes their *derived* fiber product. We note that this agrees with the usual fiber product of schemes whenever the morphisms $X \to S$ and $Y \to S$ are Tor-independent over S. In fact, the only time we need to leave the category of ordinary schemes is in §5 and Corollary 8.9, and even there derived schemes can be avoided at the cost of requiring Tor-independence of all fiber products.

2.2. **Stable ∞-categories.** We work with stable ∞-categories, as developed in [17]. Here we briefly review some key facts. We consider several classes of stable ∞-categories:

- The ∞-category $\mathrm{Cat}_{\mathrm{st}}$ of small idempotent-complete stable ∞-categories, with morphisms the exact functors. For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\mathrm{st}}$, we denote by $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ the ∞-category of exact functors from $\mathcal{C}$ to $\mathcal{D}$.
- The ∞-category $\mathrm{PrCat}_{\mathrm{st}}$ of presentable stable ∞-categories, with morphisms the cocontinuous functors (i.e. the functors that preserve small colimits). For $\mathcal{C}, \mathcal{D} \in \mathrm{PrCat}_{\mathrm{st}}$, we denote by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$ the ∞-category of cocontinuous functors from $\mathcal{C}$ to $\mathcal{D}$.
- The ∞-category $\mathrm{PrCat}_{\mathrm{st}}^{\omega}$ of compactly generated presentable stable ∞-categories, with morphisms the cocontinuous functors that preserve compact objects. For $\mathcal{C}, \mathcal{D} \in \mathrm{PrCat}_{\mathrm{st}}$, we denote by $\mathrm{Fun}^{\mathrm{L},\omega}(\mathcal{C}, \mathcal{D})$ the ∞-category of cocontinuous functors from $\mathcal{C}$ to $\mathcal{D}$ that preserve compact objects.

Remark 2.1. By definition, $\mathrm{PrCat}_{\mathrm{st}}^{\omega}$ is a (non-full) subcategory of $\mathrm{PrCat}_{\mathrm{st}}$.

We are primarily interested in $\mathrm{Cat}_{\mathrm{st}}$, but for technical reasons it is often convenient to work with $\mathrm{PrCat}_{\mathrm{st}}$; the category $\mathrm{PrCat}_{\mathrm{st}}^{\omega}$ mediates between these two. Namely, there is an *Ind-completion* functor [19, Section 5.5.7]

$$\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{st}} \to \mathrm{PrCat}_{\mathrm{st}}$$

which by construction factors through the inclusion $\mathrm{PrCat}_{\mathrm{st}}^{\omega} \to \mathrm{PrCat}_{\mathrm{st}}$, and induces an equivalence

$$\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{st}} \to \mathrm{PrCat}_{\mathrm{st}}^{\omega}$$

with inverse the functor

$$(-)^c\colon \mathrm{PrCat}_{\mathrm{st}}^{\omega} \to \mathrm{Cat}_{\mathrm{st}}$$

taking $\mathcal{C} \in \mathrm{PrCat}_{\mathrm{st}}^{\omega}$ to its subcategory $\mathcal{C}^c \subset \mathcal{C}$ of compact objects.

There is a natural way to form the "tensor product" of categories in $\mathrm{Cat}_{\mathrm{st}}, \mathrm{PrCat}_{\mathrm{st}}$, or $\mathrm{PrCat}_{\mathrm{st}}^{\omega}$ [17, Sections 4.8.1-4.8.2]. More precisely, $\mathrm{PrCat}_{\mathrm{st}}$ is a closed symmetric monoidal ∞-category, with product denoted $\mathcal{C} \otimes \mathcal{D}$ and internal mapping objects given by $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})$. The tensor product $\mathcal{C} \otimes \mathcal{D}$ is characterized by the universal property

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E}),$$

where $\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ (the category of all functors) spanned by functors which preserve small colimits separately in each variable. The category $\mathrm{PrCat}_{\mathrm{st}}^{\omega}$ inherits a closed symmetric monoidal structure from $\mathrm{PrCat}_{\mathrm{st}}$, with internal mapping objects given by $\mathrm{Fun}^{\mathrm{L},\omega}(\mathcal{C}, \mathcal{D})$. Via the equivalence $\mathrm{Cat}_{\mathrm{st}} \simeq \mathrm{PrCat}_{\mathrm{st}}$, we use this to equip $\mathrm{Cat}_{\mathrm{st}}$ with a closed symmetric monoidal structure, with internal mapping objects given by $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$. Explicitly, for $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{\mathrm{st}}$, the tensor product is given by the formula

$$\mathcal{C} \otimes \mathcal{D} \simeq (\mathrm{Ind}(\mathcal{C}) \otimes \mathrm{Ind}(\mathcal{D}))^{c}$$

and is characterized by the universal property

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C} \otimes \mathcal{D}, \mathcal{E}) \simeq \mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E}),$$

where $\mathrm{Fun}'(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ is the full subcategory of $\mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E})$ spanned by functors that are exact separately in each variable.

Derived categories of derived schemes give examples of categories in $\mathrm{PrCat}_{\mathrm{st}}$ and $\mathrm{Cat}_{\mathrm{st}}$. Namely, for X a derived scheme, there is a category $\mathrm{QCoh}(X) \in \mathrm{PrCat}_{\mathrm{st}}$ of unbounded quasi-coherent sheaves, and a full subcategory $\mathrm{Perf}(X) \in \mathrm{Cat}_{\mathrm{st}}$ of perfect complexes (see [3]); if X is a classical scheme, the homotopy categories of $\mathrm{QCoh}(X)$ and $\mathrm{Perf}(X)$ agree with their classical triangulated versions. Further, $\mathrm{QCoh}(X)$ and $\mathrm{Perf}(X)$ have natural symmetric monoidal structures corresponding to tensor products of sheaves, by which $\mathrm{QCoh}(X)$ has the structure of a commutative algebra object in $\mathrm{PrCat}_{\mathrm{st}}$ and $\mathrm{Perf}(X)$ has the structure of a commutative algebra object in $\mathrm{Cat}_{\mathrm{st}}$. We note that if X is quasi-compact and separated, then by [3, Proposition 3.19] the category $\mathrm{QCoh}(X)$ is compactly generated and there are equivalences $\mathrm{Perf}(X) \simeq \mathrm{QCoh}(X)^{c}$ and $\mathrm{Ind}(\mathrm{Perf}(X)) \simeq \mathrm{QCoh}(X)$, induced by the inclusion $\mathrm{Perf}(X) \subset \mathrm{QCoh}(X)$.

By convention, all functors between derived categories (i.e. pushforward, pullback, tensor product) will be written with underived notation (i.e. f^*, f_*, $\otimes$).

2.3. **Categories linear over a base.** In this paper, the main objects of study are categories which are linear over a base scheme S in the following sense.

Definition 2.2. Let Cat_S be the ∞-category of modules over the commutative algebra object $\mathrm{Perf}(S) \in \mathrm{Cat}_{\mathrm{st}}$, i.e

$$\mathrm{Cat}_S = \mathrm{Mod}_{\mathrm{Perf}(S)}(\mathrm{Cat}_{\mathrm{st}})$$

in the notation of [17, Section 4.5]. An object $\mathcal{C} \in \mathrm{Cat}_S$ is called an *S-linear category*. A morphism $\mathcal{C} \to \mathcal{D}$ in Cat_S is an *S-linear functor*; these morphisms form the objects of an ∞-category denoted $\mathrm{Fun}_{\mathrm{Perf}(S)}(\mathcal{C}, \mathcal{D})$.

Given an S-linear category, we use the notation

$$\begin{aligned} \mathcal{C} \times \mathrm{Perf}(S) &\to \mathcal{C} \\ (C, F) &\mapsto C \otimes F \end{aligned}$$

for the action functor of $\mathrm{Perf}(S)$ on C.

Remark 2.3. If $\pi\colon X \to S$ is a morphism of schemes, then $\mathcal{C} = \mathrm{Perf}(X)$ is naturally an S-linear category, with action functor given by $(C, F) \mapsto C \otimes \pi^*(F)$. This is the primordial example of an S-linear category. The philosophy of noncommutative algebraic geometry is to think of an arbitrary S-linear category $\mathcal{C}$ as a "noncommutative scheme with a morphism to S" (although we will not use that terminology). Throughout the paper, we will see that a number of operations associated to $X \to S$ have analogues for an arbitrary "noncommutative scheme" $\mathcal{C}$.

As above, we can also consider presentable versions of S-linear categories.

Definition 2.4. Let PrCat_S be the category of modules over the commutative algebra object $\mathrm{QCoh}(S) \in \mathrm{PrCat}_{\mathrm{st}}$, i.e.

$$\mathrm{PrCat}_S = \mathrm{Mod}_{\mathrm{QCoh}(S)}(\mathrm{PrCat}_{\mathrm{st}}).$$

An object $\mathcal{C} \in \mathrm{PrCat}_S$ is called a *presentable S-linear category*. A morphism $\mathcal{C} \to \mathcal{D}$ in PrCat_S is a *cocontinuous S-linear functor*; these morphisms form the objects of an ∞-category denoted $\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathcal{C}, \mathcal{D})$.

Recall that $\mathrm{QCoh}(S)$ is compactly generated because S is quasi-compact and separated by our standing assumptions. Hence we can also make the following definition.

Definition 2.5. Let PrCat_S^ω be the category of modules over the commutative algebra object $\mathrm{QCoh}(S) \in \mathrm{PrCat}_{\mathrm{st}}^\omega$, i.e.

$$\mathrm{PrCat}_S^\omega = \mathrm{Mod}_{\mathrm{QCoh}(S)}(\mathrm{PrCat}_{\mathrm{st}}^\omega).$$

An object $\mathcal{C} \in \mathrm{PrCat}_S^\omega$ is called a *compactly generated presentable S-linear category*. A morphism $\mathcal{C} \to \mathcal{D}$ in PrCat_S^ω is a *cocontinuous S-linear functor preserving compact objects*; these morphisms form the objects of an ∞-category denoted $\mathrm{Fun}^\omega_{\mathrm{QCoh}(S)}(\mathcal{C}, \mathcal{D})$.

The categories $\mathrm{Cat}_S, \mathrm{PrCat}_S$, and PrCat_S^ω are related in the same way as the categories $\mathrm{Cat}_{\mathrm{st}}, \mathrm{PrCat}_{\mathrm{st}}$, and $\mathrm{PrCat}_{\mathrm{st}}^\omega$. Namely, PrCat_S^ω is a non-full subcategory of PrCat_S, and Ind-completion induces an equivalence

$$\mathrm{Ind}\colon \mathrm{Cat}_S \to \mathrm{PrCat}_S^\omega$$

with inverse given by the functor

$$(-)^c\colon \mathrm{PrCat}_S^\omega \to \mathrm{Cat}_S$$

induced by taking compact objects.

Further, the categories $\mathrm{Cat}_S, \mathrm{PrCat}_S$, and PrCat_S^ω admit symmetric monoidal structures (see [17, Theorem 4.5.2.1]), with units respectively given by $\mathrm{Perf}(S)$, $\mathrm{QCoh}(S)$, and $\mathrm{QCoh}(S)$. These monoidal structures are closed with internal mapping objects given by the functor categories introduced above. For $\mathcal{C}, \mathcal{D} \in \mathrm{PrCat}_S$, we denote their tensor product by

$$\mathcal{C} \otimes_{\mathrm{QCoh}(S)} \mathcal{D} \in \mathrm{PrCat}_S.$$

For $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_S$, we denote their tensor product by

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \in \mathrm{Cat}_S,$$

which can be described by the formula

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \simeq (\mathrm{Ind}(\mathcal{C}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{D}))^c.$$

The tensor product of categories in Cat_S or PrCat_S can be characterized by a universal property. Namely, if $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathrm{Cat}_S$, then S-linear functors $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{E}$ classify the bilinear maps $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ (as defined in [17, Section 4.4.1]); a similar statement holds for PrCat_S. In particular, there is a canonical functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}.$$

Given objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$, we denote by $C \boxtimes D \in \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$ their image under this functor.

The following result gives generating objects for a tensor product of S-linear categories. We will use the following terminology. A *stable subcategory* of a stable ∞-category is a full subcategory which is also stable. If $\mathcal{C}$ is a category in $\mathrm{Cat}_{\mathrm{st}}$ and Σ is a set of objects, then we say Σ *thickly generates* $\mathcal{C}$ if the smallest idempotent-complete stable subcategory containing Σ is $\mathcal{C}$ itself.

Lemma 2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be S-linear categories. Then $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$ is thickly generated by objects of the form $C \boxtimes D$ for $C \in \mathcal{C}$ and $D \in \mathcal{D}$.*

Proof. Equivalently, $\mathrm{Ind}(\mathcal{C}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{D})$ is compactly generated by objects of the form $C \boxtimes D$ for $C \in \mathcal{C}$ and $D \in \mathcal{D}$. This can be proved as in [7, Chapter I.1, Proposition 7.4.2], which treats the analogous statement for stable ∞-categories without an S-linear structure. □

Finally, we observe that by using tensor products, we can make sense of base changes of linear categories. Namely, if $\mathcal{C}$ is an S-linear category and $S' \to S$ is a morphism of schemes, then the *base change of* $\mathcal{C}$ *along* $S' \to S$ is the S'-linear category

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S') \in \mathrm{Cat}_{S'}.$$

2.3.1. *Mapping objects.* For objects $C, D \in \mathcal{C}$ of an ∞-category $\mathcal{C}$, we denote by $\mathrm{Map}_{\mathcal{C}}(C, D)$ the space of maps from C to D. If $\mathcal{C}$ is a presentable S-linear category, then there is a mapping object

$$\mathcal{H}om_S(C, D) \in \mathrm{QCoh}(S)$$

characterized by equivalences

$$\mathrm{Map}_{\mathrm{QCoh}(S)}(F, \mathcal{H}om_S(C, D)) \simeq \mathrm{Map}_{\mathcal{C}}(C \otimes F, D)$$

for $F \in \mathrm{QCoh}(S)$. More precisely, the functor

$$\mathrm{Map}_{\mathcal{C}}(C \otimes (-), D) \colon \mathrm{QCoh}(S)^{\mathrm{op}} \to \mathcal{S},$$

where $\mathcal{S}$ denotes the ∞-category of spaces, is representable by [19, Proposition 5.5.2.2], and by definition $\mathcal{H}om_S(C, D)$ is the representing object. If $\mathcal{C}$ is an S-linear category, then $\mathcal{C}$ is a full subcategory of the presentable S-linear category $\mathrm{Ind}(\mathcal{C})$; for objects $C, D \in \mathcal{C}$ we denote by $\mathcal{H}om_S(C, D) \in \mathrm{QCoh}(S)$ the mapping object between C and D regarded as objects of $\mathrm{Ind}(\mathcal{C})$.

Remark 2.7. Let X be a scheme with a morphism $\pi \colon X \to S$, so that $\mathrm{Perf}(X)$ is S-linear. Then for $C, D \in \mathrm{Perf}(X)$, we have

$$\mathcal{H}om_S(C, D) \simeq \pi_* \mathcal{H}om_X(C, D),$$

where $\mathcal{H}om_X(C, D)$ denotes the derived sheaf Hom on X.

We have the following Künneth formula for mapping objects in tensor products of categories.

Lemma 2.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be S-linear categories. If $C_1, C_2 \in \mathcal{C}$ and $D_1, D_2 \in \mathcal{D}$, then the* $\mathrm{QCoh}(S)$*-valued mapping object between $C_1 \boxtimes D_1$ and $C_2 \boxtimes D_2$ in $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$ satisfies*

$$\mathcal{H}om_S(C_1 \boxtimes D_1, C_2 \boxtimes D_2) \simeq \mathcal{H}om_S(C_1, C_2) \otimes \mathcal{H}om_S(D_1, D_2).$$

Given an S-linear category $\mathcal{C}$, its base change along a morphism $S' \to S$ is an S'-linear category, and hence has $\mathrm{QCoh}(S')$-valued mapping objects. They satisfy the following Künneth formula.

Lemma 2.9. *Let $\mathcal{C}$ be an S-linear category. Let $S' \to S$ be a morphism of schemes. If $C_1, C_2 \in \mathcal{C}$ and $F_1, F_2 \in \mathrm{Perf}(S')$, then the* $\mathrm{QCoh}(S')$*-valued mapping object between $C_1 \boxtimes F_1$ and $C_2 \boxtimes F_2$ in $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S')$ satisfies*

$$\mathcal{H}om_{S'}(C_1 \boxtimes F_1, C_2 \boxtimes F_2) \simeq \mathcal{H}om_S(C_1, C_2) \otimes \mathcal{H}om_{S'}(F_1, F_2) \in \mathrm{QCoh}(S'),$$

where the product on the right is taken with respect to the $\mathrm{QCoh}(S)$*-module structure on* $\mathrm{QCoh}(S')$ *induced by pullback.*

2.3.2. *Adjoints.* Given a functor of stable ∞-categories $\phi \colon \mathcal{C} \to \mathcal{D}$, we typically denote by ϕ^* the left adjoint whenever it exists, and by $\phi^!$ the right adjoint whenever it exists. Further, if $\mathcal{C}$ is an S-linear category and $\phi \colon \mathcal{D}_1 \to \mathcal{D}_2$ is an S-linear exact functor of S-linear categories, by abuse of notation we denote also by

$$\phi \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_1 \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_2$$

the induced functor. The following are straightforward.

Lemma 2.10. *Let $\phi \colon \mathcal{C} \to \mathcal{D}$ be an S-linear functor between S-linear categories. If ϕ has a left adjoint $\phi^* \colon \mathcal{D} \to \mathcal{C}$ (or right adjoint $\phi^!$) when regarded as a functor of plain stable ∞-categories, then ϕ^* (or $\phi^!$) is naturally an S-linear functor.*

Lemma 2.11. *Let $\mathcal{C}$ be an S-linear category. Let $\phi \colon \mathcal{D}_1 \to \mathcal{D}_2$ be an S-linear functor between S-linear categories.*

(1) If ϕ admits a left adjoint ϕ^*, then the induced functor

$$\phi \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_1 \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_2$$

admits a left adjoint given by

$$\phi^* \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_2 \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_1.$$

(2) If ϕ admits a right adjoint $\phi^!$, then the induced functor

$$\phi \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_1 \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_2$$

admits a right adjoint given by

$$\phi^! \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_2 \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}_1.$$

Lemma 2.12. *Let $\phi \colon \mathcal{C}_1 \to \mathcal{C}_2$ be a fully faithful S-linear functor between S-linear categories. Let $\mathcal{D}$ be an S-linear category. If ϕ admits a left or right adjoint, then the induced functor*

$$\phi \colon \mathcal{C}_1 \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{C}_2 \otimes_{\mathrm{Perf}(S)} \mathcal{D}$$

is fully faithful.

Lemma 2.13. *Let $\phi_i \colon \mathcal{C} \to \mathcal{D}$ be an S-linear functor between S-linear stable ∞-categories for $i = 1, 2, 3$, and let*

$$\phi_1 \to \phi_2 \to \phi_3$$

be an exact triangle in $\mathrm{Fun}_{\mathrm{Perf}(S)}(\mathcal{C}, \mathcal{D})$.

(1) If each ϕ_i admits a left adjoint ϕ_i^*, there is an induced exact triangle

$$\phi_3^* \to \phi_2^* \to \phi_1^*.$$

(2) If each ϕ_i admits a right adjoint $\phi_i^!$, there is an induced exact triangle

$$\phi_3^! \to \phi_2^! \to \phi_1^!.$$

2.4. **Bounded coherent categories.** Given an S-scheme X, in addition to the S-linear category $\mathrm{Perf}(X)$, there is another naturally associated S-linear category: the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \subset \mathrm{QCoh}(S)$ spanned by complexes with bounded coherent cohomology. Under suitable hypotheses, $\mathrm{Perf}(X)$ in fact determines $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, by the following special case of [4, Theorem 1.1.3].

Theorem 2.14. *Let $\pi \colon X \to S$ be a proper morphism of finite presentation between derived schemes over a field of characteristic* 0*, with S locally noetherian. Then there is an equivalence*

$$\begin{aligned} \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) &\xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Perf}(S)}\Big(\mathrm{Perf}(X), \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S)\Big) \\ \mathcal{E} &\mapsto \pi_* \circ (- \otimes \mathcal{E}). \end{aligned}$$

This motivates the following definition.

Definition 2.15. Let $\mathcal{C}$ be a proper S-linear category, where S is locally noetherian over a field of characteristic 0. The *bounded coherent category* of $\mathcal{C}$ is

$$\mathcal{C}^{\mathrm{coh}} = \mathrm{Fun}_{\mathrm{Perf}(S)}\Big(\mathcal{C}, \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(S)\Big).$$

3. Semiorthogonal decompositions

In this section, we consider semiorthogonal decompositions of S-linear categories. In §3.1 we give the basic definitions, and in §3.2 we discuss the notions of admissible subcategories and mutation functors; the results here are standard in the triangulated setting (see [5, 6]), so we freely omit proofs when the usual arguments work without modification. In §3.3 we describe the relation between semiorthogonal decompositions of linear and presentable linear categories. Then we discuss induced semiorthogonal decompositions of tensor products of linear categories in §3.4, and of functor categories between linear categories in §3.5. Finally, in §3.6 we review the notion of a splitting functor from [9].

3.1. **Basic definitions.** Given an S-linear category $\mathcal{C}$, an *S-linear stable subcategory* $\mathcal{A} \subset \mathcal{C}$ is a stable subcategory of $\mathcal{C}$ which is preserved by the $\mathrm{Perf}(S)$-action on $\mathcal{C}$; this is equivalent to the data of an S-linear category $\mathcal{A}$ equipped with a fully faithful S-linear functor $\mathcal{A} \to \mathcal{C}$. Similarly, given a presentable S-linear category $\mathcal{C}$, a *presentable S-linear stable subcategory* $\mathcal{A} \subset \mathcal{C}$ is a stable subcategory of $\mathcal{C}$ which is closed under colimits and preserved by the $\mathrm{QCoh}(S)$-action on $\mathcal{C}$; this is equivalent to the data of a presentable S-linear category $\mathcal{A}$ equipped with a fully faithful cocontinuous S-linear functor $\mathcal{A} \to \mathcal{C}$.

Definition 3.1. Let $\mathcal{C}$ be an S-linear (resp. presentable S-linear) category. An *S-linear (resp. presentable S-linear) semiorthogonal decomposition* of $\mathcal{C}$ is a sequence $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ of S-linear (resp. presentable S-linear) stable subcategories of $\mathcal{C}$ — called the *components* of the decomposition — such that:

(1) $\mathcal{H}om_S(C, D) \simeq 0$ for all $C \in \mathcal{A}_i, D \in \mathcal{A}_j$, and $i > j$.
(2) For any $C \in \mathcal{C}$, there exists a diagram

$$\begin{array}{ccccccccccc} 0 & = & C_n & \longrightarrow & C_{n-1} & \longrightarrow \cdots \longrightarrow & C_1 & \longrightarrow & C_0 & = & C \\ & & \nwarrow & \swarrow & & & \nwarrow & \swarrow & & & \\ & & & A_n & & & & A_1 & & & \end{array} \tag{3.1}$$

where $A_i \in \mathcal{A}_i$ and the triangles are exact.

If only condition (1) is satisfied, we say the sequence $\mathcal{A}_1, \dots, \mathcal{A}_n$ is *semiorthogonal*.

Remark 3.2. By S-linearity of the categories $\mathcal{A}_i$, in (1) it is equivalent to require that the space $\mathrm{Map}_{\mathcal{C}}(C, D)$ is contractible.

Given a stable ∞-category $\mathcal{C}$ and a collection of subcategories $\mathcal{A}_i \subset \mathcal{C}, i = 1, \dots, n$, we denote by

$$\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle \subset \mathcal{C}$$

the stable subcategory of $\mathcal{C}$ generated by the $\mathcal{A}_i$. In particular, for a semiorthogonal decomposition as in Definition 3.1, we have

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

Lemma 3.3. *Let $\mathcal{C}$ be an S-linear (resp. presentable S-linear) category, with an S-linear (resp. presentable S-linear) semiorthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

Let $\alpha_i \colon \mathcal{A}_i \to \mathcal{C}$ denote the inclusion. Then there are S-linear (resp. cocontinuous S-linear) functors

$$\begin{aligned} \mathrm{tr}_i \colon \mathcal{C} \to \mathcal{C}, \quad & 0 \leq i \leq n, \\ \mathrm{pr}_i \colon \mathcal{C} \to \mathcal{A}_i, \quad & 1 \leq i \leq n, \end{aligned}$$

such that:

(1) There is a diagram in the category of functors $\mathrm{Fun}_{\mathrm{Perf}(S)}(\mathcal{C}, \mathcal{C})$ (resp. $\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathcal{C}, \mathcal{C})$)

$$\begin{array}{ccccccccccc} 0 & = & \mathrm{tr}_n & \longrightarrow & \mathrm{tr}_{n-1} & \longrightarrow \cdots \longrightarrow & \mathrm{tr}_1 & \longrightarrow & \mathrm{tr}_0 & = & \mathrm{id}_{\mathcal{C}} \\ & & \nwarrow & \swarrow & & & \nwarrow & \swarrow & & & \\ & & & \alpha_n \circ \mathrm{pr}_n & & & & \alpha_1 \circ \mathrm{pr}_1 & & & \end{array}$$

where the triangles are exact, which recovers (3.1) when applied to any $C \in \mathcal{C}$.
(2) The functor $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{A}_i$ is a retraction, i.e. $\mathrm{pr}_i \circ \alpha_i \simeq \mathrm{id}_{\mathcal{A}_i}$.
(3) The restriction of $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{A}_i$ to $\langle \mathcal{A}_i, \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle \subset \mathcal{C}$ is left adjoint to the inclusion $\mathcal{A}_i \to \langle \mathcal{A}_i, \dots, \mathcal{A}_n \rangle$. In particular, $\mathrm{pr}_1 = \alpha_1^*$ is left adjoint to $\alpha_1 \colon \mathcal{A}_1 \to \mathcal{C}$.
(4) The restriction of $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{A}_i$ to $\langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_i \rangle \subset \mathcal{C}$ is right adjoint to the inclusion $\mathcal{A}_i \to \langle \mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_i \rangle$. In particular, $\mathrm{pr}_n = \alpha_n^!$ is right adjoint to $\alpha_n \colon \mathcal{A}_n \to \mathcal{C}$.

Proof. This result is well-known in the triangulated setting; the same proof works in our setup. The only point which deserves explanation is that in the presentable case, the functors tr_i and pr_i are indeed cocontinuous. This claim reduces to the case where the length of the semiorthogonal decomposition is $n = 2$. Then the diagram from (1) amounts to a distinguished triangle

$$\alpha_2 \circ \mathrm{pr}_2 \to \mathrm{id}_{\mathcal{C}} \to \alpha_1 \circ \mathrm{pr}_1,$$

where $\mathrm{pr}_1 = \alpha_1^*$ and $\mathrm{pr}_2 = \alpha_2^!$. Being a left adjoint, the functor pr_1 is cocontinuous. Hence the above triangle implies $\alpha_2 \circ \mathrm{pr}_2$ is cocontinuous. Since α_2 is fully faithful and cocontinuous, we conclude pr_2 is cocontinuous. □

Definition 3.4. Given a semiorthogonal decomposition as in Lemma 3.3, the functors tr_i and pr_i are called the *truncation* and *projection* functors.

3.2. Admissible subcategories and mutation functors.

Definition 3.5. Let $\mathcal{C}$ be an S-linear category, and let $\mathcal{A} \subset \mathcal{C}$ be an S-linear stable subcategory. Let $\alpha\colon \mathcal{A} \to \mathcal{C}$ denote the inclusion. Then $\mathcal{A}$ is called:

- *left admissible* in $\mathcal{C}$ if α admits a left adjoint $\alpha^*\colon \mathcal{C} \to \mathcal{A}$;
- *right admissible* in $\mathcal{C}$ if α admits a right adjoint $\alpha^!\colon \mathcal{C} \to \mathcal{A}$;
- *admissible* in $\mathcal{C}$ if it is both left and right admissible.

Admissibility of a subcategory is related to semiorthogonal decompositions as follows. Given a subcategory $\mathcal{A}$ of an ∞-category $\mathcal{C}$, consider the full subcategories of $\mathcal{C}$ defined by

$$\mathcal{A}^{\perp} = \{\, C \in \mathcal{C} \mid \mathrm{Map}_{\mathcal{C}}(D, C) \text{ is contractible for all } D \in \mathcal{A} \,\},$$
$$^{\perp}\mathcal{A} = \{\, C \in \mathcal{C} \mid \mathrm{Map}_{\mathcal{C}}(C, D) \text{ is contractible for all } D \in \mathcal{A} \,\}.$$

We call $\mathcal{A}^{\perp}$ the *right orthogonal* to $\mathcal{A} \subset \mathcal{C}$, and $^{\perp}\mathcal{A}$ the *left orthogonal* to $\mathcal{A} \subset \mathcal{C}$. If $\mathcal{C}$ is an S-linear category and $\mathcal{A} \subset \mathcal{C}$ is an S-linear stable subcategory, then the orthogonal categories $\mathcal{A}^{\perp}$ and $^{\perp}\mathcal{A}$ are also S-linear stable subcategories of $\mathcal{C}$. Clearly, $\mathcal{A}^{\perp}, \mathcal{A}$ and $^{\perp}\mathcal{A}, \mathcal{A}$ are semiorthogonal pairs in $\mathcal{C}$.

Lemma 3.6. *Let $\mathcal{C}$ be an S-linear category, and let $\mathcal{A}, \mathcal{B}$ be a pair of S-linear stable subcategories. The following are equivalent:*

(1) $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition.
(2) $\mathcal{A} \subset \mathcal{C}$ is left admissible and $\mathcal{B} = {}^{\perp}\mathcal{A}$.
(3) $\mathcal{B} \subset \mathcal{C}$ is right admissible and $\mathcal{A} = \mathcal{B}^{\perp}$.

Definition 3.7. Assume $\mathcal{A} \subset \mathcal{C}$ is an admissible S-linear stable subcategory of an S-linear category. By Lemma 3.6 the inclusion $i\colon {}^{\perp}\mathcal{A} \to \mathcal{C}$ admits a right adjoint $i^!$, and the inclusion $j\colon \mathcal{A}^{\perp} \to \mathcal{C}$ admits a left adjoint j^*. The functor

$$\mathrm{R}_{\mathcal{A}} = i \circ i^! \colon \mathcal{C} \to \mathcal{C}$$

is called the *right mutation functor* through $\mathcal{A}$, and

$$\mathrm{L}_{\mathcal{A}} = j \circ j^* \colon \mathcal{C} \to \mathcal{C}$$

is called the *left mutation functor* through $\mathcal{A}$.

Lemma 3.8. *Let $\mathcal{C}$ be an S-linear category, and let $\mathcal{A} \subset \mathcal{C}$ be an admissible S-linear stable subcategory. Then the mutation functors $\mathrm{R}_{\mathcal{A}}$ and $\mathrm{L}_{\mathcal{A}}$ kill $\mathcal{A}$, and induce mutually inverse equivalences*

$$\mathrm{R}_{\mathcal{A}}|_{\mathcal{A}^{\perp}} \colon \mathcal{A}^{\perp} \xrightarrow{\sim} {}^{\perp}\mathcal{A} \quad \text{and} \quad \mathrm{L}_{\mathcal{A}}|_{{}^{\perp}\mathcal{A}} \colon {}^{\perp}\mathcal{A} \xrightarrow{\sim} \mathcal{A}^{\perp}.$$

Lemma 3.9. *Let $\mathcal{C}$ be an S-linear category, and let $\alpha \colon \mathcal{A} \to \mathcal{C}$ be the inclusion of an S-linear stable subcategory.*

(1) If $\mathcal{A} \subset \mathcal{C}$ is left admissible and ${}^{\perp}\mathcal{A} \subset \mathcal{C}$ is admissible, then $\alpha^* \colon \mathcal{C} \to \mathcal{A}$ has a left adjoint given by

$$\mathrm{R}_{{}^{\perp}\mathcal{A}} \circ \alpha \colon \mathcal{A} \to \mathcal{C}.$$

(2) If $\mathcal{A} \subset \mathcal{C}$ is right admissible and $\mathcal{A}^{\perp} \subset \mathcal{C}$ is admissible, then $\alpha^! \colon \mathcal{C} \to \mathcal{A}$ has a right adjoint given by

$$\mathrm{L}_{\mathcal{A}^{\perp}} \circ \alpha \colon \mathcal{A} \to \mathcal{C}.$$

Lemma 3.10. *Let $\mathcal{C}$ be an S-linear category, and let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a semiorthogonal sequence of admissible S-linear stable subcategories. Then*

$$\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle \subset \mathcal{C}$$

is an admissible S-linear stable subcategory, with left and right mutation functors given by

$$\mathrm{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle} \simeq \mathrm{L}_{\mathcal{A}_1} \circ \mathrm{L}_{\mathcal{A}_2} \circ \cdots \circ \mathrm{L}_{\mathcal{A}_n}$$
$$\mathrm{R}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle} \simeq \mathrm{R}_{\mathcal{A}_n} \circ \mathrm{R}_{\mathcal{A}_{n-1}} \circ \cdots \circ \mathrm{R}_{\mathcal{A}_1}.$$

Lemma 3.11. *Let $\mathcal{C}$ be an S-linear category with an S-linear semiorthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

(1) For $1 \leq i \leq n-1$, if $\mathcal{A}_i \subset \mathcal{C}$ is admissible there is a semiorthogonal decomposition

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathrm{L}_{\mathcal{A}_i}(\mathcal{A}_{i+1}), \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_n \rangle .$$

(2) For $2 \leq i \leq n$, if $\mathcal{A}_i \subset \mathcal{C}$ is admissible there is a semiorthogonal decomposition

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-2}, \mathcal{A}_i, \mathrm{R}_{\mathcal{A}_i}(\mathcal{A}_{i-1}), \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle .$$

3.3. **Linear versus presentable linear semiorthogonal decompositions.**

Lemma 3.12. *Let $\mathcal{C}$ be an S-linear category with an S-linear semiorthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

Then there is an induced presentable S-linear semiorthogonal decomposition

$$\mathrm{Ind}(\mathcal{C}) = \langle \mathrm{Ind}(\mathcal{A}_1), \dots, \mathrm{Ind}(\mathcal{A}_n) \rangle ,$$

where the embedding functors $\mathrm{Ind}(\mathcal{A}_i) \to \mathrm{Ind}(\mathcal{C})$ preserve compact objects.

Proof. It is easy to see that $\mathrm{Ind}(\mathcal{A}_1), \dots, \mathrm{Ind}(\mathcal{A}_n)$ are semiorthogonal in $\mathrm{Ind}(\mathcal{C})$. Further, the projection functors for the given semiorthogonal decomposition of $\mathcal{C}$ induce projection functors for $\mathrm{Ind}(\mathcal{C})$, so that Definition 3.1(2) holds. Finally, since $\mathrm{Ind}(\mathcal{A}_i)^c = \mathcal{A}_i$ and $\mathrm{Ind}(\mathcal{C})^c = \mathcal{C}$, the embeddings $\mathrm{Ind}(\mathcal{A}_i) \to \mathrm{Ind}(\mathcal{C})$ preserve compact objects. □

In general, given a presentable S-linear semiorthogonal decomposition, passing to categories of compact objects does not necessarily induce an S-linear semiorthogonal decomposition. However:

Lemma 3.13. *Let $\mathcal{C}$ be a presentable S-linear category with a presentable S-linear semiorthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

(1) The projection functor $\mathrm{pr}_1 \colon \mathcal{C} \to \mathcal{A}_1$ preserves compact objects.
(2) The embedding functor $\alpha_n \colon \mathcal{A}_n \to \mathcal{C}$ preserves compact objects.

If for all i the embedding functors $\alpha_i \colon \mathcal{A}_i \to \mathcal{C}$ preserve compact objects, then furthermore:

(1) For all i the projection functors $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{A}_i$ preserve compact objects.
(2) There is an induced semiorthogonal decomposition

$$\mathcal{C}^c = \langle \mathcal{A}_1^c, \dots, \mathcal{A}_n^c \rangle .$$

Proof. By induction we reduce to the case where $n = 2$. The functor $\mathrm{pr}_1 = \alpha_1^*$ preserves compact objects because its right adjoint α_1 is cocontinuous. Similarly, α_2 preserves compact objects because its right adjoint $\mathrm{pr}_2 = \alpha_2^!$ is cocontinuous by Lemma 3.3.

Now assume that α_1 preserves compact objects. Then there is a triangle of functors

$$\alpha_2 \circ \mathrm{pr}_2 \to \mathrm{id}_{\mathcal{C}} \to \alpha_1 \circ \mathrm{pr}_1,$$

where the second and third vertices, and hence also the first, preserve compact objects. Since α_2 is fully faithful and cocontinuous, it follows that pr_2 preserves compact objects. Since for all i the embeddings α_i and the projection functors pr_i preserve compact objects, it follows formally that there is an induced semiorthogonal decomposition $\mathcal{C}^c = \langle \mathcal{A}_1^c, \mathcal{A}_2^c \rangle$. □

Remark 3.14. Lemmas 3.12 and 3.13 can be summarized as follows: Under the equivalence $\mathrm{Ind} \colon \mathrm{Cat}_S \xrightarrow{\sim} \mathrm{PrCat}_S^{\omega}$, S-linear semiorthogonal decompostions correspond to presentable S-linear semiorthogonal decompositions such that the embedding functors of the components preserve compact objects.

3.4. **Semiorthogonal decompositions of tensor products.**

Lemma 3.15. *Let $\mathcal{C}$ be an S-linear category with an S-linear semiorthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

Let $\mathcal{D}$ be another S-linear category. Then for every i the functor

$$\mathcal{A}_i \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$$

is fully faithful, and there is an S-linear semiorthogonal decomposition

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D} = \left\langle \mathcal{A}_1 \otimes_{\mathrm{Perf}(S)} \mathcal{D}, \dots, \mathcal{A}_n \otimes_{\mathrm{Perf}(S)} \mathcal{D} \right\rangle .$$

Proof. The result reduces to the case where $n = 2$. It follows from Lemmas 3.6 and 2.12 that the functor

$$\mathcal{A}_i \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$$

is fully faithful for $i = 1, 2$. It follows from Lemmas 2.6 and 2.8 that the categories

$$\mathcal{A}_1 \otimes_{\mathrm{Perf}(S)} \mathcal{D}, \ \mathcal{A}_2 \otimes_{\mathrm{Perf}(S)} \mathcal{D}$$

are semiorthogonal in $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$. The projection functors for the original semiorthogonal decomposition induce projection functors for $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$, so that Definition 3.1(2) holds. □

Remark 3.16. If $S' \to S$ is a morphism of schemes, then taking $\mathcal{D} = \mathrm{Perf}(S')$ in Lemma 3.15 gives a base change result for semiorthogonal decompositions.

Lemma 3.17. *Let $\mathcal{C}$ be an S-linear category, and let $\mathcal{A} \subset \mathcal{C}$ be an S-linear stable subcategory. Let $\mathcal{D}$ be another S-linear category. Then if $\mathcal{A} \subset \mathcal{C}$ is left (or right) admissible, so is*

$$\mathcal{A} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \subset \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathcal{D}.$$

Proof. Follows from Lemma 3.15. □

Lemma 3.18. *Let $\mathcal{C}$ be an S-linear category, and let $\mathcal{A} \subset \mathcal{C}$ be a left admissible S-linear stable subcategory. Let $f \colon S' \to S$ be a morphism of schemes, and let*

$$\mathcal{A}' = \mathcal{A} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S') \subset \mathcal{C}' = \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S').$$

Then in terms of the functor

$$f_* \colon \mathcal{C}' = \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S') \to \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S) \simeq \mathcal{C},$$

the subcategory $\mathcal{A}' \subset \mathcal{C}'$ is given by

$$\mathcal{A}' = \left\{ C' \in \mathcal{C}' \mid f_*(C' \otimes F) \in \mathcal{A} \text{ for all } F \in \mathrm{Perf}(S') \right\}. \tag{3.2}$$

Proof. Clearly $\mathcal{A}'$ is contained in the right side of (3.2). For the reverse inclusion, we consider the left orthogonal $\mathcal{B} \subset \mathcal{C}$ to $\mathcal{A}$ and its base change

$$\mathcal{B}' = \mathcal{B} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S').$$

We have $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ by Lemma 3.6, so $\mathcal{C}' = \langle \mathcal{A}', \mathcal{B}' \rangle$ by Lemma 3.15. Hence given $C' \in \mathcal{C}'$ such that $f_*(C' \otimes F) \in \mathcal{A}$ for all $F \in \mathrm{Perf}(S')$, we must show that C' is right orthogonal to $\mathcal{B}'$. By Lemma 2.6 it suffices to show that if $B \in \mathcal{B}$ and $G \in \mathrm{Perf}(S')$, then

$$\mathcal{H}om_S(B \boxtimes G, C') \simeq 0.$$

Note that we can write $B \boxtimes G = f^*(B) \otimes G$, and hence

$$\mathcal{H}om_S(B \boxtimes G, C') \simeq \mathcal{H}om_S(f^*(B), C' \otimes G^\vee) \simeq \mathcal{H}om_S(B, f_*(C' \otimes G^\vee)).$$

This vanishes since $f_*(C' \otimes G^\vee) \in \mathcal{A}$ by assumption. □

3.5. **Semiorthogonal decompositions of functor categories.** Recall that given S-linear (resp. presentable S-linear) categories $\mathcal{C}$ and $\mathcal{D}$, the S-linear (resp. cocontinuous S-linear) functors form the objects of an S-linear (resp. presentable S-linear) category $\mathrm{Fun}_{\mathrm{Perf}(S)}(\mathcal{C}, \mathcal{D})$ (resp. $\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathcal{C}, \mathcal{D})$). In the following lemma, we use the uniform notation $\mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$ to denote $\mathrm{Fun}_{\mathrm{Perf}(S)}(\mathcal{C}, \mathcal{D})$ if $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_S$ or $\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathcal{C}, \mathcal{D})$ if $\mathcal{C}, \mathcal{D} \in \mathrm{PrCat}_S$.

Lemma 3.19. *Let $\mathcal{C}$ and $\mathcal{D}$ be S-linear (resp. presentable S-linear) categories.*

(1) Let $\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be an S-linear (resp. presentable S-linear) semiorthogonal decomposition. Then for every i the functor

$$\mathrm{Fun}_S(\mathcal{A}_i, \mathcal{D}) \to \mathrm{Fun}_S(\mathcal{C}, \mathcal{D}) \tag{3.3}$$

induced by the projection $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{A}_i$ is fully faithful. Further, there is an S-linear (resp. presentable S-linear) semiorthogonal decomposition

$$\mathrm{Fun}_S(\mathcal{C}, \mathcal{D}) = \langle \mathrm{Fun}_S(\mathcal{A}_1, \mathcal{D}), \dots, \mathrm{Fun}_S(\mathcal{A}_n, \mathcal{D}) \rangle,$$

whose projection functors

$$\mathrm{Fun}_S(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_S(\mathcal{A}_i, \mathcal{D}) \tag{3.4}$$

are induced by the embeddings $\alpha_i \colon \mathcal{A}_i \to \mathcal{D}$.

(2) Let $\mathcal{D} = \langle \mathcal{B}_1, \dots, \mathcal{B}_n \rangle$ be an S-linear (resp. presentable S-linear) semiorthogonal decomposition. Then for every i the functor

$$\mathrm{Fun}_S(\mathcal{C}, \mathcal{B}_i) \to \mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$$

induced by the emedding $\beta_i \colon \mathcal{B}_i \to \mathcal{D}$ is fully faithful. Further, there is an S-linear (resp. presentable S-linear) semiorthogonal decomposition

$$\mathrm{Fun}_S(\mathcal{C}, \mathcal{D}) = \langle \mathrm{Fun}_S(\mathcal{C}, \mathcal{B}_1), \dots, \mathrm{Fun}_S(\mathcal{C}, \mathcal{B}_n) \rangle ,$$

whose projection functors

$$\mathrm{Fun}_S(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_S(\mathcal{C}, \mathcal{B}_i)$$

are induced by the projections $\mathrm{pr}_i \colon \mathcal{D} \to \mathcal{B}_i$.

Proof. We prove (1). First assume we are in the S-linear (not presentable) situation. The result reduces to the case $n = 2$. One checks that the functor (3.4) is left adjoint to (3.3) for $i = 1$, right adjoint to (3.3) for $i = 2$, and in both cases the composition

$$\mathrm{Fun}_S(\mathcal{A}_i, \mathcal{D}) \to \mathrm{Fun}_S(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_S(\mathcal{A}_i, \mathcal{D})$$

is equivalent to the identity. This proves the claim that $\mathrm{Fun}_S(\mathcal{A}_i, \mathcal{D}) \to \mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$ is fully faithful, and the adjointness of (3.4) and (3.3) immediately implies that

$$\mathrm{Fun}_S(\mathcal{A}_1, \mathcal{D}), \ \mathrm{Fun}_S(\mathcal{A}_2, \mathcal{D})$$

are semiorthogonal in $\mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$. Finally, given any $F \in \mathrm{Fun}_S(\mathcal{C}, \mathcal{D})$, the triangle

$$\alpha_2 \circ \mathrm{pr}_2 \to \mathrm{id}_{\mathcal{C}} \to \alpha_1 \circ \mathrm{pr}_1$$

induces a triangle

$$F \circ \alpha_2 \circ \mathrm{pr}_2 \to F \to F \circ \alpha_1 \circ \mathrm{pr}_1,$$

which shows that Definition 3.1(2) holds. In the presentable S-linear case the argument above works verbatim, with the additional remark that the functors (3.3) and (3.4) are cocontinuous, because colimits of functors are computed objectwise (see [19, Section 5.1.2]).

Part (2) is proved similarly. □

Corollary 3.20. *Let $\mathcal{C}$ be a proper S-linear category, where S is locally noetherian over a field of characteristic 0. Let*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$$

be an S-linear semiorthogonal decomposition. Then there is an induced semiorthogonal decomposition

$$\mathcal{C}^{\mathrm{coh}} = \left\langle \mathcal{A}_1^{\mathrm{coh}}, \dots, \mathcal{A}_n^{\mathrm{coh}} \right\rangle .$$

3.6. **Splitting functors.** Here we review the notion of a splitting functor from [9] in the context of S-linear categories.

Definition 3.21. Let $\phi \colon \mathcal{C} \to \mathcal{D}$ be an S-linear functor between S-linear categories. The *kernel* of ϕ is the full subcategory $\ker \phi \subset \mathcal{C}$ spanned by objects $C \in \mathcal{C}$ such that $\phi(C) \simeq 0$, and the *image* of ϕ is the full subcategory $\mathrm{im}\, \phi \subset \mathcal{D}$ spanned by the objects $\phi(C) \in \mathcal{D}$ for $C \in \mathcal{C}$.

Remark 3.22. The kernel of a functor ϕ as in Definition 3.21 is automatically an S-linear stable subcategory of $\mathcal{C}$. This is true of the image of ϕ if for instance ϕ is fully faithful, but is not true in general.

The following result is (a subset of) [9, Theorem 3.3] in our setting; the proof is the same.

Theorem 3.23. *Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be an S-linear functor between S-linear categories. The following are equivalent:*

(1) $\ker \phi$ *is a left admissible subcategory of* $\mathcal{C}$*, the restriction of* ϕ *to* ${}^{\perp}(\ker\phi)$ *is fully faithful, and* $\operatorname{im}\phi$ *is a left admissible subcategory of* $\mathcal{D}$*.*

(2) ϕ *admits a left adjoint* ϕ^**, and the composition of the canonical morphism* $\phi^*\phi \to \mathrm{id}$ *with* ϕ *gives an equivalence* $\phi\phi^*\phi \simeq \phi$*.*

(3) ϕ *admits a left adjoint* ϕ^**, there are semiorthogonal decompositions*

$$\mathcal{C} = \langle \ker\phi, \operatorname{im}\phi^* \rangle ,$$
$$\mathcal{D} = \langle \operatorname{im}\phi, \ker\phi^* \rangle ,$$

and the functors ϕ *and* ϕ^* *induce mutually inverse equivalences* $\operatorname{im}\phi^* \simeq \operatorname{im}\phi$*.*

Definition 3.24. Let $\phi\colon \mathcal{C} \to \mathcal{D}$ be an S-linear functor between S-linear categories. We say ϕ is *left splitting* if the equivalent conditions of Theorem 3.23 hold.

Remark 3.25. There is also the notion of a *right splitting* functor, as in [9, Definition 3.1], but we shall only need the left version.

4. Smooth and proper categories

In this section, we study the notions of smoothness and properness of S-linear categories, which are the analogues in noncommutative algebraic geometry of the usual geometric notions of smoothness and properness of a scheme over S. In §4.1 we define smoothness and properness of linear categories, and discuss the closely related notion of dualizability. In §4.2 we explain how smoothness or properness of an S-scheme relates to the corresponding property of its category of perfect complexes. In §4.3 we show that smoothness and properness of linear categories behave well under base change. In §4.4 we show that a functor between linear categories admits adjoints if the source is smooth and proper and the target is proper. In §4.5 we analyze the behavior of smoothness and properness under semiorthogonal decompositions. Finally, in §4.6 we introduce the notion of the critical locus of a linear category.

We note that many of the results in this section are folklore, or appear in some form in the literature, cf. [1, 21, 16, 13, 25]. However, we could not find an adequate reference for the point of view taken in this work.

4.1. **Basic definitions.**

Definition 4.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit $\mathbf{1}$. An object $C \in \mathcal{C}$ is called *dualizable* if there exists an object $D \in \mathcal{C}$ and morphisms

$$\mathrm{ev}\colon D \otimes C \to \mathbf{1},$$
$$\mathrm{coev}\colon \mathbf{1} \to C \otimes D,$$

such that the compositions

$$C \xrightarrow{\mathrm{coev}\otimes\mathrm{id}} C \otimes D \otimes C \xrightarrow{\mathrm{id}\otimes\mathrm{ev}} C,$$
$$D \xrightarrow{\mathrm{id}\otimes\mathrm{coev}} D \otimes C \otimes D \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} D,$$

are equivalent to the identity. The object D is called the *dual* of C, and ev and coev are called the *evaluation* and *coevaluation* morphisms.

Remark 4.2. Equivalently, an object $C \in \mathcal{C}$ is dualizable if it is dualizable as an object of the symmetric monoidal homotopy category $\mathrm{Ho}(\mathcal{C})$. Moreover, if $C \in \mathcal{C}$ is dualizable, then the dual D and the evaluation and coevaluation morphisms are uniquely determined in $\mathrm{Ho}(\mathcal{C})$.

In particular, given a category $\mathcal{C}$ which is in Cat_S or PrCat_S, it makes sense to ask whether $\mathcal{C}$ is dualizable. If so, we denote by $\mathrm{D}_S(\mathcal{C})$ the dual.

Lemma 4.3. *Let $\mathcal{C}$ be a compactly generated presentable S-linear category. Then $\mathcal{C}$ is dualizable as an object of PrCat_S, with dual given by*

$$\mathrm{D}_S(\mathcal{C}) \simeq \mathrm{Ind}((\mathcal{C}^c)^{\mathrm{op}}),$$

where $(\mathcal{C}^c)^{\mathrm{op}}$ denotes the opposite category of $\mathcal{C}^c$.

Proof. By [7, Chapter I.1, Proposition 7.3.2], the underlying stable ∞-category of $\mathcal{C}$ is dualizable with dual $\mathrm{Ind}((\mathcal{C}^c)^{\mathrm{op}})$. Since by assumption S is quasi-compact and separated, an object of $\mathrm{QCoh}(S)$ is compact if and only if it is dualizable if and only if it is a perfect complex. Hence by [7, Chapter I.1, Lemma 9.1.5], $\mathrm{QCoh}(S)$ is a rigid symmetric monoidal ∞-category. Now the result follows from [7, Chapter I.1, Proposition 9.4.4]. □

The following is an easy consequence of the definitions.

Lemma 4.4. *Let $\mathcal{C}$ be a presentable S-linear category. If $\mathcal{C}$ is dualizable, then for any $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{PrCat}_S$ there is an equivalence*

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathcal{D}_1 \otimes_{\mathrm{QCoh}(S)} \mathcal{C}, \mathcal{D}_2) \simeq \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathcal{D}_1, \mathrm{D}_S(\mathcal{C}) \otimes_{\mathrm{QCoh}(S)} \mathcal{D}_2).$$

Now let $\mathcal{C}$ be an S-linear category. Then by Lemma 4.3, the presentable S-linear category $\mathrm{Ind}(\mathcal{C})$ is dualizable with dual $\mathrm{Ind}(\mathcal{C}^{\mathrm{op}})$. More explicitly, the evaluation morphism

$$\mathrm{ev}\colon \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{C}) \to \mathrm{QCoh}(S) \tag{4.1}$$

is induced by the functor

$$\mathcal{H}om_S(-,-)\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{QCoh}(S).$$

Under the equivalence

$$\mathrm{Ind}(\mathcal{C}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$$

given by Lemma 4.4, the coevaluation morphism

$$\mathrm{coev}\colon \mathrm{QCoh}(S) \to \mathrm{Ind}(\mathcal{C}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \tag{4.2}$$

is identified with the canonical functor

$$\mathrm{QCoh}(S) \to \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$$

which sends $\mathcal{O}_S \in \mathrm{QCoh}(S)$ to $\mathrm{id} \in \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$.

Definition 4.5. Let $\mathcal{C}$ be an S-linear category. We say:

(1) $\mathcal{C}$ is *proper* if the evaluation morphism (4.1) is a morphism in the category $\mathrm{PrCat}_S^{\omega}$, i.e. if this functor preserves compact objects.
(2) $\mathcal{C}$ is *smooth* if the coevaluation morphism (4.2) is a morphism in the category $\mathrm{PrCat}_S^{\omega}$, i.e. if this functor preserves compact objects.

Remark 4.6. The condition that $\mathcal{C}$ is smooth or proper depends on its S-linear structure. For emphasis, we shall sometimes say $\mathcal{C}$ is smooth *over* S or proper *over* S. For instance, if $T \to S$ is a morphism of schemes and $\mathcal{C}$ is a T-linear category, then we say $\mathcal{C}$ is smooth and proper over S to mean that $\mathcal{C}$ is smooth and proper with its induced S-linear structure.

Lemma 4.7. *Let $\mathcal{C}$ be an S-linear category.*

(1) $\mathcal{C}$ is proper if and only if for every $C, D \in \mathcal{C}$ the mapping object $\mathcal{H}om_S(C, D)$ lies in $\mathrm{Perf}(S) \subset \mathrm{QCoh}(S)$.
(2) $\mathcal{C}$ is smooth if and only if $\mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \in \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$ *is a compact object.*

Proof. Follows from the descriptions of the functors (4.1) and (4.2) above. □

Lemma 4.8. *Let $\mathcal{C}$ be an S-linear category. Then $\mathcal{C}$ is smooth and proper over S if and only if $\mathcal{C}$ is dualizable as an object of* Cat_S. *In this case, the dual of $\mathcal{C}$ is $\mathcal{C}^{\mathrm{op}}$.*

Proof. Since Ind induces a symmetric monoidal equivalence $\mathrm{Cat}_S \simeq \mathrm{PrCat}_S^{\omega}$, $\mathcal{C}$ is dualizable as an object of Cat_S if and only if $\mathrm{Ind}(\mathcal{C})$ is dualizable as an object of $\mathrm{PrCat}_S^{\omega}$. By definition, if $\mathcal{C}$ is smooth and proper over S, then $\mathrm{Ind}(\mathcal{C})$ is dualizable as an object of $\mathrm{PrCat}_S^{\omega}$. Conversely, if $\mathrm{Ind}(\mathcal{C})$ is dualizable as an object of $\mathrm{PrCat}_S^{\omega}$, by Remark 4.2 its duality data must be given by (4.1) and (4.2), which are hence morphisms in $\mathrm{PrCat}_S^{\omega}$. □

4.2. **Relation to geometry.** If X is an S-scheme, smoothness and properness of $\mathrm{Perf}(X)$ are related to the corresponding properties of X as follows. Recall that a morphism $X \to S$ is called *perfect* if it is pseudo-coherent and of finite Tor-dimension, see [23, Tag 0685].

Lemma 4.9. *Let $X \to S$ be a morphism of schemes.*

(1) If $X \to S$ is a perfect proper morphism, then $\mathrm{Perf}(X)$ *is proper over S.*
(2) If X and S are noetherian, $X \to S$ is separated and of finite type, and $\mathrm{Perf}(X)$ *is proper over S, then $X \to S$ is a perfect proper morphism.*
(3) $\mathrm{Perf}(X)$ *is smooth over S if and only if $\Delta_*(\mathcal{O}_X) \in \mathrm{QCoh}(X \times_S X)$ is a perfect complex, where $\Delta\colon X \to X \times_S X$ is the diagonal morphism.*
(4) If X is smooth over S, then $\mathrm{Perf}(X)$ *is smooth over S.*
(5) If $X \to S$ is flat and locally of finite presentation and $\mathrm{Perf}(X)$ *is smooth over S, then X is smooth over S.*
(6) If X is smooth and proper over S, then $\mathrm{Perf}(X)$ *is smooth and proper over S.*
(7) If X and S are noetherian, $X \to S$ is a flat, separated, and of finite type, and $\mathrm{Perf}(X)$ *is smooth and proper over S, then X is smooth and proper over S.*

Proof. By the criterion of Lemma 4.7(1) and Remark 2.7, part (1) follows from the fact that pushforward along a perfect proper morphism preserves perfect complexes (see [15, Example 2.2(a)]).

Part (2) holds by Remark 2.7 combined with [20, Lemma 0.20 and Proposition 0.21].

By the criterion of Lemma 4.7(2), $\mathrm{Perf}(X)$ is smooth over S if and only if

$$\mathrm{id} \in \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{QCoh}(X), \mathrm{QCoh}(X))$$

is a compact object. But by [3, Theorem 1.2(2)] there is an equivalence

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{QCoh}(X), \mathrm{QCoh}(X)) \simeq \mathrm{QCoh}(X \times_S X)$$

under which id corresponds to $\Delta_*(\mathcal{O}_X)$. By [3, Proposition 3.24] the fiber product $X \times_S X$ is a perfect derived scheme; in particular, $\mathrm{QCoh}(X \times_S X)^c = \mathrm{Perf}(X \times_S X)$. Hence (3) holds.

If $X \to S$ is smooth, then the derived fiber product $X \times_S X$ agrees with the usual fiber product of schemes, and Δ is a section of the smooth morphism $X \times_S X \to X$. Hence Δ is a regular immersion by [23, Tag 067R], and hence a regular closed immersion by our standing separatedness assumptions. So $\Delta_*(\mathcal{O}_X)$ is a perfect complex, which by (3) proves (4).

By Lemma 4.10(2) below, part (5) reduces to the case where $S = \mathrm{Spec}(k)$ for a field k. Moreover, it follows from (3) that the smoothness of $\mathrm{Perf}(X)$ over S is local on X, so we may assume X is affine. In this case (5) holds by [16, Proposition 4.17].

Finally, (6) follows by combining (1) and (4), and (7) by combining (2) and (5). □

4.3. **Behavior under base change.** Smoothness and properness of linear categories are stable under base change:

Lemma 4.10. *Let $\mathcal{C}$ be an S-linear category, and let $S' \to S$ be a morphism of schemes.*

(1) If $\mathcal{C}$ is proper over S, then $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S')$ is proper over S'.

(2) If $\mathcal{C}$ is smooth over S, then $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S')$ is smooth over S'.

Proof. Part (1) follows from Lemmas 4.7(1), 2.6, and 2.9. For part (2), we note that if $\mathcal{C}' = \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(S')$, then there is an equivalence

$$\mathrm{Fun}_{\mathrm{QCoh}(S')}\big(\mathrm{Ind}(\mathcal{C}'), \mathrm{Ind}(\mathcal{C}')\big) \simeq \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})) \otimes_{\mathrm{QCoh}(S)} \mathrm{QCoh}(S'),$$

under which $\mathrm{id}_{\mathrm{Ind}(\mathcal{C}')}$ corresponds to $\mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \boxtimes \mathcal{O}_{S'}$. Now the result follows from the criterion of Lemma 4.7(2). □

Lemma 4.11. *Let $T \to S$ be a morphism of schemes. Let $\mathcal{C}$ be a T-linear category which is smooth and proper over S. Let $T' \to T$ be a smooth and proper morphism. Then the base change*

$$\mathcal{C} \otimes_{\mathrm{Perf}(T)} \mathrm{Perf}(T')$$

is smooth and proper over S.

Proof. By Lemma 4.9 the category $\mathrm{Perf}(T')$ is smooth and proper over T. Now the result follows from Lemma 4.8 combined with [7, Chapter I.1, Corollary 9.5.4]. □

Remark 4.12. Lemma 4.11 is the analog of the following simple geometric fact: if T is an S-scheme, X is a T-scheme which is smooth and proper over S, and $T' \to T$ is a smooth and proper morphism, then the base change $X \times_T T'$ is smooth and proper over S. Indeed, $X \times_T T' \to S$ is the composition of the smooth and proper morphisms $X \times_T T' \to X$ and $X \to S$.

4.4. **Existence of adjoints.**

Lemma 4.13. *Let $\phi \colon \mathcal{C} \to \mathcal{D}$ be an S-linear functor between S-linear categories, where $\mathcal{C}$ is smooth and proper over S and $\mathcal{D}$ is proper over S. Then ϕ admits left and right adjoints.*

Proof. Since $\mathcal{D}$ is proper, we have a Yoneda functor

$$\mathcal{H}om_S(\phi(-), -) \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Perf}(S).$$

This induces an S-linear functor

$$\mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathrm{Perf}(S).$$

Since $\mathcal{C}$ is dualizable, there is an equivalence

$$\mathrm{Fun}_{\mathrm{Perf}(S)}\big(\mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{D}, \mathrm{Perf}(S)\big) \simeq \mathrm{Fun}_{\mathrm{Perf}(S)}(\mathcal{D}, \mathcal{C}).$$

Under this equivalence, the above functor corresponds to a functor $\phi^!\colon \mathcal{D} \to \mathcal{C}$, which is the right adjoint to ϕ; namely, by construction we have an equivalence of functors

$$\mathcal{H}om_S(\phi(-), -) \simeq \mathcal{H}om_S(-, \phi^!(-))\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Perf}(S).$$

A similar argument shows the existence of a left adjoint to ϕ. □

4.5. **Behavior under semiorthogonal decompositions.** We will need the following technical result below.

Lemma 4.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be S-linear categories.*

(1) Let $\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ be an S-linear semiorthogonal decomposition whose components are admissible. Then the embedding and projection functors for the components of the semiorthogonal decomposition

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D})) = \left\langle \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{D})), \dots, \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_n), \mathrm{Ind}(\mathcal{D})) \right\rangle$$

induced by Lemmas 3.12 and 3.19 all preserve compact objects.

(2) Let $\mathcal{D} = \langle \mathcal{B}_1, \dots, \mathcal{B}_n \rangle$ be an S-linear semiorthogonal decomposition. Then the embedding and projection functors for the components of the semiorthogonal decomposition

$$\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D})) = \left\langle \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{B}_1)), \dots, \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{B}_n)) \right\rangle$$

induced by Lemmas 3.12 and 3.19 all preserve compact objects.

Proof. We prove (1). We may assume $n = 2$. Let $\widehat{\mathrm{pr}}_i\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{A}_i)$ be the functor induced by $\mathrm{pr}_i\colon \mathcal{C} \to \mathcal{A}_i$. Then by Lemma 3.13, all we need to show is that the embedding functor

$$\begin{aligned} \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{D})) &\to \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D})) \\ F &\mapsto F \circ \widehat{\mathrm{pr}}_1 \end{aligned} \tag{4.3}$$

preserves compact objects. Note that we have an equivalence

$$\mathrm{Ind}(\mathcal{A}_1^{\mathrm{op}}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{D}))$$

induced by the functor

$$\begin{aligned} \mathcal{A}_1^{\mathrm{op}} \times \mathcal{D} &\to \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{D})) \\ (C, D) &\mapsto \mathcal{H}om_S(C, -) \otimes D, \end{aligned}$$

and similarly an equivalence

$$\mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D}))$$

induced by the functor

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} \times \mathcal{D} &\to \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D})) \\ (C, D) &\mapsto \mathcal{H}om_S(C, -) \otimes D. \end{aligned}$$

By Lemma 3.9 the functor $\mathrm{pr}_1 = \alpha_1^*\colon \mathcal{C} \to \mathcal{A}_1$ admits a left adjoint, namely $\mathrm{R}_{\mathcal{A}_2} \circ \alpha_1\colon \mathcal{A}_1 \to \mathcal{C}$. It follows that under the above equivalences, the functor (4.3) is identified with the functor

$$\mathrm{Ind}(\mathcal{A}_1^{\mathrm{op}}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes_{\mathrm{QCoh}(S)} \mathrm{Ind}(\mathcal{D})$$

given by Ind of the functor

$$\mathcal{A}_1^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{D} \to \mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$$

induced by

$$(\mathrm{R}_{\mathcal{A}_2} \circ \alpha_1) \times \mathrm{id}_{\mathcal{D}}\colon \mathcal{A}_1^{\mathrm{op}} \times \mathcal{D} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}.$$

Hence the functor (4.3) preserves compact objects.

The proof of (2) is similar. Let $\widehat{\beta}_i \colon \mathrm{Ind}(\mathcal{B}_i) \to \mathrm{Ind}(\mathcal{D})$ be the functor induced by $\beta_i \colon \mathcal{B}_i \to \mathcal{D}$. Then the functor

$$\begin{aligned} \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{B}_1)) &\to \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{D})) \\ F &\mapsto \widehat{\beta}_1 \circ F \end{aligned} \tag{4.4}$$

is identified with Ind of the functor

$$\mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{B}_1 \to \mathcal{C}^{\mathrm{op}} \otimes_{\mathrm{Perf}(S)} \mathcal{D}$$

induced by

$$\mathrm{id}_{\mathcal{C}^{\mathrm{op}}} \times \beta_1 \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{B}_1 \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}.$$

Now we conclude by the same argument as above. Note that unlike (1), we did not need the components $\mathcal{B}_i \subset \mathcal{D}$ to be admissible. □

Lemma 4.15. *Let $\mathcal{C}$ be an S-linear category with an S-linear semiorthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle .$$

(1) If $\mathcal{C}$ is proper over S, then $\mathcal{A}_i$ is proper over S for all i.
(2) If $\mathcal{C}$ is smooth over S, then $\mathcal{A}_1$ is smooth over S.
(3) If all the components $\mathcal{A}_i \subset \mathcal{C}$ are admissible, then $\mathcal{C}$ is smooth over S if and only if $\mathcal{A}_i$ is smooth over S for all i.
(4) If $\mathcal{C}$ is smooth and proper over S, then $\mathcal{A}_i \subset \mathcal{C}$ is admissible and $\mathcal{A}_i$ is smooth and proper over S for all i.

Proof. Part (1) is immediate from the definitions. Parts (2)-(4) reduce to the case $n = 2$, so we assume this for the rest of the proof.

Let $\widehat{\alpha}_i \colon \mathrm{Ind}(\mathcal{A}_i) \to \mathrm{Ind}(\mathcal{C})$ be the functor induced by the embedding $\alpha_i \colon \mathcal{A}_i \to \mathcal{C}$, and let $\widehat{\mathrm{pr}}_i \colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{A}_i)$ be the functor induced by the projection $\mathrm{pr}_i \colon \mathcal{C} \to \mathcal{A}_i$. Then by Lemmas 3.12 and 3.19 there is a semiorthogonal decomposition

$$\begin{aligned} \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})) = \langle &\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{A}_1)), \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{A}_2)), \\ &\mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_2), \mathrm{Ind}(\mathcal{A}_1)), \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_2), \mathrm{Ind}(\mathcal{A}_2)) \rangle \end{aligned}$$

with embedding functors

$$\begin{aligned} f_{ij} \colon \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_i), \mathrm{Ind}(\mathcal{A}_j)) &\to \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})) \\ F &\mapsto \widehat{\alpha}_j \circ F \circ \widehat{\mathrm{pr}}_i \end{aligned}$$

and projection functors

$$\begin{aligned} p_{ij} \colon \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C})) &\to \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_i), \mathrm{Ind}(\mathcal{A}_j)) \\ G &\mapsto \widehat{\mathrm{pr}}_j \circ G \circ \widehat{\alpha}_i. \end{aligned}$$

Observe that $p_{11}(\mathrm{id}_{\mathrm{Ind}(\mathcal{C})}) \simeq \mathrm{id}_{\mathrm{Ind}(\mathcal{A}_1)}$, $p_{22}(\mathrm{id}_{\mathrm{Ind}(\mathcal{C})}) \simeq \mathrm{id}_{\mathrm{Ind}(\mathcal{A}_2)}$, and there is an exact triangle

$$f_{22}(\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_2)}) \to \mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \to f_{11}(\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_1)}).$$

By Lemma 3.13(1) the projection p_{11} preserves compact objects. Hence if

$$\mathrm{id}_{\mathrm{Ind}(\mathcal{C})} \in \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{C}), \mathrm{Ind}(\mathcal{C}))$$

is compact then so is

$$\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_1)} \in \mathrm{Fun}_{\mathrm{QCoh}(S)}(\mathrm{Ind}(\mathcal{A}_1), \mathrm{Ind}(\mathcal{A}_1)).$$

By Lemma 4.7(2) this proves (2).

If all $\mathcal{A}_i \subset \mathcal{C}$ are admissible, then by Lemma 4.14 all of the embedding and projection functors f_{ij}, p_{ij} preserve compact objects. Hence by the observation above, $\mathrm{id}_{\mathrm{Ind}(\mathcal{C})}$ is a compact object if and only if $\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_1)}$ and $\mathrm{id}_{\mathrm{Ind}(\mathcal{A}_2)}$ are compact objects. By Lemma 4.7(2) this proves (3).

Finally, assume $\mathcal{C}$ is smooth and proper over S. Then by parts (1) and (2) the category $\mathcal{A}_1$ is smooth and proper over S. By Lemma 4.13 it follows that $\mathcal{A}_1 \subset \mathcal{C}$ is admissible. Hence by Lemma 3.6 there is a decomposition $\mathcal{C} = \langle \mathcal{A}_1^{\perp}, \mathcal{A}_1 \rangle$, and by Lemma 3.8 there is an equivalence $\mathcal{A}_2 \simeq \mathcal{A}_1^{\perp}$. So we can apply the above argument to conclude $\mathcal{A}_2$ is also smooth and proper over S. This proves (4). □

4.6. **Critical loci.**

Definition 4.16. Let $\mathcal{C}$ be an S-linear category. Let $s \in S$ be a point and let $\mathrm{Spec}(\kappa(s)) \to S$ be the corresponding morphism from the spectrum of the residue field at $s \in S$. We say s is a *critical point* for $\mathcal{C}$ if the base change

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathrm{Spec}(\kappa(s)))$$

fails to be smooth over $\mathrm{Spec}(\kappa(s))$. The *critical locus* of $\mathcal{C}$ is the set

$$\mathrm{Crit}_S(\mathcal{C}) = \{\, s \in S \mid s \text{ is a critical point for } \mathcal{C} \,\} \subset S.$$

Remark 4.17. Let $X \to S$ be a morphism of schemes which is flat and locally of finite presentation. Then by Lemma 4.9(5), we have $s \in \mathrm{Crit}_S(\mathrm{Perf}(X))$ if and only if the fiber $X_s \to \mathrm{Spec}(\kappa(s))$ fails to be smooth (the flatness of $X \to S$ guarantees $X_s \to \mathrm{Spec}(\kappa(s))$ is an ordinary, i.e. not derived, scheme).

Remark 4.18. If $X \to S$ is a morphism of schemes which is flat, proper, and of finite presentation, then $\mathrm{Crit}_S(\mathrm{Perf}(X))$ is a closed subset of S. We expect that, under suitable assumptions, this holds for general S-linear categories $\mathcal{C}$, but we have not checked this.

Lemma 4.19. *Let $\mathcal{C}$ be a smooth S-linear category. Then* $\mathrm{Crit}_S(\mathcal{C}) = \varnothing$.

Proof. Follows from Lemma 4.10. □

Lemma 4.20. *Let $\mathcal{C}$ be an S-linear category with an S-linear semiorthogonal decomposition*

$$\mathcal{C} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$$

whose components are admissible. Then we have

$$\mathrm{Crit}_S(\mathcal{C}) = \bigcup_{i=1}^{n} \mathrm{Crit}_S(\mathcal{A}_i) \subset S.$$

Proof. Follows from Lemmas 4.15(3) and 3.17. □

5. A formalism of Fourier–Mukai kernels

In this section, we introduce a formalism for describing certain functors between linear categories in terms of "Fourier–Mukai kernels". After reviewing the classical situation in §5.1, we define our notion of Fourier–Mukai kernels in the noncommutative setting in §5.2. In §5.3 we develop some basic properties of these functors.

5.1. **The geometric setting.** Let $X_1 \to T$ and $X_2 \to T$ be morphisms of schemes, such that pushforward along the projection $\mathrm{pr}_2 \colon X_2 \times_T X_1 \to X_2$ preserves perfect complexes.

Remark 5.1. By [15, Example 2.2(a)], the assumption on pr_{2*} is satisfied if pr_2 is a perfect proper morphism.

In the above situation, we call an object

$$\mathcal{E} \in \mathrm{Perf}(X_2 \times_T X_1)$$

a Fourier–Mukai kernel. To any such $\mathcal{E}$, there is an associated T-linear functor

$$\Phi_{\mathcal{E}} \colon \mathrm{Perf}(X_1) \to \mathrm{Perf}(X_2), \quad F \mapsto \mathrm{pr}_{2*}(\mathcal{E} \otimes \mathrm{pr}_1^* F).$$

Remark 5.2. Our convention that a kernel $\mathcal{E} \in \mathrm{Perf}(X_2 \times_T X_1)$ corresponds to a functor $\mathrm{Perf}(X_2) \to \mathrm{Perf}(X_1)$, instead of $\mathrm{Perf}(X_2) \to \mathrm{Perf}(X_1)$, is slightly unconventional. But it will be convenient later.

This construction $\mathcal{E} \mapsto \Phi_{\mathcal{E}}$ gives rise to a functor

$$\mathrm{Perf}(X_2 \times_T X_1) \to \mathrm{Fun}_{\mathrm{Perf}(T)}(\mathrm{Perf}(X_1), \mathrm{Perf}(X_2)) \tag{5.1}$$

from the category of Fourier–Mukai kernels to the category of T-linear functors. We have the following fundamental result; the second part gives a criterion for the above functor to be an equivalence.

Theorem 5.3 ([3, Theorem 1.2])**.** *(1) For any $X_1 \to T$ and $X_2 \to T$, there is an equivalence*

$$\mathrm{Perf}(X_1 \times_T X_2) \simeq \mathrm{Perf}(X_1) \otimes_{\mathrm{Perf}(T)} \mathrm{Perf}(X_2).$$

(2) If $X_1 \to T$ is smooth and proper, (5.1) *is an equivalence.*

5.2. **The noncommutative setting.** We consider the following situation.

Setup 5.4. For $i = 1, 2$, assume given the following data:

– A diagram of schemes

$$\begin{array}{ccc} & Z_i & \\ \swarrow & & \searrow \\ S_i & & T \end{array} \tag{5.2}$$

such that the pushforward along the projection $\mathrm{pr}_2 \colon Z_2 \times_T Z_1 \to Z_2$ preserves perfect complexes.

– An S_i-linear category $\mathcal{C}_i$.

In the above setup, the base change

$$\mathcal{C}_i \otimes_{\mathrm{Perf}(S_i)} \mathrm{Perf}(Z_i)$$

has the structure of T-linear category. In what follows, this category plays the role of the scheme X_i from our discussion in §5.1 of Fourier–Mukai functors in the geometric setting.

Definition 5.5. In Setup 5.4, we define the category of *Fourier–Mukai kernels* as

$$\mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) = \mathrm{Fun}_{\mathrm{Perf}(S_1)}\big(\mathcal{C}_1, \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2 \times_T Z_1)\big).$$

When $\mathcal{C}_i$ and S_i are clear from the context, we sometimes refer to an object of the above category as a $Z_2 \times_T Z_1$-*kernel*.

This definition is motivated by the following result.

Proposition 5.6. *In Setup 5.4, there is a natural functor*

$$\mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) \to \mathrm{Fun}_{\mathrm{Perf}(T)}\big(\mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)\big),$$

which is an equivalence if $Z_1 \to T$ is smooth and proper.

Proof. The classical functor

$$\mathrm{Perf}(Z_2 \times_T Z_1) \to \mathrm{Fun}_{\mathrm{Perf}(T)}(\mathrm{Perf}(Z_1), \mathrm{Perf}(Z_2))$$

induces a functor

$$\mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2 \times_T Z_1) \to \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Fun}_{\mathrm{Perf}(T)}(\mathrm{Perf}(Z_1), \mathrm{Perf}(Z_2)).$$

Composing with the canonical functor

$$\mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Fun}_{\mathrm{Perf}(T)}(\mathrm{Perf}(Z_1), \mathrm{Perf}(Z_2)) \to \mathrm{Fun}_{\mathrm{Perf}(T)}\big(\mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)\big),$$

this induces a functor

$$\mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) \to \mathrm{Fun}_{\mathrm{Perf}(S_1)}\big(\mathcal{C}_1, \mathrm{Fun}_{\mathrm{Perf}(T)}\big(\mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)\big)\big).$$

Composing with the canonical equivalence

$$\mathrm{Fun}_{\mathrm{Perf}(S_1)}\big(\mathcal{C}_1, \mathrm{Fun}_{\mathrm{Perf}(T)}\big(\mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)\big)\big) \\ \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{Perf}(T)}\big(\mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)\big),$$

gives the sought-after functor. It follows from Theorem 5.3 and the construction that this functor is an equivalence if $Z_1 \to T$ is smooth and proper. □

In Setup 5.4, given a kernel

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1),$$

we denote by

$$\Phi_{\mathcal{E}} \in \mathrm{Fun}_{\mathrm{Perf}(T)}\big(\mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1), \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)\big)$$

the corresponding functor given by Proposition 5.6.

5.3. **Properties of kernel functors.** For the rest of this section, we describe the the effect of certain operations on $\mathcal{E}$ in terms of the associated functor $\Phi_{\mathcal{E}}$.

Setup 5.7. In addition to the data of Setup 5.4, assume given for $i = 1, 2$ another diagram of schemes

$$\begin{array}{ccc} & Z_i' & \\ \swarrow & & \searrow \\ S_i & & T' \end{array} \tag{5.3}$$

such that the pushforward along the projection $\mathrm{pr}_2 \colon Z_2' \times_{T'} Z_1' \to Z_2'$ preserves perfect complexes.

In the above setup, if we are given a functor

$$F\colon \mathrm{Perf}(Z_2 \times_T Z_1) \to \mathrm{Perf}(Z_2' \times_{T'} Z_1')$$

which is $S_2 \times_S S_1$-linear, we denote by the same symbol the induced functor

$$F\colon \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1) \to \mathrm{FM}\big(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2' \times_{T'} Z_1'\big)$$

on kernel categories. In particular, this construction allows us to pullback and pushforward kernels along morphisms $Z_2' \times_{T'} Z_1' \to Z_2 \times_T Z_1$, and to tensor kernels with objects of $\mathrm{Perf}(Z_2 \times_T Z_1)$. As we discuss below, these operations formally behave the same as in the geometric case.

The next two results follow by unwinding the definitions.

Lemma 5.8. *Assume we are in Setup 5.7. Let*

$$F_1 \to F_2 \to F_3$$

be an exact triangle of $S_2 \times_S S_1$-linear functors $\mathrm{Perf}(Z_2 \times_T Z_1) \to \mathrm{Perf}(Z_2' \times_{T'} Z_1')$. Let

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$$

be a kernel. Then there is an induced exact triangle of $Z_2' \times_{T'} Z_1'$-kernels

$$F_1(\mathcal{E}) \to F_2(\mathcal{E}) \to F_3(\mathcal{E}).$$

Lemma 5.9. *Assume we are in Setup 5.7. Let*

$$\mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3$$

be an exact triangle of kernels in $\mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$. Let

$$F\colon \mathrm{Perf}(Z_2 \times_T Z_1) \to \mathrm{Perf}(Z_2' \times_{T'} Z_1')$$

be an $S_2 \times_S S_1$-linear functor. Then there is an induced exact triangle of kernels

$$F(\mathcal{E}_1) \to F(\mathcal{E}_2) \to F(\mathcal{E}_3).$$

in $\mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2' \times_{T'} Z_1')$.

We leave it to the reader to check that the next two results reduce to the geometric case, where they are well-known.

Lemma 5.10. *Assume we are in Setup 5.7. Let $f\colon Z_2' \times_{T'} Z_1' \to Z_2 \times_T Z_1$ be a morphism such that f_* preserves perfect complexes.*

(1) Let

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$$

be a kernel. Let $\mathcal{F}' \in \mathrm{Perf}(Z_2' \times_{T'} Z_1')$. Then there is an equivalence of kernels

$$f_*(f^*(\mathcal{E}) \otimes \mathcal{F}') \simeq \mathcal{E} \otimes f_*(\mathcal{F}').$$

(2) Let

$$\mathcal{E}' \in \mathrm{FM}\big(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2' \times_{T'} Z_1'\big)$$

be a kernel. Let $\mathcal{F} \in \mathrm{Perf}(Z_2 \times_T Z_1)$. Then there is an equivalence of kernels

$$f_*(\mathcal{E}' \otimes f^*(\mathcal{F})) \simeq f_*(\mathcal{E}') \otimes \mathcal{F}.$$

Lemma 5.11. *Let*

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$$

be a kernel. Let $\mathcal{F}_2 \in \mathrm{Perf}(Z_2), \mathcal{F}_1 \in \mathrm{Perf}(Z_1)$*, and let* $\mathcal{F}_2 \boxtimes \mathcal{F}_1 \in \mathrm{Perf}(Z_2 \times_T Z_1)$ *be their exterior product. Then there is an equivalence*

$$\Phi_{\mathcal{E} \otimes (\mathcal{F}_2 \boxtimes \mathcal{F}_1)} \simeq (- \otimes \mathcal{F}_2) \circ \Phi_{\mathcal{E}} \circ (- \otimes \mathcal{F}_1).$$

Setup 5.12. Assume that in Setup 5.7, for $i = 1, 2$ the diagrams (5.2) and (5.3) fit into a commutative diagram

$$\begin{array}{ccccc} & & Z_i' & \longrightarrow & T' \\ & \swarrow & \downarrow{\scriptstyle f_i} & & \downarrow \\ S_i & \longleftarrow & Z_i & \longrightarrow & T \end{array} \tag{5.4}$$

In the above setup, we have a morphism

$$f_2 \times f_1 \colon Z_2' \times_{T'} Z_1' \to Z_2 \times_T Z_1,$$

which by pushforward (when it preserves perfect complexes) and pullback induces functors between kernel categories as above. Further, the morphism f_i induces by pushforward (when it preserves perfect complexes) and pullback functors

$$f_{i*} \colon \mathcal{C}_i \otimes_{\mathrm{Perf}(S_i)} \mathrm{Perf}(Z_i') \to \mathcal{C}_i \otimes_{\mathrm{Perf}(S_i)} \mathrm{Perf}(Z_i),$$
$$f_i^* \colon \mathcal{C}_i \otimes_{\mathrm{Perf}(S_i)} \mathrm{Perf}(Z_i) \to \mathcal{C}_i \otimes_{\mathrm{Perf}(S_i)} \mathrm{Perf}(Z_i').$$

The next two results easily reduce to the geometric case, where they are well-known.

Lemma 5.13. *In Setup 5.12, assume further that* $(f_2 \times f_1)_*$ *preserves perfect complexes. Let*

$$\mathcal{E}' \in \mathrm{FM}\big(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2' \times_{T'} Z_1'\big)$$

be a kernel, and

$$\mathcal{E} = (f_2 \times f_1)_* \mathcal{E}' \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1).$$

Then there is an equivalence

$$\Phi_{\mathcal{E}} \simeq f_{2*} \circ \Phi_{\mathcal{E}'} \circ f_1^*.$$

Lemma 5.14. *Assume we are in Setup 5.12. Let*

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$$

be a kernel, and

$$\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E} \in \mathrm{FM}\big(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2' \times_{T'} Z_1'\big).$$

(1) If the square

$$\begin{array}{ccc} Z_2' & \longleftarrow & Z_2' \times_{T'} Z_1' \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_2 \times f_1} \\ Z_2 & \longleftarrow & Z_2 \times_T Z_1 \end{array}$$

is cartesian, then there is an equivalence

$$\Phi_{\mathcal{E}'} \circ f_1^* \simeq f_2^* \circ \Phi_{\mathcal{E}}. \tag{5.5}$$

(2) If the square

$$\begin{array}{ccc} Z_2' \times_{T'} Z_1' & \longrightarrow & Z_1' \\ {\scriptstyle f_2\times f_1}\big\downarrow & & \big\downarrow{\scriptstyle f_1} \\ Z_2 \times_T Z_1 & \longrightarrow & Z_1 \end{array}$$

is cartesian, then there is an equivalence

$$f_{2*} \circ \Phi_{\mathcal{E}'} \simeq \Phi_{\mathcal{E}} \circ f_{1*}. \tag{5.6}$$

Remark 5.15. For $i = 1, 2$, the assumption of Lemma 5.16(i) automatically holds if the square in diagram (5.4) is cartesian.

Lemma 5.16. *In Setup 5.12, assume further that that $T' = T$. Let*

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$$

be a kernel, and

$$\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E} \in \mathrm{FM}\big(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2' \times_T Z_1'\big).$$

Then there is an equivalence

$$\Phi_{\mathcal{E}'} \simeq f_2^* \circ \Phi_{\mathcal{E}} \circ f_{1*}. \tag{5.7}$$

Proof. Factor $f_2 \times f_1$ as the composition

$$Z_2' \times_T Z_1' \xrightarrow{\mathrm{id}\times f_1} Z_2' \times_T Z_1 \xrightarrow{f_2 \times \mathrm{id}} Z_2 \times_T Z_1$$

and apply successively Lemma 5.16 parts (1) and (2). □

Proposition 5.17. *In Setup 5.12, assume further that for $i = 1, 2$ the square in diagram* (5.4) *is cartesian. Let*

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$$

be a kernel, and let $\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E}$ be its base change. Assume the associated functors

$$\Phi_{\mathcal{E}} \colon \mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1) \to \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2),$$
$$\Phi_{\mathcal{E}'} \colon \mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1') \to \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2'),$$

admit left adjoints $\Phi_{\mathcal{E}}^*$ and $\Phi_{\mathcal{E}'}^*$.

(1) If $\Phi_{\mathcal{E}}$ is fully faithful, so is $\Phi_{\mathcal{E}'}$.

(2) If $\Phi_{\mathcal{E}}^*$ is fully faithful, so is $\Phi_{\mathcal{E}'}^*$.

Proof. For (1), we must check that the canonical morphism

$$\Phi_{\mathcal{E}'}^* \Phi_{\mathcal{E}'}(C) \to C$$

is an equivalence for any $C \in \mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1')$. Note that by Theorem 5.3(1), there is an equivalence

$$\mathrm{Perf}(Z_1') \simeq \mathrm{Perf}(Z_1) \otimes_{\mathrm{Perf}(T)} \mathrm{Perf}(T').$$

Hence Lemma 2.6 implies it is enough to check the morphism above is an equivalence for

$$C = C_1 \boxtimes F_1 \boxtimes G,$$

where $C_1 \in \mathcal{C}_1, F_1 \in \mathrm{Perf}(Z_1), G \in \mathrm{Perf}(T')$. This follows from T'-linearity of $\Phi_{\mathcal{E}'}$ and $\Phi_{\mathcal{E}'}^*$, the equivalence (5.7), and the equivalence $\Phi_{\mathcal{E}'}^* \circ f_2^* \simeq f_1^* \circ \Phi_{\mathcal{E}}^*$ obtained from (5.6) by taking left adjoints.

Part (2) is proved similarly. □

Proposition 5.18. *In Setup 5.12, assume further that for $i = 1, 2$ the diagram* (5.4) *is cartesian. Let*

$$\mathcal{E} \in \mathrm{FM}(\mathcal{C}_1/S_1, \mathcal{C}_2/S_2, Z_2 \times_T Z_1)$$

be a kernel such that the associated functor

$$\Phi_{\mathcal{E}} \colon \mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1) \to \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2)$$

is left splitting. Let $\mathcal{E}' = (f_2 \times f_1)^* \mathcal{E}$ be the base changed kernel, and assume the associated functor

$$\Phi_{\mathcal{E}'} \colon \mathcal{C}_1 \otimes_{\mathrm{Perf}(S_1)} \mathrm{Perf}(Z_1') \to \mathcal{C}_2 \otimes_{\mathrm{Perf}(S_2)} \mathrm{Perf}(Z_2')$$

admits a left adjoint. Then $\Phi_{\mathcal{E}'}$ is left splitting.

Proof. By the same argument used to prove Proposition 5.17, it follows that $\Phi_{\mathcal{E}'}$ satisfies the condition of Theorem 3.23(2). □

Part II. Homological projective duality

6. Lefschetz categories

In this section, we introduce the notion of a Lefschetz category over a projective space, which plays the role of an embedded projective variety in "homological projective geometry". We start by studying in §6.1 the notion of a Lefschetz decomposition of a linear category equipped with an autoequivalence. A Lefschetz category over a projective space (or bundle) $\mathbf{P}(V)$ is then defined as a $\mathbf{P}(V)$-linear category equipped with a Lefschetz decomposition with respect to the autoequivalence given by tensoring with $\mathcal{O}_{\mathbf{P}(V)}(1)$. In §6.2 we show that there are induced semiorthogonal decompositions of the "linear sections" of a Lefschetz category over $\mathbf{P}(V)$. Our definitions are modeled on those of [9], which treats the geometric case.

6.1. Lefschetz sequences.

Definition 6.1. Let $\mathcal{C}$ be an S-linear category, and let $\mathrm{T}\colon \mathcal{C} \to \mathcal{C}$ be an S-linear autoequivalence. A *right Lefschetz sequence* (of length m) in $\mathcal{C}$ with respect to T is a sequence

$$\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1}$$

of admissible S-linear subcategories of $\mathcal{C}$, such that the sequence

$$\mathcal{A}_0, \mathrm{T}(\mathcal{A}_1), \ldots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \tag{6.1}$$

is semiorthogonal in $\mathcal{C}$. A right Lefschetz sequence is called:

– *rectangular* if

$$\mathcal{A}_0 = \mathcal{A}_1 = \cdots = \mathcal{A}_{m-1}.$$

– *full* if (6.1) gives a semiorthogonal decomposition

$$\mathcal{C} = \left\langle \mathcal{A}_0, \mathrm{T}(\mathcal{A}_1), \ldots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \right\rangle, \tag{6.2}$$

which is called a *right Lefschetz decomposition*.

Given a right Lefschetz sequence $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1}$, we define the *$i$-th primitive component* $\mathfrak{a}_i$ to be the S-linear subcategory of $\mathcal{A}_i$ given as the right orthogonal to $\mathcal{A}_{i+1}$ in $\mathcal{A}_i$. For convenience, we also set $\mathcal{A}_i = \mathfrak{a}_i = 0$ for $i \geq m$. By construction, we have an S-linear semiorthogonal decomposition

$$\mathcal{A}_i = \langle \mathfrak{a}_i, \mathfrak{a}_{i+1}, \ldots, \mathfrak{a}_{m-1} \rangle . \tag{6.3}$$

Next we prove that in the case of a full Lefschetz sequence, $\mathcal{A}_0$ also admits a decomposition into certain "twisted" primitive components. Let $\alpha_0\colon \mathcal{A}_0 \to \mathcal{C}$ denote the inclusion functor, and let $\alpha_0^*\colon \mathcal{C} \to \mathcal{A}_0$ denote its left adjoint. We define the *i-th twisted primitive component* by

$$\mathfrak{a}'_i = \alpha_0^*(\mathrm{T}^{i+1}(\mathfrak{a}_i)). \tag{6.4}$$

Lemma 6.2. *Let $\mathcal{C}$ be an S-linear category, with an S-linear autoequivalence T and a full right Lefschetz sequence $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1}$ with respect to T. Then:*

(1) The functor $\alpha_0^*\colon \mathcal{C} \to \mathcal{A}_0$ is fully faithful on the category $\mathrm{T}^{i+1}(\mathfrak{a}_i)$ for all i, hence $\mathfrak{a}'_i$ is a stable subcategory of $\mathcal{A}_0$.

(2) There is an S-linear semiorthogonal decomposition

$$\mathcal{A}_0 = \left\langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_{m-1} \right\rangle . \tag{6.5}$$

Proof. Applying the autoequivalence T to (6.2) and using $\mathcal{A}_i = \langle \mathfrak{a}_i, \mathcal{A}_{i+1} \rangle$, we obtain a decomposition

$$\mathcal{C} = \left\langle \mathrm{T}(\mathfrak{a}_0), \mathrm{T}(\mathcal{A}_1), \mathrm{T}^2(\mathfrak{a}_1), \mathrm{T}^2(\mathcal{A}_2), \dots, \mathrm{T}^{m-1}(\mathfrak{a}_{m-2}), \mathrm{T}^{m-1}(\mathcal{A}_{m-1}), \mathrm{T}^m(\mathfrak{a}_{m-1}) \right\rangle . \quad (6.6)$$

Hence using Lemma 3.11, we find a decomposition

$$\mathcal{C} = \left\langle \mathrm{T}(\mathfrak{a}_0), \mathrm{L}_{\mathrm{T}(\mathcal{A}_1)}(\mathrm{T}^2(\mathfrak{a}_1)), \dots, \mathrm{L}_{\langle \mathrm{T}(\mathcal{A}_1), \dots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \rangle}(\mathrm{T}^m(\mathfrak{a}_{m-1})), \mathrm{T}(\mathcal{A}_1), \dots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \right\rangle .$$

Comparing with (6.2), this implies there is a decomposition

$$\mathcal{A}_0 = \left\langle \mathrm{T}(\mathfrak{a}_0), \mathrm{L}_{\mathrm{T}(\mathcal{A}_1)}(\mathrm{T}^2(\mathfrak{a}_1)), \dots, \mathrm{L}_{\langle \mathrm{T}(\mathcal{A}_1), \dots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \rangle}(\mathrm{T}^m(\mathfrak{a}_{m-1})) \right\rangle .$$

Hence by Lemma 3.8, to finish it suffices to show α_0^* is given on $\mathrm{T}^{i+1}(\mathfrak{a}_i)$ by the mutation functor $\mathrm{L}_{\langle \mathrm{T}(\mathcal{A}_1), \dots, \mathrm{T}^i(\mathcal{A}_i) \rangle}$. But this holds since by (6.6) we have

$$\mathrm{T}^{i+1}(\mathfrak{a}_i) \subset \left\langle \mathrm{T}^{i+1}(\mathcal{A}_i), \mathrm{T}^{i+2}(\mathcal{A}_{i+2}), \dots, \mathrm{T}^{m-1}(\mathcal{A}_{m-1}) \right\rangle^{\perp} . \qquad \square$$

There is a "left" analgoue of right Lefschetz sequences.

Definition 6.3. Let $\mathcal{D}$ be an S-linear category, and let $\mathrm{T} \colon \mathcal{D} \to \mathcal{D}$ be an S-linear autoequivalence. A *left Lefschetz sequence* (of length n) in $\mathcal{D}$ with respect to T is a sequence

$$\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$$

of admissible S-linear subcategories of $\mathcal{D}$, such that the sequence

$$\mathrm{T}^{-(n-1)}(\mathcal{B}_{n-1}), \dots, \mathrm{T}^{-1}(\mathcal{B}_1), \mathcal{B}_0 \quad (6.7)$$

is semiorthogonal in $\mathcal{D}$. A left Lefschetz sequence is called:

– *rectangular* if

$$\mathcal{B}_{n-1} = \cdots = \mathcal{B}_1 = \mathcal{B}_0.$$

– *full* if (6.7) gives a semiorthogonal decomposition

$$\mathcal{D} = \left\langle \mathrm{T}^{-(n-1)}(\mathcal{B}_{n-1}), \dots, \mathrm{T}^{-1}(\mathcal{B}_1), \mathcal{B}_0 \right\rangle , \quad (6.8)$$

which is called a *left Lefschetz decomposition*.

Given a left Lefschetz sequence $\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$, we define the j-th primitive component $\mathfrak{b}_j$ to be the left orthogonal to $\mathcal{B}_{j+1}$ in $\mathcal{B}_j$, so that there is a semiorthogonal decomposition

$$\mathcal{B}_j = \langle \mathfrak{b}_{n-1}, \dots, \mathfrak{b}_{j+1}, \mathfrak{b}_j \rangle . \quad (6.9)$$

If we set

$$\mathfrak{b}'_j = \beta_0^!(\mathrm{T}^{-(j+1)}(\mathfrak{b}_j))$$

where $\beta_0 \colon \mathcal{B}_0 \to \mathcal{D}$ denotes the inclusion functor and $\beta_0^! \colon \mathcal{D} \to \mathcal{B}_0$ its right adjoint, then there is an obvious analogue of Lemma 6.2; we will not need it.

Finally, we note that a full Lefschetz sequence can be reconstructed from its biggest component:

Lemma 6.4. *(1) Let $\mathcal{C}$ be an S-linear category with an S-linear autoequivalence* $\mathrm{T} \colon \mathcal{C} \to \mathcal{C}$. *Let $\mathcal{A}_0 \supset \cdots \supset \mathcal{A}_{m-1}$ be a full right Lefschetz sequence with respect to* T. *Then $\mathcal{A}_0$ determines $\mathcal{A}_i$ inductively for all i by the formula*

$$\mathcal{A}_i = \mathrm{T}^{-i}({}^{\perp}\mathcal{A}_0) \cap \mathcal{A}_{i-1}.$$

(2) Let $\mathcal{D}$ be an S-linear category with an S-linear autoequivalence $\mathrm{T}\colon \mathcal{D} \to \mathcal{D}$. Let $\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_0$ be a full left Lefschetz sequence with respect to T. Then $\mathcal{B}_0$ determines $\mathcal{B}_j$ inductively for all j by the formula

$$\mathcal{B}_j = \mathrm{T}^j(\mathcal{B}_0^{\perp}) \cap \mathcal{B}_{j-1}.$$

Proof. The same proof as in [11, Lemma 2.18] works. □

6.2. **Lefschetz categories over a projective bundle.** Let V be a rank N vector bundle over our base scheme S, and let H denote the relative hyperplane class on the projective bundle $\mathbf{P}(V)$. Given a $\mathbf{P}(V)$-linear category $\mathcal{C}$ and $C \in \mathcal{C}$, we use the notation $C(H) = C \otimes \mathcal{O}_{\mathbf{P}(V)}(H)$.

We will be concerned with Lefschetz sequences that are compatible with a $\mathbf{P}(V)$-linear structure, in the following sense.

Definition 6.5. A *right Lefschetz category over* $\mathbf{P}(V)$ is a $\mathbf{P}(V)$-linear category $\mathcal{C}$ equipped with a full right Lefschetz sequence $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1}$ with respect to the (S-linear) autoequivalence $- \otimes \mathcal{O}_{\mathbf{P}(V)}(H)\colon \mathcal{C} \to \mathcal{C}$. We say that $\mathcal{C}$ is *nondegenerate* if the length of its Lefschetz sequence is less than the rank of V, i.e. if $m < N$.

A *left Lefschetz category over* $\mathbf{P}(V)$ (and nondegeneracy of such) is defined analogously.

Remark 6.6. We will often simply write "$\mathcal{C}$ is a Lefschetz category over $\mathbf{P}(V)$", leaving implicit the Lefschetz sequence $\mathcal{A}_0 \supset \mathcal{A}_1 \supset \cdots \supset \mathcal{A}_{m-1}$.

See [12] for many interesting examples of Lefschetz decompositions. Let us record one:

Example 6.7. Let $L \subset V$ be a subbundle of rank r, and let $\pi\colon \mathbf{P}(L) \to S$ be the corresponding projective bundle. Then the standard semiorthogonal decomposition

$$\mathrm{Perf}(\mathbf{P}(L)) = \langle \pi^*\mathrm{Perf}(S), \pi^*\mathrm{Perf}(S)(H), \dots, \pi^*\mathrm{Perf}(S)((r-1)H) \rangle$$

gives $\mathrm{Perf}(\mathbf{P}(L))$ the structure of a right Lefschetz category over $\mathbf{P}(V)$, which is nondegenerate unless $L = V$.

Remark 6.8. The hypothesis that a Lefschetz category $\mathcal{C}$ is nondegenerate is quite mild. First of all, we can always embed V into a larger rank vector bundle V' so that $\mathcal{C}$ is a nondegenerate Lefschetz category over $\mathbf{P}(V')$. Second, the Lefschetz categories that arise in practice are essentially always nondegenerate. In fact, we show in Corollary 6.14 below that the length of the Lefschetz decomposition of $\mathcal{C}$ is at most N, so that the only way for nondegeneracy to fail is if $m = N$. Moreover, if $m = N$, we show $\mathcal{C}$ is closely related to $\mathrm{Perf}(\mathbf{P}(V))$.

Definition 6.9. An equivalence between two Lefschetz categories $\mathcal{C}$ and $\mathcal{C}'$ over $\mathbf{P}(V)$ is a $\mathbf{P}(V)$-linear equivalence of categories $\mathcal{C} \simeq \mathcal{C}'$ that identifies their Lefschetz sequences.

Next we describe the behavior of Lefschetz categories under passage to linear sections. From now on, we simplify our notation by writing $\mathbf{P} = \mathbf{P}(V)$. Further, for $0 \leq r \leq N$, let $\mathbf{G}_r = \mathrm{Gr}(r, V)$ be the Grassmannian of rank r subbundles of V, and let $\mathcal{U}_r$ be the universal rank r bundle on $\mathbf{G}_r$. Let $\mathbf{L}_r = \mathbf{P}_{\mathbf{G}_r}(\mathcal{U}_r)$ be the corresponding universal family of (projective) linear subspaces. These spaces fit into a commutative diagram

$$\begin{array}{ccc} \mathbf{L}_r & \xrightarrow{f_r} & \mathbf{G}_r \\ {\scriptstyle p_r}\downarrow & & \downarrow \\ \mathbf{P} & \longrightarrow & S \end{array} \tag{6.10}$$

Note that the induced morphism

$$\iota_r \colon \mathbf{L}_r \to \mathbf{P} \times_S \mathbf{G}_r$$

is an embedding.

Given a $\mathbf{P}$-linear category $\mathcal{C}$ (e.g. a Lefschetz category over $\mathbf{P}$), we define the corresponding family of linear sections by

$$\mathbf{L}_r(\mathcal{C}) = \mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{L}_r).$$

Remark 6.10. If there exists a morphism of schemes $X \to \mathbf{P}$ and a $\mathbf{P}$-linear equivalence $\mathcal{C} \simeq \mathrm{Perf}(X)$, then by Theorem 5.3(1)

$$\mathbf{L}_r(\mathcal{C}) \simeq \mathrm{Perf}(X) \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{L}_r) \simeq \mathrm{Perf}(X \times_{\mathbf{P}} \mathbf{L}_r).$$

Hence, at the level of perfect complexes, the category $\mathbf{L}_r(\mathcal{C})$ recovers the universal family of linear sections $X \times_{\mathbf{P}} \mathbf{L}_r$ of X.

Remark 6.11. Let $L \subset V$ be a fixed rank r subbundle of V. Let $x_L \colon S \to \mathbf{G}_r$ be the corresponding morphism, so that the pullback of $\mathbf{L}_r \to \mathbf{G}_r$ along x_L is $\mathbf{P}(L) \to S$. Then by Theorem 5.3(1), base change along $x_L^* \colon \mathrm{Perf}(\mathbf{G}_r) \to \mathrm{Perf}(S)$ gives

$$\mathbf{L}_r(\mathcal{C}) \otimes_{\mathrm{Perf}(\mathbf{G}_r)} \mathrm{Perf}(S) \simeq \mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P}(L)).$$

The category on the right should be thought of as the "linear section" of $\mathcal{C}$ by $\mathbf{P}(L)$. Indeed, if there exists a morphism of schemes $X \to \mathbf{P}$ and a $\mathbf{P}$-linear equivalence $\mathcal{C} \simeq \mathrm{Perf}(X)$, then by Theorem 5.3(1)

$$\mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P}(L)) \simeq \mathrm{Perf}(X \times_{\mathbf{P}} \mathbf{P}(L)).$$

Hence in general the "fibers" of the natural $\mathbf{G}_r$-linear structure on $\mathbf{L}_r(\mathcal{C})$ are the "linear sections" of $\mathcal{C}$.

Remark 6.12. The category $\mathbf{L}_r(\mathcal{C})$ is easy to describe for some extreme values of r:
- $\mathbf{G}_0 = S$ and $\mathbf{L}_0 = \varnothing$, so $\mathbf{L}_0(\mathcal{C}) = 0$.
- $\mathbf{G}_1 = \mathbf{L}_1 = \mathbf{P}$, so $\mathbf{L}_1(\mathcal{C}) = \mathcal{C}$.
- $\mathbf{G}_N = S$ and $\mathbf{L}_N = \mathbf{P}$, so $\mathbf{L}_N(\mathcal{C}) = \mathcal{C}$.

We can also form the product of the $\mathbf{P}$-linear category $\mathcal{C}$ with $\mathbf{G}_r$, i.e.

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r).$$

If $\mathcal{C}$ is a right Lefschetz category over $\mathbf{P}$, then by Lemma 3.15 there is a $\mathbf{G}_r$-linear semiorthogonal decomposition

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) = \left\langle \mathcal{A}_0 \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \right\rangle. \tag{6.11}$$

We are going to show that part of this semiorthogonal decomposition embeds into $\mathbf{L}_r(\mathcal{C})$. For this, we consider the functor

$$f_r^* \colon \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \to \mathbf{L}_r(\mathcal{C}). \tag{6.12}$$

induced by pullback along the morphism $f_r \colon \mathbf{L}_r \to \mathbf{G}_r$.

Lemma 6.13. *Let $\mathcal{C}$ be $\mathbf{P}$-linear category equipped with a right Lefschetz sequence $\mathcal{A}_0 \supset \cdots \supset \mathcal{A}_{m-1}$ with respect to $- \otimes \mathcal{O}_{\mathbf{P}}(H)$. Fix nonnegative integers r and s such that $r+s = N$. Then the functor* (6.12) *is fully faithful on the subcategory*

$$\mathcal{A}_i \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \subset \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r)$$

for $s \le i \le m-1$. Moreover, the images under (6.12) *of the categories*

$$\mathcal{A}_s(sH) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \mathcal{A}_{s+1}((s+1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots \\ \dots, \mathcal{A}_{m-1}((m-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \quad (6.13)$$

form a semiorthogonal sequence of admissible $\mathbf{G}_r$-linear stable subcategories of $\mathbf{L}_r(\mathcal{C})$.

Proof. Note that there is an equivalence

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \simeq \mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P} \times_S \mathbf{G}_r).$$

induced by pullback along the projection $\mathbf{P} \times_S \mathbf{G}_r \to \mathbf{G}_r$. Under this equivalence, the functor (6.12) is identified with the $\mathbf{G}_r$-linear functor

$$\iota_r^* \colon \mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P} \times_S \mathbf{G}_r) \to \mathbf{L}_r(\mathcal{C}), \quad (6.14)$$

which we will work with instead. By the semiorthogonal decomposition (6.11), to prove the result it suffices to show that if $C \in \mathcal{A}_i(iH)$, $D \in \mathcal{A}_j(jH)$, $s \le i \le j \le m-1$, and F, G are in the image of the pullback functor $\mathrm{Perf}(\mathbf{G}_r) \to \mathrm{Perf}(\mathbf{P} \times_S \mathbf{G}_r)$, then

$$\mathcal{H}om_{\mathbf{G}_r}(\iota_r^*(D \boxtimes G), \iota_r^*(C \boxtimes F)) \simeq \mathcal{H}om_{\mathbf{G}_r}(D \boxtimes G, C \boxtimes F).$$

For this, first observe that by adjunction

$$\mathcal{H}om_{\mathbf{G}_r}(\iota_r^*(D \boxtimes G), \iota_r^*(C \boxtimes F)) \simeq \mathcal{H}om_{\mathbf{G}_r}(D \boxtimes G, C \boxtimes \iota_{r*}\iota_r^* F).$$

Next note that $\iota_r \colon \mathbf{L}_r \hookrightarrow \mathbf{P} \times_S \mathbf{G}_r$ is a codimension s subscheme, cut out by the tautological section of the vector bundle $\mathcal{Q}_r(H)$, where $\mathcal{Q}_r = V/\mathcal{U}_r$ is the universal quotient bundle. Hence there is a Koszul resolution

$$0 \to (\wedge^s \mathcal{Q}_r^\vee)(-sH) \to \cdots \to \mathcal{Q}_r^\vee(-H) \to \mathcal{O}_{\mathbf{P} \times_S \mathbf{G}_r} \to \mathcal{O}_{\mathbf{L}_r} \to 0. \quad (6.15)$$

In view of the equivalence $\iota_{r*}\iota_r^* F \simeq F \otimes \mathcal{O}_{\mathbf{L}_r}$, it thus suffices to show that

$$\mathcal{H}om_{\mathbf{G}_r}(D \boxtimes G, C(-tH) \boxtimes (F \otimes \wedge^t \mathcal{Q}_r^\vee))$$

vanishes for $1 \le t \le s$. But $C(-tH) \in \mathcal{A}_i((i-t)H) \subset \mathcal{A}_{i-t}((i-t)H)$, so the vanishing holds by (6.11). This proves the fully faithful and semiorthogonal statements of the lemma. The subcategories (6.13) are admissible because each $\mathcal{A}_i \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \subset \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r)$ is (by Lemma 3.17) and the functor (6.14) has left and right adjoints (by Lemma 2.11). □

The following result was anticipated in Remark 6.8.

Corollary 6.14. *Let $\mathcal{C}$ be $\mathbf{P}$-linear category equipped with a right Lefschetz sequence $\mathcal{A}_0 \supset \cdots \supset \mathcal{A}_{m-1}$ with respect to $- \otimes \mathcal{O}_{\mathbf{P}}(H)$.*

(1) $\mathcal{A}_i = 0$ for $i \ge N$.

(2) The action functor $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}) \to \mathcal{C}$ is fully faithful on the subcategory

$$\mathcal{A}_{N-1} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}) \subset \mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}).$$

(3) If $\mathcal{A}_i = \mathcal{A}$ is constant for $0 \le i \le N-1$, then the action functor $\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}) \to \mathcal{C}$ induces an equivalence of Lefschetz categories

$$\mathcal{A} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}) \simeq \mathcal{C},$$

where the Lefschetz decomposition on the left side is the one induced by the standard semiorthogonal decomposition of $\mathrm{Perf}(\mathbf{P})$*, namely*

$$\mathcal{A} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}) = \langle \mathcal{A}, \mathcal{A}(H), \dots, \mathcal{A}((N-1)H) \rangle .$$

Proof. Taking $r = 0$ in Lemma 6.13 gives (1), taking $r = 1$ gives (2), and then (3) follows from the definitions. □

In the situation of Lemma 6.13, we denote by $\mathcal{K}_r(\mathcal{C})$ the right orthogonal in $\mathbf{L}_r(\mathcal{C})$ to the twist by $\mathcal{O}(-(s-1)H)$ of the image of the sequence of categories (6.13). The twist by $\mathcal{O}(-(s-1)H)$ in our definition does not affect $\mathcal{K}_r(\mathcal{C})$ up to equivalence, but this normalization will be convenient later. Since f_r^* is an equivalence when restricted to any of the categories in (6.13), we omit it in our notation for their images. Thus we have a $\mathbf{G}_r$-linear semiorthogonal decomposition

$$\mathbf{L}_r(\mathcal{C}) = \left\langle \mathcal{K}_r(\mathcal{C}), \mathcal{A}_s(H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-s)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \right\rangle. \quad (6.16)$$

For $s \geq m$ the sequence (6.13) is empty, so by definition $\mathbf{L}_r(\mathcal{C}) = \mathcal{K}_r(\mathcal{C})$ in this case. Further, if $m < N$ (i.e. if $\mathcal{C}$ is a nondegenerate Lefschetz category), then $\mathcal{K}_1(\mathcal{C}) = \mathbf{L}_1(\mathcal{C}) = \mathcal{C}$.

The $\mathbf{G}_r$-linear category $\mathcal{K}_r(\mathcal{C})$ should be thought of as the "interesting" component of the family of linear sections $\mathbf{L}_r(\mathcal{C})$, since the remaining components in (6.16) come from the ambient Lefschetz sequence. Our ultimate goal is to show that, when $\mathcal{C}$ is smooth and proper over S, the category $\mathcal{K}_{N-1}(\mathcal{C})$ controls all of the other $\mathcal{K}_r(\mathcal{C})$. Given a rank r subbundle $L \subset V$, we define

$$\mathcal{K}_L(\mathcal{C}) = \mathcal{K}_r(\mathcal{C}) \otimes_{\mathrm{Perf}(\mathbf{G}_r)} \mathrm{Perf}(S) \quad (6.17)$$

as the base change along the morphism $S \to \mathbf{G}_r$ classifying $L \subset V$.

Lemma 6.15. *Let $\mathcal{C}$ be $\mathbf{P}$-linear category equipped with a right Lefschetz sequence $\mathcal{A}_0 \supset \cdots \supset \mathcal{A}_{m-1}$ with respect to $- \otimes \mathcal{O}_{\mathbf{P}}(H)$. Fix nonnegative integers r and s such that $r + s = N$. Let $L \subset V$ be a rank r subbundle. Then there is a semiorthogonal decomposition*

$$\mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P}(L)) = \langle \mathcal{K}_L(\mathcal{C}), \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle.$$

Proof. Follows from Remark 6.11, Lemma 3.15, and the decomposition (6.16). □

Remark 6.16. Lemma 6.15 says that for a range of degrees, the components of a Lefschetz sequence in $\mathcal{C}$ embed into any linear section of $\mathcal{C}$. This is analogous to the behavior of the cohomology of a projective variety under passage to linear sections, as governed by the Lefschetz hyperplane theorem. This analogy is the source of our terminology, which goes back to [9].

We end this section by recording the obvious "left" analogue of Lemma 6.13.

Lemma 6.17. *Let $\mathcal{D}$ be $\mathbf{P}$-linear category equipped with a left Lefschetz sequence $\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_0$ with respect to $- \otimes \mathcal{O}_{\mathbf{P}}(H)$. Fix nonnegative integers r and s such that $r + s = N$. Then the functor $f_r^* \colon \mathcal{D} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \to \mathbf{L}_r(\mathcal{D})$ is fully faithful on the subcategory*

$$\mathcal{B}_j \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \subset \mathcal{D} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r)$$

for $s \leq j \leq n-1$. Moreover, the images of the categories

$$\begin{aligned} &\mathcal{B}_{n-1}(-(n-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots \\ &\qquad \dots, \mathcal{B}_{s+1}(-(s+1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \mathcal{B}_s(-sH) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \end{aligned} \quad (6.18)$$

is a semiorthogonal sequence of admissible $\mathbf{G}_r$-linear stable subcategories of $\mathbf{L}_r(\mathcal{D})$.

In the situation of Lemma 6.17, we define a $\mathbf{G}_r$-linear category $\mathcal{K}_r(\mathcal{D})$ by the semiorthogonal decomposition

$$\mathbf{L}_r(\mathcal{D}) = \left\langle \mathcal{B}_{n-1}(-(n-s)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots, \mathcal{B}_s(-H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \mathcal{K}_r(\mathcal{D}) \right\rangle.$$

7. The homological projective dual of a Lefschetz category

Note that there is an isomorphism $\mathbf{G}_{N-1} \cong \mathbf{P}(V^\vee)$. Parallel to our notation above, we write $\mathbf{P}^\vee = \mathbf{P}(V^\vee)$, and denote by H' the relative hyperplane class on $\mathbf{P}^\vee$. Moreover, we use the notation $\mathbf{H} = \mathbf{L}_{N-1}$ for the universal hyperplane in $\mathbf{P}$. We also drop the subscripts in our notation for the maps $p_{N-1}, f_{N-1}, \iota_{N-1}$, so that diagram (6.10) takes the form

$$\begin{array}{ccc} \mathbf{H} & \xrightarrow{f} & \mathbf{P}^\vee \\ {\scriptstyle p}\downarrow & & \downarrow \\ \mathbf{P} & \longrightarrow & S \end{array}$$

and we have an embedding $\iota\colon \mathbf{H} \to \mathbf{P} \times_S \mathbf{P}^\vee$. Finally, given a $\mathbf{P}$-linear category $\mathcal{C}$, we write $\mathbf{H}(\mathcal{C}) = \mathbf{L}_{N-1}(\mathcal{C})$ and call this category the *universal hyperplane section* of $\mathcal{C}$.

Definition 7.1. Let $\mathcal{C}$ be a right Lefschetz category over $\mathbf{P}$. The *homological projective dual (HPD) category* $\mathcal{C}^\vee$ of $\mathcal{C}$ is the $\mathbf{P}^\vee$-linear category $\mathcal{C}^\vee = \mathcal{K}_{N-1}(\mathcal{C})$. Explicitly, $\mathcal{C}^\vee$ is defined by the semiorthogonal decomposition

$$\mathbf{H}(\mathcal{C}) = \left\langle \mathcal{C}^\vee, \mathcal{A}_1(H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee), \dots, \mathcal{A}_{m-1}((m-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee) \right\rangle. \qquad (7.1)$$

In §7.1 we describe the relationship between the HPD category and classical projective duality. Our aim for the rest of this section is to show that, if $\mathcal{C}$ is a nondegenerate right Lefschetz category over $\mathbf{P}$ which is smooth and proper over S, then the HPD category $\mathcal{C}^\vee$ is equipped with a natural left Lefschetz sequence with respect to the autoequivalence $- \otimes \mathcal{O}_{\mathbf{P}^\vee}(H')$ (Proposition 7.10).[2] Later we will prove that the Lefschetz sequence we construct is full (Theorem 8.7(2)), so that $\mathcal{C}^\vee$ is naturally a (nondegenerate) left Lefschetz category over $\mathbf{P}^\vee$.

Remark 7.2. All of the results in this section admit obvious analogues for left Lefschetz categories, which we leave to the reader to formulate.

7.1. Relation to classical projective duality.

Recall that in Definition 4.16 we introduced the notion of the critical locus of a linear category.

Definition 7.3. Let $\mathcal{C}$ be a $\mathbf{P}$-linear category which is smooth and proper over S. The *classical projective dual* $\mathrm{CPD}(\mathcal{C}) \subset \mathbf{P}^\vee$ is the set

$$\mathrm{CPD}(\mathcal{C}) = \mathrm{Crit}_{\mathbf{P}^\vee}(\mathbf{H}(\mathcal{C})) \subset \mathbf{P}^\vee.$$

Remark 7.4. Definition 7.3 can also be made without the assumption that $\mathcal{C}$ is smooth and proper over S. We impose these hypotheses so that our definition recovers the classical notion in the geometric case (see Corollary 7.6 below).

Lemma 7.5. *Assume our base $S = \mathrm{Spec}(k)$ where k is an algebraically closed field. Let X be an integral scheme of finite type over S equipped with a closed immersion $X \to \mathbf{P}$, such that X is not contained in any hyperplane in $\mathbf{P}$. Then*

$$\mathrm{Crit}_{\mathbf{P}^\vee}(\mathbf{H}(\mathrm{Perf}(X))) = \{\, H \in \mathbf{P}^\vee \mid X \times_{\mathbf{P}} H \text{ is singular} \,\}.$$

[2] In fact, modulo admissibility of the components of the Lefschetz sequence, we prove the same without the assumption that $\mathcal{C}$ is smooth and proper over S.

Proof. Since X is not contained in any hyperplane in $\mathbf{P}$, the universal hyperplane section $X \times_{\mathbf{P}} \mathbf{H}$ of X is flat over $\mathbf{P}^\vee$. So since $\mathbf{H}(\mathrm{Perf}(X)) \simeq \mathrm{Perf}(X \times_{\mathbf{P}} \mathbf{H})$, the result follows from Remark 4.17. □

Recall that if X as in Lemma 7.5 is *in addition smooth*, then its *classical projective dual* is given by

$$X^\vee = \{ H \in \mathbf{P}^\vee \mid X \times_{\mathbf{P}} H \text{ is singular} \}.$$

Hence we have:

Corollary 7.6. *Assume our base $S = \mathrm{Spec}(k)$ where k is an algebraically closed field. Let X be a smooth integral scheme of finite type over S equipped with a closed immersion $X \to \mathbf{P}$, such that X is not contained in any hyperplane in $\mathbf{P}$. Then*

$$\mathrm{CPD}(\mathrm{Perf}(X)) = X^\vee \subset \mathbf{P}^\vee.$$

Lemma 7.7. *Let $\mathcal{C}$ be a $\mathbf{P}$-linear category which is smooth and proper over S. Then the universal hyperplane section category $\mathbf{H}(\mathcal{C})$ is smooth and proper over S.*

Proof. Follows from Lemma 4.11, since the morphism $\mathbf{H} \to \mathbf{P}$ is smooth and proper. □

Proposition 7.8. *Let $\mathcal{C}$ be a right Lefschetz category over $\mathbf{P}$, which is smooth and proper over S. Then*

$$\mathrm{CPD}(\mathcal{C}) = \mathrm{Crit}_{\mathbf{P}^\vee}(\mathcal{C}^\vee) \subset \mathbf{P}^\vee.$$

Proof. By Lemmas 4.15(4) and 7.7 the components of the semiorthogonal decomposition (7.1) are admissible. Hence by Lemma 4.20 we have

$$\mathrm{CPD}(\mathcal{C}) = \mathrm{Crit}_{\mathbf{P}^\vee}(\mathcal{C}^\vee) \cup \bigcup_{i=1}^{m} \mathrm{Crit}_{\mathbf{P}^\vee}\big(\mathcal{A}_i(iH) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee)\big)$$

But Lemma 4.15 also implies that the components $\mathcal{A}_i(iH)$ of the Lefschetz decomposition of $\mathcal{C}$ are smooth and proper over S. Hence by Lemma 4.10 their base changes

$$\mathcal{A}_i(iH) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee)$$

are smooth and proper over $\mathbf{P}^\vee$. Hence

$$\mathrm{Crit}_{\mathbf{P}^\vee}\big(\mathcal{A}_i(iH) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee)\big) = \varnothing$$

by Lemma 4.19. □

7.2. **Construction of the Lefschetz sequence.** Let $\mathcal{C}$ be a nondegenerate right Lefschetz category over $\mathbf{P}$. Let $\gamma\colon \mathcal{C}^\vee \to \mathbf{H}(\mathcal{C})$ denote the inclusion functor. Let $\gamma^*\colon \mathbf{H}(\mathcal{C}) \to \mathcal{C}^\vee$ denote its left adjoint, which exists by Lemma 3.6. The pullback functor $p^*\colon \mathrm{Perf}(\mathbf{P}) \to \mathrm{Perf}(\mathbf{H})$ induces a functor

$$p^*\colon \mathcal{C} \simeq \mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P}) \to \mathbf{H}(\mathcal{C})$$

which is abusively denoted by the same symbol. Further, recall that by Lemma 6.2 there are twisted primitive components

$$\mathfrak{a}'_i = \alpha_0^*(\mathfrak{a}_i((i+1)H))$$

which give a semiorthogonal decomposition

$$\mathcal{A}_0 = \left\langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_{m-1} \right\rangle. \tag{7.2}$$

The following is the key ingredient in constructing the desired Lefschetz sequence in $\mathcal{C}^\vee$. We postpone its proof to the end of this section.

Lemma 7.9. *Let $\mathcal{C}$ be a nondegenerate right Lefschetz category over $\mathbf{P}$.*

(1) The functor

$$\gamma^* \circ p^* \colon \mathcal{C} \to \mathcal{C}^\vee$$

is fully faithful on the subcategory $\mathcal{A}_0 \subset \mathcal{C}$.

(2) Let $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \dots, \mathfrak{a}'_i \rangle \subset \mathcal{A}_0$ for some $0 \leq i \leq m-1$. Then for any $D \in \mathcal{C}$ and $1 \leq t \leq N-2-i$, we have

$$\mathcal{H}om_{\mathbf{P}^\vee}(\gamma^* p^*(D), \gamma^* p^*(C)(-tH')) \simeq 0.$$

For $\mathcal{C}$ a nondegenerate right Lefschetz category over $\mathbf{P}$, we define $\mathcal{A}_0^\vee = \gamma^* p^* \mathcal{A}_0$, which by virtue of Lemma 7.9(1) is a stable subcategory of $\mathcal{C}^\vee$ equivalent to $\mathcal{A}_0$. This will be the biggest component in the promised left Lefschetz sequence in $\mathcal{C}^\vee$.

By nondegeneracy of $\mathcal{C}$, the number m of components in (7.2) satisfies $m < N$. Hence, since $\mathfrak{a}'_i = 0$ for $i \geq m$ by our convention that $\mathfrak{a}_i = 0$ for $i \geq m$, we can rewrite (6.5) as

$$\mathcal{A}_0 = \left\langle \mathfrak{a}'_0, \mathfrak{a}'_1, \dots, \mathfrak{a}'_{N-2} \right\rangle.$$

For $0 \leq j \leq N-2$, we define

$$\mathfrak{a}^\vee_j = \gamma^* p^* (\mathfrak{a}'_{N-2-j}).$$

Applying $\gamma^* p^*$ to the above semiorthogonal decomposition then gives

$$\mathcal{A}_0^\vee = \left\langle \mathfrak{a}^\vee_{N-2}, \dots, \mathfrak{a}^\vee_1, \mathfrak{a}^\vee_0 \right\rangle.$$

Further, we define

$$n = N - \#\{\, i \mid \mathcal{A}_i = \mathcal{A}_0 \,\}.$$

Then it follows from the definitions that $\mathfrak{a}^\vee_j = 0$ for $j \geq n$ and $\mathfrak{a}^\vee_{n-1} \neq 0$. Hence we have

$$\mathcal{A}_0^\vee = \left\langle \mathfrak{a}^\vee_{n-1}, \dots, \mathfrak{a}^\vee_1, \mathfrak{a}^\vee_0 \right\rangle.$$

The $\mathfrak{a}^\vee_j$ determine a sequence of S-linear categories $\mathcal{A}^\vee_{n-1} \subset \cdots \subset \mathcal{A}^\vee_1 \subset \mathcal{A}^\vee_0$ by the formula

$$\mathcal{A}^\vee_j = \left\langle \mathfrak{a}^\vee_{n-1}, \dots, \mathfrak{a}^\vee_{j+1}, \mathfrak{a}^\vee_j \right\rangle. \tag{7.3}$$

Proposition 7.10. *Let $\mathcal{C}$ be a nondegenerate right Lefschetz category over $\mathbf{P}$. Then the sequence of categories*

$$\mathcal{A}^\vee_{n-1}(-(n-1)H'), \dots, \mathcal{A}^\vee_1(-H'), \mathcal{A}^\vee_0$$

is semiorthogonal in $\mathcal{C}^\vee$. Further, if $\mathcal{C}$ is smooth and proper over S, then the subcategories $\mathcal{A}^\vee_j \subset \mathcal{C}^\vee$ are admissible, i.e. $\mathcal{A}^\vee_{n-1} \subset \cdots \subset \mathcal{A}^\vee_1 \subset \mathcal{A}^\vee_0$ forms a left Lefschetz sequence in $\mathcal{C}^\vee$ with respect to the autoequivalence $- \otimes \mathcal{O}_{\mathbf{P}^\vee}(H')$.

Proof. The semiorthogonality assertion follows from Lemma 7.9(2) and the definition of $\mathcal{A}^\vee_j$.

To prove $\mathcal{A}^\vee_j \subset \mathcal{C}^\vee$ is admissible, by Lemma 3.10 it suffices to show $\mathfrak{a}^\vee_j = \gamma^* p^*(\mathfrak{a}'_{N-2-j}) \subset \mathcal{C}^\vee$ is admissible for all j. The functors $\gamma^* \colon \mathbf{H}(\mathcal{C}) \to \mathcal{C}^\vee$ and $p^* \colon \mathcal{C} \to \mathbf{H}(\mathcal{C})$ admit right adjoints (the second by Lemma 2.11), and $\mathfrak{a}'_{N-2-j}$ is admissible, so $\mathfrak{a}^\vee_j \subset \mathcal{C}^\vee$ is right admissible. But if $\mathcal{C}$ is smooth and proper over S, then $\mathcal{C}^\vee$ is also smooth and proper over S by Lemmas 7.7 and 4.15(4). So again by Lemma 4.15(4), we conclude $\mathfrak{a}^\vee_j \subset \mathcal{C}^\vee$ is in fact admissible. □

7.3. **A complementary result.** We record here another useful consequence of Lemma 7.9.

Lemma 7.11. *The image and kernel of the functor* $p_* \circ \gamma \colon \mathcal{C}^\vee \to \mathcal{C}$ *and of its left adjoint* $\gamma^* \circ p^* \colon \mathcal{C} \to \mathcal{C}^\vee$ *are given by*

$$\mathrm{im}(p_* \circ \gamma) = \mathcal{A}_0, \quad \ker(p_* \circ \gamma) = \mathcal{A}_0^{\vee\perp},$$
$$\mathrm{im}(\gamma^* \circ p^*) = \mathcal{A}_0^\vee, \quad \ker(\gamma^* \circ p^*) = {}^\perp\mathcal{A}_0.$$

Proof. First we prove $\mathrm{im}(p_* \circ \gamma) \subset \mathcal{A}_0$. For $C \in \mathcal{C}$ and $D \in \mathcal{C}^\vee$, we have

$$\mathcal{H}om_S(C, p_*\gamma(D)) \simeq \mathcal{H}om_S(p^*(C), \gamma(D)),$$

which vanishes for $C \in \mathcal{A}_i(iH)$, $1 \leq i \leq m-1$, by the defining semiorthogonal decomposition (7.1) of $\mathcal{C}^\vee$. Hence

$$p_*\gamma(D) \in \langle \mathcal{A}_1(H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle^\perp = \mathcal{A}_0,$$

as desired. Since $\gamma^* \circ p^*$ is fully faithful on $\mathcal{A}_0$ by Lemma 7.9(1), it follows that in fact $\mathrm{im}(p_* \circ \gamma) = \mathcal{A}_0$. From this, $\ker(\gamma^* \circ p^*) = {}^\perp\mathcal{A}_0$ follows by adjunction. Hence $\mathrm{im}(\gamma^* \circ p^*)$ coincides with $\mathcal{A}_0^\vee = (\gamma^* \circ p^*)(\mathcal{A}_0)$. Again by adjunction, $\ker(p_* \circ \gamma) = \mathcal{A}_0^{\vee\perp}$ follows formally from this. □

7.4. **Proof of Lemma 7.9.** We will need some auxiliary lemmas. Let $p_* \colon \mathbf{H}(\mathcal{C}) \to \mathcal{C}$ denote the functor induced by $p_* \colon \mathrm{Perf}(\mathbf{H}) \to \mathrm{Perf}(\mathbf{P})$.

Lemma 7.12. *Let* $\mathcal{C}$ *be a* $\mathbf{P}$*-linear category. Then:*

(1) The functor $p^* \colon \mathcal{C} \to \mathbf{H}(\mathcal{C})$ *is fully faithful.*
(2) The functor $p_* \colon \mathbf{H}(\mathcal{C}) \to \mathcal{C}$ *kills the subcategory* $p^*(\mathcal{C})(-tH')$ *for* $1 \leq t \leq N-2$.

Proof. The morphism $p \colon \mathbf{H} \to \mathbf{P}$ is the projectivization of a rank $N-1$ vector bundle on $\mathbf{P}$. Namely, $\mathbf{H} = \mathbf{P}(\mathcal{K})$ where $\mathcal{K}$ is the kernel of the canonical surjection $V^\vee \otimes \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_{\mathbf{P}}(H)$. It follows that $p^* \colon \mathrm{Perf}(\mathbf{P}) \to \mathrm{Perf}(\mathbf{H})$ is fully faithful, and $p_* \colon \mathrm{Perf}(\mathbf{H}) \to \mathrm{Perf}(\mathbf{P})$ kills the subcategory

$$p^*(\mathrm{Perf}(\mathbf{P}))(-tH') \subset \mathrm{Perf}(\mathbf{H})$$

for $1 \leq t \leq N-2$. From this, the lemma follows formally. □

Lemma 7.12 shows in particular that $p^* \colon \mathcal{C} \to \mathbf{H}(\mathcal{C})$ embeds $\mathcal{A}_k$ into $\mathbf{H}(\mathcal{C})$ for all k; we abusively denote the image also by $\mathcal{A}_k \subset \mathbf{H}(\mathcal{C})$. The following result controls the morphisms between objects in various twists of these categories.

Lemma 7.13. *Let* $\mathcal{C}$ *be a right Lefschetz category over* $\mathbf{P}$.

(1) The pair of categories

$$\mathcal{A}_k, \, \mathcal{A}_\ell(aH + bH')$$

is semiorthogonal in $\mathbf{H}(\mathcal{C})$ *provided one of the following conditions hold:*
- $1 \leq a \leq \ell - 1$.
- $1 \leq b \leq N-2$.
- $a = 0$ *and* $b = N-1$.
- $a = \ell$ *and* $b = 0$.

(2) The pair of categories

$$p^*(\langle \mathfrak{a}'_0, \mathfrak{a}'_1, \dots, \mathfrak{a}'_i \rangle), \, \mathcal{A}_\ell(\ell H + bH')$$

is semiorthogonal in $\mathbf{H}(\mathcal{C})$ *provided* $i < \ell$.

Proof. Let $C, D \in \mathcal{C}$. Note that we have equivalences

$$p^*(D)(aH + bH') \simeq \iota^*(D(aH) \boxtimes \mathcal{O}(bH')),$$
$$p^*(C) \simeq \iota^*(C \boxtimes \mathcal{O}),$$

where $\iota\colon \mathbf{H} \to \mathbf{P} \times_S \mathbf{P}^\vee$ denotes the embedding and $D(aH) \boxtimes \mathcal{O}(bH')$ and $C \boxtimes \mathcal{O}$ are regarded as objects of

$$\mathcal{C} \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee) \simeq \mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P} \times_S \mathbf{P}^\vee).$$

Specializing the argument in Lemma 6.13 to the case $r = N - 1$, we thus obtain an exact triangle

$$\begin{aligned} \mathcal{H}om_{\mathbf{P}^\vee}(D((a+1)H) \boxtimes \mathcal{O}((b+1)H'), C \boxtimes \mathcal{O}) \to \mathcal{H}om_{\mathbf{P}^\vee}(D(aH) \boxtimes \mathcal{O}(bH'), C \boxtimes \mathcal{O}) \\ \to \mathcal{H}om_{\mathbf{P}^\vee}(p^*(D)(aH + bH'), p^*(C)). \end{aligned} \tag{7.4}$$

By Lemma 2.9 the first two terms of this triangle can be written as

$$\mathcal{H}om_S(D((a+1)H), C) \otimes \mathcal{H}om_{\mathbf{P}^\vee}(\mathcal{O}((b+1)H'), \mathcal{O}) \tag{7.5}$$
$$\mathcal{H}om_S(D(aH), C) \otimes \mathcal{H}om_{\mathbf{P}^\vee}(\mathcal{O}(bH'), \mathcal{O}). \tag{7.6}$$

To prove (1), assume $C \in \mathcal{A}_k$ and $D \in \mathcal{A}_\ell$. We must show that

$$\mathcal{H}om_{\mathbf{P}^\vee}(p^*(D)(aH + bH'), p^*(C))$$

vanishes for a, b satisfying any of the stated conditions. By the exact triangle (7.4), it suffices to show that the terms (7.5) and (7.6) vanish. But (7.5) vanishes provided either $0 \le a \le \ell - 1$ or $0 \le b \le N - 2$, and (7.6) vanishes provided either $1 \le a \le \ell$ or $1 \le b \le N - 1$.

To prove (2), assume $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \dots, \mathfrak{a}'_i \rangle$, $D \in \mathcal{A}_\ell$, $a = \ell$, and $i < \ell$. As above, it suffices to show that (7.5) and (7.6) vanish. This is clear for (7.6). Note that since $C \in \mathcal{A}_k \subset \mathcal{A}_0$, we have

$$\mathcal{H}om_S(D((\ell+1)H), C) \simeq \mathcal{H}om_S(\alpha_0^*(D((\ell+1)H)), C). \tag{7.7}$$

It suffices to show that this morphism space vanishes to show that (7.5) does. Observe that α_0^* kills the second term in the semiorthogonal decomposition

$$\mathcal{A}_\ell((\ell+1)H) = \langle \mathfrak{a}_\ell((\ell+1)H), \mathcal{A}_{\ell+1}((\ell+1)H) \rangle,$$

hence $\alpha_0^*(D((\ell+1)H)) \in \mathfrak{a}'_\ell$. Therefore (7.7) vanishes by the semiorthogonal decomposition (7.2). □

For any object $X \in \mathbf{H}(\mathcal{C})$, there is an exact triangle

$$\mathrm{R}_{\mathcal{C}^\vee}(X) \to X \to \gamma\gamma^*(X). \tag{7.8}$$

A priori $\mathrm{R}_{\mathcal{C}^\vee}(X)$ can be any object in the subcategory generated by the categories to the right of $\mathcal{C}^\vee$ in (7.1). The following lemma shows that for X pulled back from the subcategory generated by a subset of the twisted primitive components of $\mathcal{A}_0$, the object $\mathrm{R}_{\mathcal{C}^\vee}(X)$ lies in a restricted subcategory.

Lemma 7.14. *Let $\mathcal{C}$ be a right Lefschetz category over $\mathbf{P}$. Let $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \dots, \mathfrak{a}'_i \rangle \subset \mathcal{A}_0$ for some $0 \le i \le m - 1$. Then $\mathrm{R}_{\mathcal{C}^\vee} p^*(C)$ lies in the subcategory of $\mathbf{H}(\mathcal{C})$ generated by the categories*

$$p^*(\langle \mathcal{A}_1(H), \mathcal{A}_2(2H), \dots, \mathcal{A}_{i-t+1}((i-t+1)H) \rangle) \otimes \mathcal{O}(-tH'), \quad 1 \le t \le i.$$

Proof. Let $X = p^*(C)$. By definition, the functor $\gamma\gamma^*$ coincides with the left mutation functor through the subcategory

$$^\perp\mathcal{C}^\vee = \big\langle \mathcal{A}_1(H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee), \dots, \mathcal{A}_{m-1}((m-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee) \big\rangle \subset \mathbf{H}(\mathcal{C}),$$

so that $\mathrm{R}_{\mathcal{C}^\vee} p^*(C)$ is determined by an exact triangle

$$\mathrm{R}_{\mathcal{C}^\vee} p^*(C) \to X \to \mathrm{L}_{^\perp\mathcal{C}^\vee}(X).$$

To prove the lemma, we factor $\mathrm{L}_{^\perp\mathcal{C}^\vee}$ according to the above semiorthogonal decomposition of $^\perp\mathcal{C}^\vee$, and inductively control the corresponding exact triangle at each step.

Namely, for $1 \le \ell \le m-1$, we define

$$\mathcal{D}_\ell = \big\langle \mathcal{A}_\ell(H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee), \dots, \mathcal{A}_{m-1}((m-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee) \big\rangle \subset \mathbf{H}(\mathcal{C}),$$

and define X_ℓ by the exact triangle

$$X_\ell \to X \to \mathrm{L}_{\mathcal{D}_\ell}(X).$$

Then we claim that X_ℓ is contained in the subcategory of $\mathbf{H}(\mathcal{C})$ generated by

$$\mathcal{A}_k(kH - tH'),\ \ell \le k \le i,\ 1 \le t \le i-k+1, \tag{7.9}$$

where for $\ell > i$ this subcategory is by definition 0. The case $\ell = 1$ gives the statement of the lemma.

Note that for any ℓ, there is a semiorthogonal decomposition

$$\mathcal{A}_\ell(\ell H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee) = \big\langle \mathcal{A}_\ell(\ell H), \mathcal{A}_\ell(\ell H + H'), \dots, \mathcal{A}_\ell(\ell H + (N-1)H') \big\rangle$$

induced via base change by the standard decomposition of $\mathrm{Perf}(\mathbf{P}^\vee)$. By Lemma 7.13(2) it thus follows that for $\ell > i$ the category $\mathcal{D}_\ell$ is left orthogonal to X, hence $\mathrm{L}_{\mathcal{D}_\ell}(X) = X$ and $X_\ell = 0$. This proves the claim for $\ell > i$.

For $1 \le \ell \le i+1$, we argue by descending induction on ℓ. The base case $\ell = i+1$ was handled above. Assume the claim holds for ℓ. By Lemma 3.10, we have

$$\mathrm{L}_{\mathcal{D}_{\ell-1}}(X) \simeq \mathrm{L}_{\mathcal{A}_{\ell-1}((\ell-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee)} \mathrm{L}_{\mathcal{D}_\ell}(X). \tag{7.10}$$

It follows from Lemma 7.13(1) that $\mathcal{A}_{\ell-1}((\ell-1)H + bH')$ is left orthogonal to X for $0 \le b \le N-2$ and to X_ℓ for $0 \le b \le N-i+\ell-3$ (here we used the induction assumption). Thus, by the exact triangle defining X_ℓ, we find that $\mathcal{A}_{\ell-1}((\ell-1)H + bH')$ is left orthogonal to $\mathrm{L}_{\mathcal{D}_\ell}(X)$ for $0 \le b \le N-i+\ell-3$. That is, in the decomposition of $\mathcal{A}_{\ell-1}((\ell-1)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{P}^\vee)$ given by

$$\begin{aligned} \langle \mathcal{A}_{\ell-1}((\ell-1)H - (i-\ell+2)H'), \dots, \mathcal{A}_{\ell-1}((\ell-1)H - H'), \\ \mathcal{A}_{\ell-1}((\ell-1)H), \dots, \mathcal{A}_{\ell-1}((\ell-1)H + (N-i+\ell-3)H') \rangle, \end{aligned}$$

the second row is left orthogonal to $\mathrm{L}_{\mathcal{D}_\ell}(X)$. It follows that the right side of (7.10) can be rewritten as

$$\mathrm{L}_{\langle \mathcal{A}_{\ell-1}((\ell-1)H - (i-\ell+2)H'), \dots, \mathcal{A}_{\ell-1}((\ell-1)H - H') \rangle} \mathrm{L}_{\mathcal{D}_\ell}(X),$$

and therefore the cone of the canonical morphism $\mathrm{L}_{\mathcal{D}_\ell}(X) \to \mathrm{L}_{\mathcal{D}_{\ell-1}}(X)$ is contained in the subcategory of $\mathbf{H}(\mathcal{C})$ generated by

$$\mathcal{A}_{\ell-1}((\ell-1)H - tH'),\ 1 \le t \le i-\ell+2.$$

By the induction assumption, the cone of $X \to \mathrm{L}_{\mathcal{D}_\ell}(X)$ is contained in the subcategory generated by the categories (7.9). We conclude that the cone of the composite morphism

$X \to \mathrm{L}_{\mathcal{D}_{\ell-1}}(X)$ — which agrees with $X_{\ell-1}$ up to a shift — is contained in the claimed subcategory of $\mathbf{H}(\mathcal{C})$. □

Now we prove Lemma 7.9. Suppose $\mathcal{C}$ is a nondegenerate right Lefschetz category over $\mathbf{P}$. Then by adjunction, Lemma 7.9 can be rephrased as saying that

$$p_*(\gamma\gamma^* p^*(C)(-tH'))$$

is canonically equivalent to:

(1) C for $C \in \mathcal{A}_0$ and $t = 0$,
(2) 0 for $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_i \rangle$, $0 \le i \le m-1$, and $1 \le t \le N-2-i$.

Taking $X = p^*(C)$ in (7.8) and tensoring by $\mathcal{O}(-tH')$, we get an exact triangle

$$\mathrm{R}_{\mathcal{C}^\vee}(p^*(C))(-tH') \to p^*(C)(-tH') \to (\gamma\gamma^* p^*(C))(-tH').$$

Assume $C \in \mathcal{A}_0$ and $t = 0$ as in (1). Then by Lemma 7.14 the first term of this triangle lies in the subcategory of $\mathbf{H}(\mathcal{C})$ generated by $p^*(\mathcal{C})(-uH')$ for $1 \le u \le m-1$. By nondegeneracy of $\mathcal{C}$ we have $m-1 \le N-2$, so by Lemma 7.12(2) the functor $p_* \colon \mathbf{H}(\mathcal{C}) \to \mathcal{C}$ kills all of these categories, and thus also the first term of the above triangle. Further, by Lemma 7.12(1) the functor p_* applied to the second term $p^*(C)$ is canonically equivalent to C. Hence applying p_* to the above triangle proves (1).

Assume $C \in \langle \mathfrak{a}'_0, \mathfrak{a}'_1, \ldots, \mathfrak{a}'_i \rangle$ and $1 \le t \le N-2-i$ as in (2). Then arguing as above we find that p_* kills the first two terms of the above triangle, and hence also the last, proving (2). □

8. The main theorem of HPD

Let $\mathcal{C}$ be a nondegenerate right Lefschetz category over $\mathbf{P}$. Then in the previous section we constructed the $\mathbf{P}^\vee$-linear HPD category $\mathcal{C}^\vee$, and showed it has a natural left Lefschetz sequence with respect to $- \otimes \mathcal{O}_{\mathbf{P}^\vee}(H')$ if $\mathcal{C}$ is smooth and proper over S.

In this section, we assume that $\mathcal{C}$ is smooth and proper over S. Under this assumption, we prove that the Lefschetz sequence of $\mathcal{C}^\vee$ is full, so that $\mathcal{C}^\vee$ is a nondegenerate left Lefschetz category over $\mathbf{P}^\vee$. Further, we prove that taking the homological projective dual once more recovers $\mathcal{C}$, i.e. we prove that $(\mathcal{C}^\vee)^\vee$ is equivalent to $\mathcal{C}$ as a right Lefschetz category over $\mathbf{P}$. This is the analogue of the fact that classical projective duality is indeed a duality.

In fact, we will deduce these statements from a significantly stronger result (Theorem 8.4), which describes the series of categories $\mathcal{K}_r(\mathcal{C})$ defined by (6.16) in terms of an analogous series of categories associated to $\mathcal{C}^\vee$.

Our arguments generalize those of [9, §6], by placing them in the framework set up in the previous sections.

8.1. The main theorem and its corollaries.

Throughout this section, we work in the following setting.

Setup 8.1.

(1) $\mathcal{C}$ is a $\mathbf{P}$-linear category which is smooth and proper over S.
(2) $\mathcal{C}$ is equipped with the structure of a nondegenerate right Lefschetz category over $\mathbf{P}$, i.e. is equipped with a Lefschetz decomposition

$$\mathcal{C} = \langle \mathcal{A}_0, \mathcal{A}_1(H), \ldots, \mathcal{A}_{m-1}((m-1)H) \rangle$$

where $m < N = \mathrm{rank}(V)$.
(3) $\mathcal{D}$ is a $\mathbf{P}^\vee$-linear category and $\phi \colon \mathcal{D} \to \mathcal{C}^\vee$ is a $\mathbf{P}^\vee$-linear functor such that:

(a) ϕ is fully faithful and admits a left adjoint $\phi^* \colon \mathcal{C}^\vee \to \mathcal{D}$,
(b) the categories $\mathcal{B}_j = \phi^* \mathcal{A}_j^\vee$ form a left Lefschetz sequence $\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$ in $\mathcal{D}$ with respect to $- \otimes \mathcal{O}_{\mathbf{P}^\vee}(H')$,
(c) the image of the composition $p_* \circ \gamma \circ \phi \colon \mathcal{D} \to \mathcal{C}$ is $\mathcal{A}_0$.

Remark 8.2. The category $\mathcal{D} = \mathcal{C}^\vee$ satisfies the assumptions with $\phi = \mathrm{id}_{\mathcal{C}^\vee}$. Indeed, (3a) is automatic, (3b) holds by Proposition 7.10, and (3c) holds by Proposition 7.11. In fact, we will see below that in Setup 8.1, the functor ϕ is automatically an equivalence. This gives useful criteria for checking that a category $\mathcal{D}$ is equivalent to the HPD category $\mathcal{C}^\vee$.

Lemma 8.3. *The category $\mathcal{D}$ is smooth and proper over S.*

Proof. By our assumption that $\phi \colon \mathcal{D} \to \mathcal{C}^\vee$ is fully faithful and admits a left adjoint, it follows that $\mathcal{D}$ is a component in a $\mathbf{P}^\vee$-linear (and in particular S-linear) semiorthogonal decomposition of $\mathbf{H}(\mathcal{C})$. But $\mathbf{H}(\mathcal{C})$ is smooth and proper over S by Lemma 7.7, hence $\mathcal{D}$ is smooth and proper by Lemma 4.15(4). □

Let $\mathbf{G}_s^\vee = \mathrm{Gr}(s, V^\vee)$ be the Grassmannian of rank s subbundles of $V^\vee$, and let $\mathbf{L}_s^\vee$ be the corresponding projectivized universal family. Then Lemma 6.17 (with $\mathbf{P}$ replaced with the dual $\mathbf{P}^\vee$) applies to $\mathcal{D}$, and we get $\mathbf{G}_s^\vee$-linear categories $\mathcal{K}_s(\mathcal{D})$ for $0 \leq s \leq N$, characterized by semiorthogonal decompositions

$$\mathbf{L}_s^\vee(\mathcal{D}) = \left\langle \mathcal{B}_{n-1}(-(n-r)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_s^\vee), \dots, \mathcal{B}_r(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_s^\vee), \mathcal{K}_s(\mathcal{D}) \right\rangle \tag{8.1}$$

where $r = N - s$. There is an identification $\mathbf{G}_r \cong \mathbf{G}_s^\vee$, by which we regard $\mathcal{K}_s(\mathcal{D})$ as a $\mathbf{G}_r$-linear category. We also note that there is a commutative diagram

$$\begin{array}{ccc} \mathbf{L}_s^\vee & \xrightarrow{g_s} & \mathbf{G}_r \\ {\scriptstyle q_s}\big\downarrow & & \big\downarrow \\ \mathbf{P}^\vee & \longrightarrow & S \end{array}$$

We aim to prove that there is a $\mathbf{G}_r$-linear equivalence $\mathcal{K}_s(\mathcal{D}) \simeq \mathcal{K}_r(\mathcal{C})$. By definition we have $\mathcal{K}_r(\mathcal{C}) \subset \mathbf{L}_r(\mathcal{C})$ and $\mathcal{K}_s(\mathcal{D}) \subset \mathbf{L}_s^\vee(\mathcal{D})$. The desired equivalence will be induced by a functor $\mathbf{L}_s^\vee(\mathcal{D}) \to \mathbf{L}_r(\mathcal{C})$, which can be described using the kernel formalism of §5 as follows. Consider the $\mathbf{P}^\vee$-linear composition $\gamma \circ \phi \colon \mathcal{D} \to \mathbf{H}(\mathcal{C})$, where recall $\mathbf{H} = \mathbf{L}_{N-1}$ and $\gamma \colon \mathcal{C}^\vee \to \mathbf{H}(\mathcal{C})$ is the inclusion. By Proposition 5.6, there is a kernel

$$\mathcal{E} \in \mathrm{FM}(\mathcal{D}/\mathbf{P}^\vee, \mathcal{C}/\mathbf{P}, \mathbf{H} \times_{\mathbf{P}^\vee} \mathbf{P}^\vee)$$

such that $\Phi_{\mathcal{E}} = \gamma \circ \phi$. Let

$$\zeta_r \colon \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee \to \mathbf{H}$$

be the natural morphism. Set

$$\mathcal{E}_r = \zeta_r^* \mathcal{E} \in \mathrm{FM}(\mathcal{D}/\mathbf{P}^\vee, \mathcal{C}/\mathbf{P}, \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee)$$

and define

$$\Phi_r = \Phi_{\mathcal{E}_r} \colon \mathbf{L}_s^\vee(\mathcal{D}) \to \mathbf{L}_r(\mathcal{C})$$

to be the associated $\mathbf{G}_r$-linear functor. This is the functor we are after. Note that $\Phi_{\mathcal{E}} = \Phi_{N-1}$ under the identifications $\mathbf{P}^\vee = \mathbf{L}_1^\vee$, $\mathbf{H} = \mathbf{L}_{N-1}$.

The key result of this section is the following.

Theorem 8.4. *In Setup 8.1, for all nonnegative integers r and s such that $r+s=N$, there is a $\mathbf{G}_r$-linear equivalence*

$$\phi_r \colon \mathcal{K}_s(\mathcal{D}) \xrightarrow{\sim} \mathcal{K}_r(\mathcal{C})$$

induced by the restriction of Φ_r to $\mathcal{K}_s(\mathcal{D})$.

The proof of Theorem 8.4 will be given below. Here we derive some consequences.

Corollary 8.5. *In Setup 8.1, the functor $\phi\colon \mathcal{D} \to \mathcal{C}^\vee$ is an equivalence.*

Proof. By construction, the functor $\phi\colon \mathcal{D} \to \mathcal{C}^\vee$ coincides with the equivalence ϕ_{N-1} of Theorem 8.4 (note that $\mathcal{K}_1(\mathcal{D}) = \mathbf{L}_1^\vee(\mathcal{D}) = \mathcal{D}$). □

Corollary 8.6. *In Setup 8.1, the Lefschetz sequence $\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$ in $\mathcal{D}$ is full.*

Proof. The defining semiorthogonal decomposition (8.1) of $\mathcal{K}_s(\mathcal{D})$ for $s=N$ can be written as

$$\mathcal{D} = \left\langle \mathcal{B}_{n-1}(-nH'), \dots, \mathcal{B}_1(-2H'), \mathcal{B}_0(-H'), \mathcal{K}_N(\mathcal{D}) \right\rangle.$$

Twisting by $\mathcal{O}(H')$, we get

$$\mathcal{D} = \left\langle \mathcal{B}_{n-1}(-(n-1)H'), \dots, \mathcal{B}_1(-H'), \mathcal{B}_0, \mathcal{K}_N(\mathcal{D})(H') \right\rangle.$$

Hence the Lefschetz sequence $\mathcal{B}_{n-1} \subset \cdots \subset \mathcal{B}_1 \subset \mathcal{B}_0$ in $\mathcal{D}$ is full if and only if $\mathcal{K}_N(\mathcal{D}) \simeq 0$. But by Theorem 8.4 we have $\mathcal{K}_N(\mathcal{D}) \simeq \mathcal{K}_0(\mathcal{C})$, and $\mathcal{K}_0(\mathcal{C}) \simeq 0$ by Remark 6.12. □

By combining the above results in the case $\mathcal{D} = \mathcal{C}^\vee$, we obtain the main theorem of HPD.

Theorem 8.7. *Let $\mathcal{C}$ be a nondegenerate right Lefschetz category over $\mathbf{P}$, which is smooth and proper over S. Then the following hold:*

(1) $\mathcal{C}^\vee$ is smooth and proper over S.

(2) There is a left Lefschetz decomposition

$$\mathcal{C}^\vee = \left\langle \mathcal{A}_{n-1}^\vee(-(n-1)H'), \dots, \mathcal{A}_1^\vee(-H'), \mathcal{A}_0^\vee \right\rangle,$$

where the components $\mathcal{A}_j^\vee$ are given by (7.3).

(3) For all nonnegative integers r and s satisfying $r+s=N$, there is a $\mathbf{G}_r$-linear equivalence

$$\phi_r \colon \mathcal{K}_s(\mathcal{C}^\vee) \xrightarrow{\sim} \mathcal{K}_r(\mathcal{C}).$$

(4) The functor $\phi_1 \colon (\mathcal{C}^\vee)^\vee \xrightarrow{\sim} \mathcal{C}$ is an equivalence of right Lefschetz categories over $\mathbf{P}$.

Proof. Parts (1)-(3) are immediate from the above, and (4) follows by unwinding the definitions. □

By base change, we obtain the following result for individual "linear sections", which establishes Theorem 1.3 from §1.

Corollary 8.8. *Let $\mathcal{C}$ be a nondegenerate right Lefschetz category over $\mathbf{P}$, which is smooth and proper over S. Let $L \subset V$ be a subbundle, let $L^\perp = \ker(V^\vee \to L^\vee)$ be its orthogonal, and let $r = \operatorname{rank}(L)$ and $s = \operatorname{rank}(L^\perp)$. Then there are semiorthogonal decompositions*

$$\mathcal{C} \otimes_{\mathrm{Perf}(\mathbf{P})} \mathrm{Perf}(\mathbf{P}(L)) = \langle \mathcal{K}_L(\mathcal{C}), \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle,$$

$$\mathcal{C}^\vee \otimes_{\mathrm{Perf}(\mathbf{P}^\vee)} \mathrm{Perf}(\mathbf{P}(L^\perp)) = \left\langle \mathcal{A}_{n-1}^\vee(-(n-r)H'), \dots, \mathcal{A}_r^\vee(-H'), \mathcal{K}_{L^\perp}(\mathcal{C}^\vee) \right\rangle,$$

and an S-linear equivalence $\mathcal{K}_L(\mathcal{C}) \simeq \mathcal{K}_{L^\perp}(\mathcal{C}^\vee)$.

Proof. There are semiorthogonal decompositions of the claimed form by Lemma 6.15 (and the obvious analogue for $\mathcal{C}^\vee$). The equivalence $\mathcal{K}_L(\mathcal{C}) \simeq \mathcal{K}_{L^\perp}(\mathcal{C}^\vee)$ is the base change of the equivalence from Theorem 8.7(3) along the morphism $S \to \mathbf{G}_r$ classifying $L \subset V$. □

Now we specialize to the case where $\mathcal{C}$ and $\mathcal{C}^\vee$ are geometric. If $\mathcal{C} = \mathrm{Perf}(X)$ with $\mathbf{P}$-linear structure induced by a morphism $X \to \mathbf{P}$, we write $\mathcal{K}_L(X)$ for $\mathcal{K}_L(\mathcal{C})$, and we use the analogous notation if $\mathcal{C}^\vee$ is geometric. The following result recovers Theorem 1.1 from §1.

Corollary 8.9. *Let X be a smooth and proper scheme over S with a morphism $X \to \mathbf{P}$. Assume* $\mathrm{Perf}(X)$ *is equipped with a nondegenerate right Lefschetz decomposition*

$$\mathrm{Perf}(X) = \langle \mathcal{A}_0, \mathcal{A}_1(H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle .$$

Assume there is a morphism of schemes $Y \to \mathbf{P}^\vee$ and a $\mathbf{P}^\vee$-linear equivalence

$$\mathrm{Perf}(Y) \simeq \mathrm{Perf}(X)^\vee.$$

1. *If $Y \to S$ is flat and locally of finite presentation, then Y is smooth over S.*
2. *If Y and S are noetherian and $Y \to S$ is separated and of finite type, then Y is proper over S.*
3. *Let $\mathcal{B}_j \subset \mathrm{Perf}(Y)$ be the subcategory corresponding to $\mathcal{A}_j^\vee \subset \mathrm{Perf}(X)^\vee$ under the equivalence $\mathrm{Perf}(Y) \simeq \mathrm{Perf}(X)^\vee$. Then there is a left Lefschetz decomposition*
$$\mathrm{Perf}(Y) = \left\langle \mathcal{B}_{n-1}(-(n-1)H'), \dots, \mathcal{B}_1(-H'), \mathcal{B}_0 \right\rangle .$$
4. *Let $L \subset V$ be a subbundle, let $L^\perp = \ker(V^\vee \to L^\vee)$ be its orthogonal, and let $r = \mathrm{rank}(L)$ and $s = \mathrm{rank}(L^\perp)$. Then there are semiorthogonal decompositions*
$$\mathrm{Perf}(X \times_{\mathbf{P}} \mathbf{P}(L)) = \langle \mathcal{K}_L(X), \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle ,$$
$$\mathrm{Perf}(Y \times_{\mathbf{P}^\vee} \mathbf{P}(L^\perp)) = \left\langle \mathcal{B}_{n-1}(-(n-r)H'), \dots, \mathcal{B}_r(-H'), \mathcal{K}_{L^\perp}(Y) \right\rangle ,$$
and an S-linear equivalence $\mathcal{K}_L(X) \simeq \mathcal{K}_{L^\perp}(Y)$.
5. *Assume that Y and S are noetherian and defined over a field of characteristic 0, and that $Y \to S$ is separated and of finite type. Then for $L \subset V$ and $L^\perp \subset V^\vee$ as in (4), there are semiorthogonal decompositions*
$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X \times_{\mathbf{P}} \mathbf{P}(L)) = \left\langle \mathcal{K}_L(X)^{\mathrm{coh}}, \mathcal{A}_s^{\mathrm{coh}}(H), \dots, \mathcal{A}_{m-1}^{\mathrm{coh}}((m-s)H)) \right\rangle ,$$
$$\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(Y \times_{\mathbf{P}^\vee} \mathbf{P}(L^\perp)) = \left\langle \mathcal{B}_{n-1}^{\mathrm{coh}}(-(n-r)H'), \dots, \mathcal{B}_r^{\mathrm{coh}}(-H'), \mathcal{K}_{L^\perp}(Y)^{\mathrm{coh}} \right\rangle ,$$
and an S-linear equivalence $\mathcal{K}_L(X)^{\mathrm{coh}} \simeq \mathcal{K}_{L^\perp}(Y)^{\mathrm{coh}}$.

Proof. Parts (1) and (2) follow from Theorem 8.7(1) combined with Lemma 4.9. Part (3) follows from Theorem 8.7(2). Part (4) follows from Corollary 8.8 combined with Theorem 5.3(1). Part (5) follows from parts (2) and (4) by applying Theorem 2.14 and Corollary 3.20. □

Remark 8.10. In Corollary 8.9, we emphasize that the fiber products $X \times_{\mathbf{P}} \mathbf{P}(L)$ and $Y \times_{\mathbf{P}^\vee} \mathbf{P}(L^\perp)$ are taken in the derived sense, according to our conventions from §2.1. This addresses the question of [9, Remark 6.26], i.e. Corollary 8.9 removes the "admissibility" assumption on $L \subset V$ from [9, Theorem 6.3].

8.2. **Notation.** Throughout, r and s will denote nonnegative integers such that $r + s = N$. We will prove Theorem 8.4 by an induction argument. For this, we need to relate the functors

$$\begin{aligned} \Phi_r \colon \mathbf{L}_s^\vee(\mathcal{D}) &\to \mathbf{L}_r(\mathcal{C}), \\ \Phi_{r-1} \colon \mathbf{L}_{s+1}^\vee(\mathcal{D}) &\to \mathbf{L}_{r-1}(\mathcal{C}). \end{aligned}$$

Here and below, when we consider the functor Φ_{r-1} or the categories $\mathbf{L}_{s+1}^\vee(\mathcal{D})$ and $\mathbf{L}_{r-1}(\mathcal{C})$, we implicitly assume $r \geq 1$.

To this end, we introduce some auxiliary spaces. Let $\mathbf{Fl}_{r-1,r}$ denote the variety of flags of subspaces $L_{r-1} \subset L_r \subset V$ where $\dim L_i = i$. This variety comes with forgetful maps

$$\begin{array}{ccc} & \mathbf{Fl}_{r-1,r} & \\ {}^{\pi_{r-1}}\swarrow & & \searrow^{\pi_r} \\ \mathbf{G}_{r-1} & & \mathbf{G}_r \end{array} \tag{8.2}$$

Further, define

$$\begin{aligned} \mathbf{L}_{r-1}^+ &= \mathbf{L}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{Fl}_{r-1,r}, \\ \mathbf{L}_r^- &= \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{Fl}_{r-1,r}. \end{aligned}$$

These spaces fit into a commutative diagram

$$\begin{array}{ccccccc} \mathbf{Fl}_{r-1,r} & \xleftarrow{f_{r-1}^+} & \mathbf{L}_{r-1}^+ & \xrightarrow{a_r} & \mathbf{L}_r^- & \xrightarrow{f_r^-} & \mathbf{Fl}_{r-1,r} \\ {\scriptstyle \pi_{r-1}}\downarrow & & \downarrow{\scriptstyle \pi_{r-1}^+} & & {\scriptstyle \pi_r^-}\downarrow & & \downarrow{\scriptstyle \pi_r} \\ \mathbf{G}_{r-1} & \xleftarrow{f_{r-1}} & \mathbf{L}_{r-1} & & \mathbf{L}_r & \xrightarrow{f_r} & \mathbf{G}_r \end{array} \tag{8.3}$$

where the squares are cartesian and a_r is a closed embedding. We also introduce the notation

$$\begin{aligned} \alpha_r = \pi_r^- \circ a_r &\colon \mathbf{L}_{r-1}^+ \to \mathbf{L}_r, \\ p_{r-1}^+ = p_{r-1} \circ \pi_{r-1}^+ &\colon \mathbf{L}_{r-1}^+ \to \mathbf{P}, \\ p_r^- = p_r \circ \pi_r^- &\colon \mathbf{L}_r^- \to \mathbf{P}, \end{aligned}$$

where recall $p_r \colon \mathbf{L}_r \to \mathbf{P}$ denotes the projection.

Dually, let $\mathbf{Fl}_{s,s+1}^\vee$ denote the the variety of flags of subspaces $M_s \subset M_{s+1} \subset V^\vee$, where $\dim M_i = i$. This variety comes with forgetful maps

$$\begin{array}{ccc} & \mathbf{Fl}_{s,s+1}^\vee & \\ {}^{\pi_{s+1}^\vee}\swarrow & & \searrow^{\pi_s^\vee} \\ \mathbf{G}_{s+1}^\vee & & \mathbf{G}_s^\vee \end{array} \tag{8.4}$$

As above, we define

$$\begin{aligned} \mathbf{L}_{s+1}^{\vee,-} &= \mathbf{L}_{s+1}^\vee \times_{\mathbf{G}_{s+1}^\vee} \mathbf{Fl}_{s,s+1}^\vee, \\ \mathbf{L}_s^{\vee,+} &= \mathbf{L}_s^\vee \times_{\mathbf{G}_s^\vee} \mathbf{Fl}_{s,s+1}^\vee. \end{aligned}$$

Under the canonical isomorphisms $\mathbf{Fl}^{\vee}_{s,s+1} \cong \mathbf{Fl}_{r-1,r}$, $\mathbf{G}^{\vee}_{s+1} \cong \mathbf{G}_{r-1}$, and $\mathbf{G}^{\vee}_{s} \cong \mathbf{G}_{r}$, the diagram (8.4) is identified with (8.2). Hence the above spaces fit into a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{Fl}_{r-1,r} & \xleftarrow{g^{-}_{s+1}} & \mathbf{L}^{\vee,-}_{s+1} & \xleftarrow{b_{s+1}} & \mathbf{L}^{\vee,+}_{s} & \xrightarrow{g^{+}_{s}} & \mathbf{Fl}_{r-1,r} \\
{\scriptstyle \pi_{r-1}}\downarrow & & \downarrow{\scriptstyle \pi^{\vee,-}_{s+1}} & & {\scriptstyle \pi^{\vee,+}_{s}}\downarrow & & \downarrow{\scriptstyle \pi_{r}} \\
\mathbf{G}_{r-1} & \xleftarrow{g_{s+1}} & \mathbf{L}^{\vee}_{s+1} & & \mathbf{L}^{\vee}_{s} & \xrightarrow{g_{s}} & \mathbf{G}_{r}
\end{array} \tag{8.5}$$

As above, we also set

$$\begin{aligned}
\beta_{s+1} &= \pi^{\vee,-}_{s+1} \circ b_{s+1} \colon \mathbf{L}^{\vee,+}_{s} \to \mathbf{L}^{\vee}_{s+1}, \\
q^{-}_{s+1} &= q_{s+1} \circ \pi^{\vee,-}_{s+1} \colon \mathbf{L}^{\vee,-}_{s+1} \to \mathbf{P}^{\vee}, \\
q^{+}_{s} &= q_{s} \circ \pi^{\vee,+}_{s} \colon \mathbf{L}^{\vee,+}_{s} \to \mathbf{P}^{\vee},
\end{aligned}$$

where recall $q_s \colon \mathbf{L}^{\vee}_{s} \to \mathbf{P}^{\vee}$ denotes the projection.

We have morphisms

$$\begin{aligned}
\pi^{-}_{r} \times \pi^{\vee,+}_{s} &\colon \mathbf{L}^{-}_{r} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,+}_{s} \to \mathbf{L}_{r} \times_{\mathbf{G}_{r}} \mathbf{L}^{\vee}_{s}, \\
\pi^{+}_{r-1} \times \pi^{\vee,-}_{s+1} &\colon \mathbf{L}^{+}_{r-1} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,-}_{s+1} \to \mathbf{L}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{L}^{\vee}_{s+1}.
\end{aligned}$$

Pulling back the kernel $\mathcal{E}_r$ for Φ_r along these morphisms, we obtain kernels

$$\begin{aligned}
\mathcal{E}^{-}_{r} &= (\pi^{-}_{r} \times \pi^{\vee,+}_{s})^{*}\mathcal{E}_{r} \in \mathrm{FM}\big(\mathcal{D}/\mathbf{P}^{\vee}, \mathcal{C}/\mathbf{P}, \mathbf{L}^{-}_{r} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,+}_{s}\big), \\
\mathcal{E}^{+}_{r-1} &= (\pi^{+}_{r-1} \times \pi^{\vee,-}_{s+1})^{*}\mathcal{E}_{r-1} \in \mathrm{FM}\big(\mathcal{D}/\mathbf{P}^{\vee}, \mathcal{C}/\mathbf{P}, \mathbf{L}^{+}_{r-1} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,-}_{s+1}\big).
\end{aligned}$$

We denote by

$$\begin{aligned}
\Phi^{-}_{r} &\colon \mathbf{L}^{\vee,+}_{s}(\mathcal{D}) \to \mathbf{L}^{-}_{r}(\mathcal{C}), \\
\Phi^{+}_{r-1} &\colon \mathbf{L}^{\vee,-}_{s+1}(\mathcal{D}) \to \mathbf{L}^{+}_{r-1}(\mathcal{C}),
\end{aligned}$$

the associated $\mathbf{Fl}_{r-1,r}$-linear functors.

Finally, we denote by $\mathcal{U}_r$ the rank r tautological subbundle of $V \otimes \mathcal{O}$ on $\mathbf{G}_r$, and by $\mathcal{W}_s$ the rank s tautological subbundle of $V^{\vee} \otimes \mathcal{O}$ on $\mathbf{G}^{\vee}_{s}$. By abuse of notation, we use the same symbol to denote the pullback of $\mathcal{U}_r$ or $\mathcal{W}_s$ to any space mapping to $\mathbf{G}_r$ or $\mathbf{G}^{\vee}_{s}$. Note that under the isomorphism $\mathbf{G}_r \cong \mathbf{G}^{\vee}_{s}$, the bundle $\mathcal{W}_s$ corresponds to the orthogonal bundle $\mathcal{U}^{\perp}_{r}$.

8.3. **Geometric lemmas.** Here we gather some results describing the geometry of the spaces introduced above. The proofs are left to the reader.

Lemma 8.11. *(1) The morphism $a_r \colon \mathbf{L}^{+}_{r-1} \to \mathbf{L}^{-}_{r}$ embeds $\mathbf{L}^{+}_{r-1}$ as a divisor, cut out by a section of the line bundle $(\mathcal{U}_r/\mathcal{U}_{r-1})(H)$.*
(2) The morphism $b_{s+1} \colon \mathbf{L}^{\vee,+}_{s} \to \mathbf{L}^{\vee,-}_{s+1}$ embeds $\mathbf{L}^{\vee,+}_{s}$ as a divisor, cut out by a section of the line bundle $(\mathcal{U}_r/\mathcal{U}_{r-1})^{\vee}(H')$.

Lemma 8.12. *(1) The morphism $(\zeta_r, \mathrm{pr}_2) \colon \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}^{\vee}_{s} \to \mathbf{H} \times_{\mathbf{P}^{\vee}} \mathbf{L}^{\vee}_{s}$ is a closed immersion, with image cut out by a regular section of the vector bundle $(\mathcal{W}_s/\mathcal{O}(-H'))^{\vee}(H)$.*
(2) The morphism $(\mathrm{pr}_1, \zeta_r) \colon \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}^{\vee}_{s} \to \mathbf{L}_r \times_{\mathbf{P}} \mathbf{H}$ is a closed immersion, with image cut out by a regular section of the vector bundle $(\mathcal{U}_r/\mathcal{O}(-H))^{\vee}(H')$.

Lemma 8.13. *The morphism $(p^{-}_{r}, f^{-}_{r}) \colon \mathbf{L}^{-}_{r} \to \mathbf{P} \times_S \mathbf{Fl}_{r-1,r}$ is a closed immersion, with image cut out by a regular section of the vector bundle $((V \otimes \mathcal{O})/\mathcal{U}_r)(H)$.*

Lemma 8.14. *The morphism* $\beta_{s+1} = \pi^{\vee,-}_{s+1} \circ b_{s+1} \colon \mathbf{L}^{\vee,+}_s \to \mathbf{L}^{\vee}_{s+1}$ *is the projectivization of the vector bundle* $(\mathcal{W}_{s+1}/\mathcal{O}(-H'))^\vee$.

8.4. **Relations between the functors.** We aim here to relate the various kernel functors introduced above. The main statement we are after is Proposition 8.19. We start with some preparations.

Lemma 8.15. *The functors*

$$\Phi_r \colon \mathbf{L}^\vee_s(\mathcal{D}) \to \mathbf{L}_r(\mathcal{C}), \qquad \Phi_{r-1} \colon \mathbf{L}^\vee_{s+1}(\mathcal{D}) \to \mathbf{L}_{r-1}(\mathcal{C}),$$
$$\Phi^-_r \colon \mathbf{L}^{\vee,+}_s(\mathcal{D}) \to \mathbf{L}^-_r(\mathcal{C}), \qquad \Phi^+_{r-1} \colon \mathbf{L}^{\vee,-}_{s+1}(\mathcal{D}) \to \mathbf{L}^+_{r-1}(\mathcal{C}),$$

admit left adjoints $\Phi^*_r, (\Phi^-_r)^*, \Phi^*_{r-1}, (\Phi^+_{r-1})^*$, *and right adjoints* $\Phi^!_r, (\Phi^-_r)^!, \Phi^!_{r-1}, (\Phi^+_{r-1})^!$.

Proof. By Lemma 8.3 the category $\mathcal{D}$ is smooth and proper over S. Hence by Lemma 4.11 we find that the source and target categories of all the functors in question are smooth and proper over S. Now the result follows from Lemma 4.13. □

Lemma 8.16. *There are equivalences of functors*

$$\Phi^-_r \circ (\pi^{\vee,+}_s)^* \simeq (\pi^-_r)^* \circ \Phi_r, \qquad \Phi^+_{r-1} \circ (\pi^{\vee,-}_{s+1})^* \simeq (\pi^+_{r-1})^* \circ \Phi_{r-1},$$
$$(\pi^-_r)_* \circ \Phi^-_r \simeq \Phi_r \circ (\pi^{\vee,+}_s)_*, \qquad (\pi^+_{r-1})_* \circ \Phi^+_{r-1} \simeq \Phi_{r-1} \circ (\pi^{\vee,-}_{s+1})_*,$$
$$(\pi^{\vee,+}_s)_* \circ (\Phi^-_r)^* \simeq (\Phi_r)^* \circ (\pi^-_r)_*, \qquad (\pi^{\vee,-}_{s+1})_* \circ (\Phi^+_{r-1})^* \simeq \Phi^*_{r-1} \circ (\pi^+_{r-1})_*,$$
$$(\pi^{\vee,+}_s)^{*L} \circ (\Phi^-_r)^* \simeq (\Phi_r)^* \circ (\pi^-_r)^{*L}, \qquad (\pi^{\vee,-}_{s+1})^{*L} \circ (\Phi^+_{r-1})^* \simeq \Phi^*_{r-1} \circ (\pi^+_{r-1})^{*L},$$

where $(\pi^{\vee,+}_s)^{*L}$ *denotes the left adjoint of* $(\pi^{\vee,+}_s)^*$, *and similarly for* $(\pi^-_r)^{*L}, (\pi^{\vee,-}_{s+1})^{*L}, (\pi^+_{r-1})^{*L}$.

Proof. The first two rows follow from Lemma 5.16 and Remark 5.15. By adjunction the third row is equivalent to the assertion

$$\Phi^-_r \circ (\pi^{\vee,+}_s)^! \simeq (\pi^-_r)^! \circ \Phi_r, \qquad \Phi^+_{r-1} \circ (\pi^{\vee,-}_{s+1})^! \simeq (\pi^+_{r-1})^! \circ \Phi_{r-1}.$$

We have

$$(\pi^-_r)^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_r)]) \circ (\pi^-_r)^*,$$
$$(\pi^{\vee,+}_s)^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_r)]) \circ (\pi^{\vee,+}_s)^*,$$
$$(\pi^+_{r-1})^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1})]) \circ (\pi^+_{r-1})^*,$$
$$(\pi^{\vee,-}_{s+1})^! \simeq (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1})]) \circ (\pi^{\vee,-}_{s+1})^*.$$

Hence by $\mathbf{Fl}_{r-1,r}$-linearity the assertion reduces to the first row. Similarly, we have

$$(\pi^-_r)^{*L} \simeq (\pi^-_r)_* \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_r)]),$$
$$(\pi^{\vee,+}_s)^{*L} \simeq (\pi^{\vee,+}_s)_* \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_r}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_r)]),$$
$$(\pi^+_{r-1})^{*L} \simeq (\pi^+_{r-1})_* \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1})]),$$
$$(\pi^{\vee,-}_{s+1})^{*L} \simeq (\pi^{\vee,-}_{s+1})_* \circ (- \otimes \omega_{\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1}}[\dim(\mathbf{Fl}_{r-1,r}/\mathbf{G}_{r-1})]).$$

Hence by $\mathbf{Fl}_{r-1,r}$-linearity the fourth row follows from the third. □

We define

$$\mathbf{M}_r^- = \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_s^{\vee,+},$$
$$\mathbf{M}_{r-1}^+ = \mathbf{L}_{r-1}^+ \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}.$$

Then by definition Φ_r^- is defined by an $\mathbf{M}_r^-$-kernel and Φ_{r-1}^+ by an $\mathbf{M}_{r-1}^+$-kernel. These spaces fit into a cartesian commutative diagram

$$\begin{array}{ccc} \mathbf{L}_{r-1}^+ \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_s^{\vee,+} & \xrightarrow{\mathrm{id}\times b_{s+1}} & \mathbf{M}_{r-1}^+ \\ \downarrow{\scriptstyle a_r\times\mathrm{id}} & & \downarrow{\scriptstyle a_r\times\mathrm{id}} \\ \mathbf{M}_r^- & \xrightarrow{\mathrm{id}\times b_{s+1}} & \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} \end{array} \tag{8.6}$$

By Lemma 8.11 all of the morphisms in this diagram are divisorial embeddings.

The projections $p_r^- \colon \mathbf{L}_r^- \to \mathbf{P}^\vee$ and $q_{s+1}^- \colon \mathbf{L}_{s+1}^{\vee,-} \to \mathbf{P}^\vee$ induce a morphism

$$\nu_r = p_r^- \times q_{s+1}^- \colon \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} \to \mathbf{P} \times_S \mathbf{P}^\vee. \tag{8.7}$$

Let $\widetilde{\mathbf{M}}_r$ be defined by the fiber product diagram

$$\begin{array}{ccc} \widetilde{\mathbf{M}}_r & \xrightarrow{\iota_r} & \mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-} \\ \downarrow{\scriptstyle \widetilde{\zeta}_r} & & \downarrow{\scriptstyle \nu_r} \\ \mathbf{H} & \xrightarrow{\iota} & \mathbf{P} \times_S \mathbf{P}^\vee \end{array} \tag{8.8}$$

The compositions

$$\zeta_r^- \colon \mathbf{M}_r^- \xrightarrow{\pi_r^- \times \pi_s^{\vee,+}} \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee \xrightarrow{\zeta_r} \mathbf{H},$$
$$\zeta_{r-1}^+ \colon \mathbf{M}_{r-1}^+ \xrightarrow{\pi_{r-1}^+ \times \pi_{s+1}^{\vee,-}} \mathbf{L}_{r-1} \times_{\mathbf{G}_{r-1}} \mathbf{L}_{s+1}^\vee \xrightarrow{\zeta_{r-1}} \mathbf{H},$$

factor through closed embeddings

$$\delta_r^- \colon \mathbf{M}_r^- \hookrightarrow \widetilde{\mathbf{M}}_r \tag{8.9}$$
$$\delta_{r-1}^+ \colon \mathbf{M}_{r-1}^+ \hookrightarrow \widetilde{\mathbf{M}}_r. \tag{8.10}$$

Lemma 8.17. *We have*

$$\widetilde{\mathbf{M}}_r = \mathbf{M}_r^- \cup \mathbf{M}_{r-1}^+,$$

where the right side is the scheme-theoretic union inside $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$.

Proof. Note that by definition $\widetilde{\mathbf{M}}_r$ is cut out in $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$ by a section of the line bundle $\mathcal{O}(H + H')$. On the other hand, by Lemma 8.11 the union $\mathbf{M}_r^- \cup \mathbf{M}_{r-1}^+$ is cut out in $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$ by a section of the line bundle $(\mathcal{U}_r/\mathcal{U}_{r-1})(H) \otimes (\mathcal{U}_r/\mathcal{U}_{r-1})^\vee(H') \cong \mathcal{O}(H+H')$. These two sections of $\mathcal{O}(H + H')$ coincide, cf. [9, Lemma 6.13]. □

Let $\widetilde{\mathcal{E}}_r = \widetilde{\zeta}_r^* \mathcal{E}$ be the $\widetilde{\mathbf{M}}_r$-kernel obtained by pulling back the $\mathbf{H}$-kernel $\mathcal{E}$. We denote by D_r the Cartier divisor on $\mathbf{L}_{s+1}^{\vee,-}$ corresponding to the line bundle $(\mathcal{U}_r/\mathcal{U}_{r-1})^\vee(H')$. By Lemma 8.11 the scheme $\mathbf{M}_r^-$ is cut out in $\mathbf{L}_r^- \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}_{s+1}^{\vee,-}$ by a section of $\mathcal{O}(D_r)$.

Lemma 8.18. *(1) There is an exact triangle*

$$(a_r \times \mathrm{id})_* \mathcal{E}^+_{r-1} \otimes \mathcal{O}(-D_r) \to \iota_{r*}\widetilde{\mathcal{E}}_r \to (\mathrm{id} \times b_{s+1})_* \mathcal{E}^-_r$$

of $\mathbf{L}^-_r \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,-}_{s+1}$*-kernels.*

(2) There is an equivalence

$$(a_r \times \mathrm{id})^* \mathcal{E}^-_r \simeq (\mathrm{id} \times b_{s+1})^* \mathcal{E}^+_{r-1}$$

of $\mathbf{L}^+_{r-1} \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,+}_{s}$*-kernels.*

(3) There is an equivalence

$$\iota_{r*}\widetilde{\mathcal{E}}_r \simeq \nu_r^* \iota_* \mathcal{E}$$

of $\mathbf{L}^-_r \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,-}_{s+1}$*-kernels.*

Proof. It follows from Lemma 8.17 that there is an exact sequence

$$0 \to (\delta^+_{r-1})_* \mathcal{O}_{\mathbf{M}^+_{r-1}}(-D_r) \to \mathcal{O}_{\widetilde{\mathbf{M}}_r} \to (\delta^-_r)_* \mathcal{O}_{\mathbf{M}^-_r} \to 0$$

of sheaves on $\widetilde{\mathbf{M}}_r$, where δ^+_{r-1} and δ^-_r are the embeddings (8.10) and (8.9). Now (1) follows by kernel formalism. More precisely, tensoring the above exact sequence by $\widetilde{\mathcal{E}}_r$ gives an exact triangle of kernels by Lemma 5.8. But by definition $(\delta^+_{r-1})^*(\widetilde{\mathcal{E}}_r) \simeq \mathcal{E}^+_{r-1}$ and $(\delta^-_r)^*(\widetilde{\mathcal{E}}_r) \simeq \mathcal{E}^-_r$, so by the projection formula for kernels (Lemma 5.10), the resulting exact triangle can be written

$$(\delta^+_{r-1})_* \mathcal{E}^+_{r-1} \otimes \mathcal{O}(-D_r) \to \widetilde{\mathcal{E}}_r \to (\delta^-_r)_* \mathcal{E}^-_r$$

Now the result follows by pushing forward via $\iota_r \colon \widetilde{\mathbf{M}}_r \to \mathbf{L}^-_r \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,-}_{s+1}$ and using the projection formula once more.

It follows from the definitions that both kernels appearing in (2) are equivalent to the pullback of $\widetilde{\mathcal{E}}_r$ along the natural map $\mathbf{L}^-_r \times_{\mathbf{Fl}_{r-1,r}} \mathbf{L}^{\vee,-}_{s+1} \to \widetilde{\mathbf{M}}_r$, hence (2) holds.

Finally, (3) follows from the cartesian square (8.8) and base change. □

Proposition 8.19. *(1) There is an exact triangle*

$$\Phi^-_r \circ b^!_{s+1} \to (a_r)_* \circ \Phi^+_{r-1} \to \Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)}$$

of functors $\mathbf{L}^{\vee,-}_{s+1}(\mathcal{D}) \to \mathbf{L}^-_r(\mathcal{C})$*, where* $b^!_{s+1} \colon \mathbf{L}^{\vee,-}_{s+1}(\mathcal{D}) \to \mathbf{L}^{\vee,+}_s(\mathcal{D})$ *is the functor induced by the right adjoint to* $(b_{s+1})_* \colon \mathrm{Perf}(\mathbf{L}^{\vee,+}_s) \to \mathrm{Perf}(\mathbf{L}^{\vee,-}_{s+1})$

(2) There is an exact triangle

$$(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^* \to (\Phi^+_{r-1})^* \circ a_r^* \to (b_{s+1})_* \circ (\Phi^-_r)^*$$

of functors $\mathbf{L}^-_r(\mathcal{C}) \to \mathbf{L}^{\vee,-}_{s+1}(\mathcal{D})$*.*

(3) There is an equivalence

$$a_r^* \circ \Phi^-_r \simeq \Phi^+_{r-1} \circ (b_{s+1})_*$$

of functors $\mathbf{L}^{\vee,+}_s(\mathcal{D}) \to \mathbf{L}^+_{r-1}(\mathcal{C})$*.*

Proof. It follows from Lemma 8.11(2) that $b^!_{s+1} = b^*_{s+1} \circ (- \otimes \mathcal{O}(D_r))[-1]$. Now (1) follows by rotating the triangle of kernels from Lemma 8.18(1), twisting by $\mathcal{O}(D_r)$, passing to the associated kernel functors, and using Lemmas 5.11 and 5.13. By Lemmas 2.13 and 8.15, (2) follows from (1) by passing to left adjoints. Finally, (3) follows from Lemma 8.18(2) by passing to the associated kernel functors and using Lemma 5.16. □

8.5. **Semiorthogonal sequences in** $\ker \Phi_r^*$ **and** $\ker \Phi_r$**.** Our next goal is to show that the semiorthogonal sequence to the right of $\mathcal{K}_r(\mathcal{C})$ in the decomposition (6.16) is contained in $\ker \Phi_r^*$, and the semiorthogonal sequence to the left of $\mathcal{K}_s(\mathcal{D})$ in (8.1) is contained $\ker \Phi_r$. Later we will combine this result with the fact that the Φ_r are left splitting (Proposition 8.28) to reduce Theorem 8.4 to a statement about the generation of $\mathbf{L}_r(\mathcal{C})$ and $\mathbf{L}_s^\vee(\mathcal{D})$ by certain semiorthogonal sequences.

Lemma 8.20. *For $0 \le i \le s-1$ there are kernels*

$$\mathcal{F}_{r,i},\ \mathcal{K}_{r,i} \in \mathrm{FM}(\mathcal{D}/\mathbf{P}^\vee, \mathcal{C}/\mathbf{P}, \mathbf{P} \times_S \mathbf{L}_s^\vee)$$

such that:

(1) $\Phi_{\mathcal{F}_{r,0}} \simeq p_{r*} \circ \Phi_r$*.*

(2) For all i there is an exact triangle

$$\mathcal{F}_{r,i+1} \to \mathcal{K}_{r,i} \to \mathcal{F}_{r,i}$$

where for $i = s-1$ we set $\mathcal{F}_{r,s} = 0$.

(3) There is an equivalence

$$\Phi_{\mathcal{K}_{r,i}} \simeq (- \otimes \mathcal{O}(-iH)) \circ p_* \circ \Phi_{\mathcal{E}} \circ q_{s*} \circ (- \otimes \wedge^i(\mathcal{W}_s/\mathcal{O}(-H'))).$$

Proof. By definition the functor Φ_r is given by the kernel

$$\mathcal{E}_r = \zeta_r^* \mathcal{E} \in \mathrm{FM}(\mathcal{D}/\mathbf{P}^\vee, \mathcal{C}/\mathbf{P}, \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee).$$

Hence by Lemma 5.13 the functor $p_{r*} \circ \Phi_r$ is given by the pushforward of this kernel along

$$p_r \times \mathrm{id} \colon \mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee \to \mathbf{P} \times_S \mathbf{L}_s^\vee,$$

i.e. by the kernel $(p_r \times \mathrm{id})_* \mathcal{E}_r$. To understand this kernel, we consider the commutative diagram

$$\begin{array}{ccc}
\mathbf{L}_r \times_{\mathbf{G}_r} \mathbf{L}_s^\vee & \xrightarrow{\ p_r \times \mathrm{id}\ } & \mathbf{P} \times_S \mathbf{L}_s^\vee \\
\downarrow{\scriptstyle (\zeta_r, \mathrm{pr}_2)} & & \\
\mathbf{H} \times_{\mathbf{P}^\vee} \mathbf{L}_s^\vee & \xrightarrow{\ p \times \mathrm{id}\ } & \mathbf{P} \times_S \mathbf{L}_s^\vee \\
\downarrow{\scriptstyle \mathrm{pr}_1} & & \downarrow{\scriptstyle \mathrm{id} \times q_s} \\
\mathbf{H} & \xrightarrow{\ \iota\ } & \mathbf{P} \times_S \mathbf{P}^\vee
\end{array} \tag{8.11}$$

where the square is cartesian. We have

$$\begin{aligned}
(p_r \times \mathrm{id})_* \mathcal{E}_r &\simeq (p \times \mathrm{id})_* (\zeta_r, \mathrm{pr}_2)_* (\zeta_r, \mathrm{pr}_2)^* \mathrm{pr}_1^* \mathcal{E} \\
&\simeq (p \times \mathrm{id})_* (\mathrm{pr}_1^* \mathcal{E} \otimes (\zeta_r, \mathrm{pr}_2)_* \mathcal{O}).
\end{aligned}$$

It follows from Lemma 8.12(1) that there is a resolution of $(\zeta_r, \mathrm{pr}_2)_* \mathcal{O}$ of the form

$$0 \to \wedge^{s-1}(\mathcal{W}_s/\mathcal{O}(-H'))(-(s-1)H) \to \cdots \to (\mathcal{W}_s/\mathcal{O}(-H'))(-H) \to \mathcal{O} \to (\zeta_r, \mathrm{pr}_2)_* \mathcal{O} \to 0. \tag{8.12}$$

Let

$$\mathcal{K}_{r,i} = (p \times \mathrm{id})_* (\mathrm{pr}_1^* \mathcal{E} \otimes \wedge^i(\mathcal{W}_s/\mathcal{O}(-H'))(-iH))$$

denote the kernel corresponding to the i-th term in this resolution. Then

$$\begin{aligned}
\mathcal{K}_{r,i} &\simeq ((p \times \mathrm{id})_* \mathrm{pr}_1^* \mathcal{E}) \otimes \wedge^i(\mathcal{W}_s/\mathcal{O}(-H'))(-iH) \\
&\simeq ((\mathrm{id} \times q_s)^* \iota_* \mathcal{E}) \otimes \wedge^i(\mathcal{W}_s/\mathcal{O}(-H'))(-iH)
\end{aligned}$$

where the first line holds since $\wedge^i(\mathcal{W}_s/\mathcal{O}(-H'))$ is pulled back from $\mathbf{L}_s^\vee$ and $\mathcal{O}(-iH)$ from $\mathbf{P}$, and the second since the square in (8.11) is cartesian. The kernel formalism then shows that $\mathcal{K}_i$ satisfies part (3) of the lemma, and the rest follows by splitting the resolution (8.12) into short exact sequences. □

Proposition 8.21. *There is an inclusion*

$$\left\langle \mathcal{A}_s(H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-s)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \right\rangle \subset \ker \Phi_r^*.$$

Proof. By adjunction, the assertion is equivalent to

$$\operatorname{im} \Phi_r \subset \left\langle \mathcal{A}_s(H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-s)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \right\rangle^{\perp},$$

where the orthogonal is taken inside $\mathbf{L}_r(\mathcal{C})$. By $\mathbf{G}_r$-linearity of Φ_r, this is equivalent to

$$\operatorname{im}(p_{r*} \circ \Phi_r) \subset \langle \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle^{\perp}$$

where the orthogonal is taken inside $\mathcal{C}$. The functor $\Phi_{\mathcal{K}_{r,i}}$ from Lemma 8.20 satisfies

$$\operatorname{im} \Phi_{\mathcal{K}_{r,i}} \subset \mathcal{A}_0(-iH). \tag{8.13}$$

Indeed, using the description of this functor from Lemma 8.20(3), this follows from Lemma 7.11. Hence we find

$$\operatorname{im}(p_{r*} \circ \Phi_r) \subset \langle \mathcal{A}_0(-(s-1)H), \mathcal{A}_0(-(s-2)) \dots, \mathcal{A}_0 \rangle,$$

so we are done by Lemma 8.22 below. □

Lemma 8.22. *For any $i \geq 0$ the following subcategories of $\mathcal{C}$ coincide:*

$$\langle \mathcal{A}_0, \mathcal{A}_0(H), \dots, \mathcal{A}_0((i-1)H) \rangle,$$
$$\langle \mathcal{A}_0, \mathcal{A}_1(H), \dots, \mathcal{A}_{i-1}((i-1)H) \rangle,$$
$$\langle \mathcal{A}_i(iH), \mathcal{A}_{i+1}((i+1)H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle^{\perp}.$$

Proof. The semiorthogonal decomposition

$$\mathcal{C} = \langle \mathcal{A}_0, \mathcal{A}_1(H), \dots, \mathcal{A}_{m-1}((m-1)H) \rangle$$

implies the last two categories coincide, and the first category contains the second and is contained in the third. □

Proposition 8.23. *There is an inclusion*

$$\left\langle \mathcal{B}_{n-1}(-(n-r)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_s^\vee), \dots, \mathcal{B}_r(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_s^\vee) \right\rangle \subset \ker \Phi_r.$$

Proof. Analogous to Proposition 8.21 and left to the reader. □

8.6. **Technical results about the functor $\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)}$.** Here we prove two related results about the (left adjoint of the) functor $\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)}$ appearing in Proposition 8.19, which will be needed later in our induction arguments. Namely, we describe the image of a certain subcategory under $(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^*$ (Proposition 8.24) and show that certain composite functors involving $(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^*$ vanish (Proposition 8.27).

Proposition 8.24. *The image*

$$(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^* \left((\langle \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes_S \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^{\perp} \right)$$

lies inside the subcategory

$$\mathcal{B}_{r-1}(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}) \subset \mathbf{L}_{s+1}^{\vee,-}(\mathcal{D}).$$

Proof. By Lemma 5.11 we have

$$(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r\otimes\mathcal{O}(D_r)})^* \simeq (-\otimes\mathcal{O}(-D_r))\circ(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r})^*.$$

Since by definition $\mathcal{O}(D_r)=(\mathcal{U}_r/\mathcal{U}_{r-1})^\vee(H')$, it therefore suffices to show

$$(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r})^*\left((\langle\mathcal{A}_s(H),\dots,\mathcal{A}_{m-1}((m-s)H)\rangle\otimes_S\mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^\perp\right)\subset\mathcal{B}_{r-1}\otimes_{\mathrm{Perf}(S)}\mathrm{Perf}(\mathbf{Fl}_{r-1,r}).$$

For this, we consider the diagram

$$\begin{array}{ccccc}
 & & \mathbf{L}_r^-\times_{\mathbf{Fl}_{r-1,r}}\mathbf{L}_{s+1}^{\vee,-} & \longrightarrow & \mathbf{L}_{s+1}^{\vee,-} \\
 & & \downarrow{\scriptstyle \mathrm{id}\times j} & & \downarrow{\scriptstyle j} \\
\mathbf{L}_r^- & \longleftarrow & \mathbf{L}_r^-\times_{\mathbf{Fl}_{r-1,r}}(\mathbf{Fl}_{r-1,r}\times_S\mathbf{P}^\vee) & \longrightarrow & \mathbf{Fl}_{r-1,r}\times_S\mathbf{P}^\vee \\
\downarrow{\scriptstyle p_r^-} & & \downarrow{\scriptstyle p_r^-\times\mathrm{pr}_2} & & \\
\mathbf{P} & \longleftarrow & \mathbf{P}\times_S\mathbf{P}^\vee & &
\end{array}$$

where $j\colon\mathbf{L}_{s+1}^{\vee,-}\to\mathbf{Fl}_{r-1,r}\times_S\mathbf{P}^\vee$ is the embedding and the squares are cartesian. By definition the morphism $\nu_r\colon\mathbf{L}_r^-\times_{\mathbf{Fl}_{r-1,r}}\mathbf{L}_{s+1}^{\vee,-}\to\mathbf{P}\times_S\mathbf{P}^\vee$ defined by (8.7) satisfies $\nu_r=(p_r^-\times\mathrm{pr}_2)\circ(\mathrm{id}\times j)$. Hence by Lemma 8.18(3) there is an equivalence

$$\iota_{r*}\widetilde{\mathcal{E}}_r\simeq\nu_r^*\iota_*\mathcal{E}\simeq(\mathrm{id}\times j)^*(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}.$$

It follows from Lemma 5.16 that

$$\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r}\simeq\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}}\circ j_*$$

and hence

$$(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r})^*\simeq j^*\circ(\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}})^*$$

To prove the lemma, it thus suffices to show the image

$$(\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}})^*\left((\langle\mathcal{A}_s(H),\dots,\mathcal{A}_{m-1}((m-s)H)\rangle\otimes_S\mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^\perp\right)$$

is contained in

$$\mathcal{B}_{r-1}\otimes_{\mathrm{Perf}(S)}\mathrm{Perf}(\mathbf{Fl}_{r-1,r})\subset\mathcal{D}\otimes_{\mathrm{Perf}(\mathbf{P}^\vee)}\mathrm{Perf}(\mathbf{Fl}_{r-1,r}\times_S\mathbf{P}^\vee).$$

Since the image in question is $\mathbf{Fl}_{r-1,r}$-linear, it suffices to show

$$\mathrm{pr}_{2*}(\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}})^*\left((\langle\mathcal{A}_s(H),\dots,\mathcal{A}_{m-1}((m-s)H)\rangle\otimes_S\mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^\perp\right)\subset\mathcal{B}_{r-1}\quad(8.14)$$

as subcategories of $\mathcal{D}$. By the above diagram and Lemma 5.16, there is an equivalence

$$\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}}\circ\mathrm{pr}_2^*\simeq(p_r^-)^*\circ\Phi_{\iota_*\mathcal{E}}.$$

Hence, if pr_2^{*L} and $(p_r^-)^{*L}$ denote the left adjoints of the pullback functors of the morphisms $\mathrm{pr}_2\colon\mathbf{Fl}_{r-1,r}\times_S\mathbf{P}^\vee\to\mathbf{P}^\vee$ and $p_r^-\colon\mathbf{L}_r^-\to\mathbf{P}$, then we have

$$\mathrm{pr}_2^{*L}\circ(\Phi_{(p_r^-\times\mathrm{pr}_2)^*\iota_*\mathcal{E}})^*\simeq(\Phi_{\iota_*\mathcal{E}})^*\circ(p_r^-)^{*L}.\quad(8.15)$$

It is easy to see

$$\mathrm{pr}_2^{*L}\simeq\mathrm{pr}_{2*}\circ(-\otimes\omega_{\mathbf{Fl}_{r-1,r}/S}[\dim(\mathbf{Fl}_{r-1,r}/S)])\quad(8.16)$$

$$(p_r^-)^{*L}\simeq(p_r^-)_*\circ(-\otimes\mathcal{O}(sH)\otimes\det(\mathcal{U}_r)^{-1}\otimes\omega_{\mathbf{Fl}_{r-1,r}/S}[\dim(\mathbf{Fl}_{r-1,r}/S)-s])\quad(8.17)$$

where we have used Lemma 8.13 in deriving the second equivalence. Combining (8.15), (8.16), (8.17), and using that $\omega_{\mathbf{Fl}_{r-1,r}/S}$ and $\det(\mathcal{U}_r)^{-1}$ are pulled back from $\mathbf{Fl}_{r-1,r}$, the claim (8.14) reduces to showing

$$(\Phi_{\iota_*\mathcal{E}})^* \left(\langle \mathcal{A}_s((s+1)H), \dots, \mathcal{A}_{m-1}(mH) \rangle^{\perp} \right) \subset \mathcal{B}_{r-1}.$$

But $(\Phi_{\iota_*\mathcal{E}})^* \simeq \phi^* \circ \gamma^* \circ p^*$ and

$$\langle \mathcal{A}_s((s+1)H), \dots, \mathcal{A}_{m-1}(mH) \rangle^{\perp} = \langle \mathcal{A}_0(H), \dots, \mathcal{A}_{s-1}(sH) \rangle ,$$

so the desired inclusion follows from Lemma 8.25 below and the definition of $\mathcal{B}_{r-1}$. □

Lemma 8.25. *For any $i \geq 0$ the functor $\gamma^* \circ p^* \colon \mathcal{C} \to \mathcal{C}^{\vee}$ satisfies*

$$\gamma^* p^* \left(\langle \mathcal{A}_0(H), \dots, \mathcal{A}_{i-1}(i) \rangle \right) \subset \mathcal{A}^{\vee}_{N-i-1}.$$

Proof. By Lemma 7.11 the functor $\gamma^* \circ p^*$ kills ${}^{\perp}\mathcal{A}_0$. Hence $\gamma^* p^*(C) \simeq \gamma^* p^* \alpha_0^*(C)$ for any $C \in \mathcal{C}$, where recall $\alpha_0 \colon \mathcal{A}_0 \to \mathcal{C}$ is the inclusion. But it follows immediately from the definitions that

$$\alpha_0^* \left(\langle \mathcal{A}_0(H), \dots, \mathcal{A}_{i-1}(iH) \rangle \right) = \left\langle \mathfrak{a}'_0, \dots, \mathfrak{a}'_{i-1} \right\rangle ,$$

and then the result follows from the definition of $\mathcal{A}^{\vee}_{N-i-1}$. □

We will need the following observation.

Lemma 8.26. *There are inclusions*

$$\operatorname{im} \Phi_r^- \subset \left(\langle \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes_{\operatorname{Perf}(S)} \operatorname{Perf}(\mathbf{Fl}_{r-1,r}) \right)^{\perp}$$
$$\left\langle \mathcal{B}_{n-1}(-(n-r+1)H'), \dots, \mathcal{B}_{r-1}(-H') \right\rangle \otimes_{\operatorname{Perf}(S)} \operatorname{Perf}(\mathbf{Fl}_{r-1,r}) \subset \ker \Phi^+_{r-1}.$$

Proof. By Lemma 8.16 we have

$$(\pi_r^-)_* \circ \Phi_r^- \simeq \Phi_r \circ (\pi_s^{\vee,+})_*,$$
$$\Phi^+_{r-1} \circ (\pi^{\vee,-}_{s+1})^* \simeq (\pi^+_{r-1})^* \circ \Phi_{r-1},$$

from which the result follows using $\mathbf{Fl}_{r-1,r}$-linearity and Propositions 8.21 and 8.23. □

Proposition 8.27. *We have*

$$\Phi^+_{r-1} \circ (\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^* \circ \Phi_r^- \simeq 0. \tag{8.18}$$

Moreover, we have the stronger vanishings

$$(\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^* \circ \Phi_r^- \simeq 0 \quad \text{if } r-1 \geq n, \tag{8.19}$$
$$\Phi^+_{r-1} \circ (\Phi_{\iota_{r*}\widetilde{\mathcal{E}}_r \otimes \mathcal{O}(D_r)})^* \simeq 0 \quad \text{if } r \leq N-m. \tag{8.20}$$

Proof. Follows from Lemmas 8.26 and 8.24 and the observations that $\mathcal{B}_{r-1} = 0$ if $r-1 \geq n$ and

$$\left(\langle \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes_{\operatorname{Perf}(S)} \operatorname{Perf}(\mathbf{Fl}_{r-1,r}) \right)^{\perp} = \mathbf{L}_r^-(\mathcal{C})$$

if $s \geq m$ or equivalently $r \leq N-m$. □

8.7. The functors Φ_r are left splitting.

Proposition 8.28. *The functor* $\Phi_r \colon \mathbf{L}_s^\vee(\mathcal{D}) \to \mathbf{L}_r(\mathcal{C})$ *is left splitting for any* r*, and hence*

$$\mathbf{L}_r(\mathcal{C}) = \langle \operatorname{im} \Phi_r, \ker \Phi_r^* \rangle,$$
$$\mathbf{L}_s^\vee(\mathcal{D}) = \langle \ker \Phi_r, \operatorname{im} \Phi_r^* \rangle.$$

Proof. We argue by descending induction on r. For $r = N$ the result is trivial as $\mathbf{L}_0(\mathcal{D}) = 0$, and for $r = N-1$ the result holds since by assumption the functor Φ_{N-1} is fully faithful with admissible image. Now assume the result holds for Φ_r. We will prove that Φ_{r-1} satisfies criterion (2) of Theorem 3.23.

Note that it follows from Lemma 8.14 that the functor

$$\beta_{s+1}^* \colon \mathbf{L}_{s+1}^\vee(\mathcal{D}) \to \mathbf{L}_s^{\vee,+}(\mathcal{D})$$

is fully faithful, and hence

$$(\pi_{s+1}^{\vee,-})_*(b_{s+1})_*\beta_{s+1}^* \simeq (\beta_{s+1})_*\beta_{s+1}^* \simeq \mathrm{id}.$$

Thus applying Lemma 8.16 we find

$$\begin{aligned} \Phi_{r-1}\Phi_{r-1}^*\Phi_{r-1} &\simeq \Phi_{r-1}\Phi_{r-1}^*\Phi_{r-1}(\pi_{s+1}^{\vee,-})_*(b_{s+1})_*\beta_{s+1}^* \\ &\simeq (\pi_{r-1}^+)_*\Phi_{r-1}^+(\Phi_{r-1}^+)^*\Phi_{r-1}^+(b_{s+1})_*\beta_{s+1}^*. \end{aligned} \tag{8.21}$$

Now we examine the inner term $\Phi_{r-1}^+(\Phi_{r-1}^+)^*\Phi_{r-1}^+(b_{s+1})_*$. First from Proposition 8.19(3) we deduce

$$\Phi_{r-1}^+(\Phi_{r-1}^+)^*\Phi_{r-1}^+(b_{s+1})_* \simeq \Phi_{r-1}^+(\Phi_{r-1}^+)^*a_r^*\Phi_r^-.$$

By (8.18) if we compose the the triangle of Proposition 8.19(2) on the left with Φ_{r-1}^+ and on the right with Φ_r^-, then the first term vanishes and we obtain an equivalence

$$\Phi_{r-1}^+(\Phi_{r-1}^+)^*a_r^*\Phi_r^- \simeq \Phi_{r-1}^+(b_{s+1})_*(\Phi_r^-)^*\Phi_r^-.$$

Applying Proposition 8.19(3) again, we find

$$\Phi_{r-1}^+(b_{s+1})_*(\Phi_r^-)^*\Phi_r^- \simeq a_r^*\Phi_r^-(\Phi_r^-)^*\Phi_r^-$$

Since by assumption Φ_r is left splitting, by Proposition 5.18 so is Φ_r^-, and hence

$$a_r^*\Phi_r^-(\Phi_r^-)^*\Phi_r^- \simeq a_r^*\Phi_r^-.$$

Combining the above and applying Proposition 8.19(3) one more time, all together we have shown

$$\Phi_{r-1}^+(\Phi_{r-1}^+)^*\Phi_{r-1}^+(b_{s+1})_* \simeq \Phi_{r-1}^+(b_{s+1})_*.$$

Returning to (8.21), we have thus shown

$$\Phi_{r-1}\Phi_{r-1}^*\Phi_{r-1} \simeq (\pi_{r-1}^+)_*\Phi_{r-1}^+(b_{s+1})_*\beta_{s+1}^*.$$

So using Lemma 8.16 we conclude

$$\Phi_{r-1}\Phi_{r-1}^*\Phi_{r-1} \simeq \Phi_{r-1}(\pi_{s+1}^{\vee,-})_*(b_{s+1})_*\beta_{s+1}^* \simeq \Phi_{r-1}. \qquad \square$$

8.8. **Proof of Theorem 8.4.** By combining Proposition 8.28, Theorem 3.23, and Propositions 8.21 and 8.23, we obtain semiorthogonal sequences

$$\mathbf{L}_r(\mathcal{C}) \supset \left\langle \operatorname{im}\Phi_r, \mathcal{A}_s(H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-s)H) \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}_r) \right\rangle,$$
$$\mathbf{L}^\vee_s(\mathcal{D}) \supset \left\langle \mathcal{B}_{n-1}(-(n-r)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}^\vee_s), \dots, \mathcal{B}_r(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}^\vee_s), \operatorname{im}\Phi^*_r \right\rangle,$$

such that the functors Φ_r and Φ^*_r induce mutually inverse equivalences $\operatorname{im}\Phi^*_r \simeq \operatorname{im}\Phi_r$. Hence to prove Theorem 8.4, it remains to prove that the above semiorthogonal sequences generate $\mathbf{L}_r(\mathcal{C})$ and $\mathbf{L}^\vee_s(\mathcal{D})$. We do this in Propositions 8.29 and 8.31 via an inductive argument. We start with $\mathbf{L}^\vee_s(\mathcal{D})$, where the base case of the induction holds by our assumptions on $\mathcal{D}$.

Proposition 8.29. *We have*

$$\mathbf{L}^\vee_s(\mathcal{D}) = \left\langle \mathcal{B}_{n-1}(-(n-r)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}^\vee_s), \dots, \mathcal{B}_r(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}^\vee_s), \operatorname{im}\Phi^*_r \right\rangle.$$

Proof. If $r \geq n$ the right side is simply $\operatorname{im}\Phi^*_r$. Since Φ_r is left splitting by Proposition 8.28, the result amounts to the claim that $\ker\Phi_r = 0$, i.e. that Φ_r is fully faithful. This can be proved by descending induction in r. The base case $r = N-1$ holds by Setup 8.1(3a). We leave it to the reader to check that the same induction argument from Proposition 8.28, but using (8.19) in place of (8.18) and Proposition 5.17 in place of Proposition 5.18, shows that $\Phi^*_r\Phi_r \simeq \mathrm{id}$, i.e. Φ_r is fully faithful, for all $r \geq n$.

For the case $r \leq n$, we again use descending induction. Assume the result holds for some $r \leq n$. To prove the result for $r-1$, suppose $D \in \mathbf{L}^\vee_{s+1}(\mathcal{D})$ lies in

$$\left\langle \mathcal{B}_{n-1}(-(n-r+1)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}^\vee_s), \dots, \mathcal{B}_{r-1}(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{G}^\vee_s), \operatorname{im}\Phi^*_{r-1} \right\rangle^\perp. \tag{8.22}$$

We must show $D \simeq 0$. For this, it is enough to show $b^!_{s+1}(\pi^{\vee,-}_{s+1})^*D \simeq 0$. Indeed, the functor $b^!_{s+1}\circ(\pi^{\vee,-}_{s+1})^*$ differs from $\beta^*_{s+1} = b^*_{s+1}\circ(\pi^{\vee,-}_{s+1})^*$ by the twist by a line bundle by Lemma 8.11(2), and the functor β^*_{s+1} is fully faithful by Lemma 8.14. By base change the induction hypothesis implies

$$\mathbf{L}^{\vee,-}_s(\mathcal{D}) = \langle \mathcal{B}_{n-1}(-(n-r)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}), \dots$$
$$\dots, \mathcal{B}_r(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}), \operatorname{im}(\Phi^-_r)^* \rangle. \tag{8.23}$$

Hence it suffices to prove $b^!_{s+1}(\pi^{\vee,-}_{s+1})^*D$ lies in the right orthogonal to the right side.

First we claim

$$(\pi^{\vee,-}_{s+1})^*D \in \langle \mathcal{B}_{n-1}(-(n-r+1)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}), \dots$$
$$\dots, \mathcal{B}_{r-1}(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}), \operatorname{im}(\Phi^+_{r-1})^* \rangle^\perp. \tag{8.24}$$

If $(\pi^{\vee,-}_{s+1})^{*L}$ denotes the left adjoint of $(\pi^{\vee,-}_{s+1})^*$, then by Lemma 8.16 we have

$$(\pi^{\vee,-}_{s+1})^{*L} \circ (\Phi^+_{r-1})^* \simeq \Phi^*_{r-1} \circ (\pi^+_{r-1})^{*L}.$$

Now the claim (8.24) follows by adjunction and our assumption that D lies in (8.22).

It follows from (8.24) that $(\pi^{\vee,-}_{s+1})^*D$ lies in the right orthogonal to the images of the subcategory

$$(\langle \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes_S \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^\perp$$

under the first two terms of the triangle from Proposition 8.19(2), and hence $(\pi_{s+1}^{\vee,-})^* D$ lies in the right orthogonal to

$$(b_{s+1})_*(\Phi_r^-)^* \left((\langle \mathcal{A}_s(H), \dots, \mathcal{A}_{m-1}((m-s)H) \rangle \otimes_S \mathrm{Perf}(\mathbf{Fl}_{r-1,r}))^\perp \right).$$

It then follows from Lemma 8.26 that in fact $(\pi_{s+1}^{\vee,-})^* D$ lies in the right orthogonal to the entire image of the functor $(b_{s+1})_*(\Phi_r^-)^*$. Hence

$$b_{s+1}^!(\pi_{s+1}^{\vee,-})^* D \in (\mathrm{im}\,(\Phi_r^-)^*)^\perp. \tag{8.25}$$

Next note that by Lemma 8.11(2) there is an exact sequence of sheaves on $\mathbf{L}_{s+1}^{\vee,-}$

$$0 \to (\mathcal{U}_r/\mathcal{U}_{r-1})(-H') \to \mathcal{O}_{\mathbf{L}_{s+1}^{\vee,-}} \to (b_{s+1})_* \mathcal{O}_{\mathbf{L}_s^{\vee,+}} \to 0.$$

This implies that the image of the functor $(b_{s+1})_* \colon \mathbf{L}_s^{\vee,+}(\mathcal{D}) \to \mathbf{L}_{s+1}^{\vee,-}(\mathcal{D})$ applied to

$$\left\langle \mathcal{B}_{n-1}(-(n-r)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}), \dots, \mathcal{B}_r(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}) \right\rangle$$

is contained in

$$\left\langle \mathcal{B}_{n-1}(-(n-r+1)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}), \dots, \mathcal{B}_{r-1}(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}) \right\rangle.$$

By (8.24) it thus follows that $b_{s+1}^!(\pi_{s+1}^{\vee,-})^* D$ lies in

$$\left\langle \mathcal{B}_{n-1}(-(n-r)H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}), \dots \dots, \mathcal{B}_r(-H') \otimes_{\mathrm{Perf}(S)} \mathrm{Perf}(\mathbf{Fl}_{r-1,r}) \right\rangle^\perp.$$

Together with (8.25), this shows $b_{s+1}^!(\pi_{s+1}^{\vee,-})^* D$ lies in the right orthogonal to (8.23), as required. □

The following gives the base case for the induction in the proof of generation of $\mathbf{L}_r(\mathcal{C})$.

Lemma 8.30. *We have* $\mathrm{im}\,\Phi_1 = \mathbf{L}_1(\mathcal{C})$.

Proof. By definition $\mathbf{L}_1 = \mathbf{P}$ and $p_1 \colon \mathbf{L}_1 \to \mathbf{P}$ is the identity morphism. In particular, Φ_1 coincides with the functor $p_{1*} \circ \Phi_1$, which we will analyze using Lemma 8.20. Note that $q_{N-1} \colon \mathbf{L}_{N-1}^\vee \to \mathbf{P}^\vee$ is isomorphic to the projective bundle $\mathbf{P}(\mathcal{S}) \to \mathbf{P}^\vee$, where $\mathcal{S}$ is the vector bundle on $\mathbf{P}^\vee$ defined by the short exact sequence

$$0 \to \mathcal{S} \to V \otimes \mathcal{O} \to \mathcal{O}(H') \to 0.$$

Under this identification, the bundle $\mathcal{W}_{N-1}/\mathcal{O}(-H')$ on $\mathbf{L}_{N-1}^\vee$ corresponds to $\Omega_{\mathbf{P}(\mathcal{S})/\mathbf{P}^\vee}(H)$. By a computation on the projective bundle $\mathbf{P}(\mathcal{S}) \to \mathbf{P}^\vee$, we thus obtain for $0 \le i, t \le N-2$ an isomorphism

$$(q_{N-1})_*(\wedge^i(\mathcal{W}_{N-1}/\mathcal{O}(-H'))(-tH)) \simeq \begin{cases} \mathcal{O}_{\mathbf{P}^\vee}[-i] & \text{if } i = t, \\ 0 & \text{else.} \end{cases}$$

It follows that for $0 \le i \le N-2$ the functor $\Phi_{\mathcal{K}_{1,i}}$ from Lemma 8.20 satisfies

$$\Phi_{\mathcal{K}_{1,i}} \circ (- \otimes \mathcal{O}(-tH)) \circ q_{N-1}^* \simeq \begin{cases} [-i] \circ (- \otimes \mathcal{O}(-iH)) \circ p_* \circ \Phi_{\mathcal{E}} & \text{if } i = t, \\ 0 & \text{else,} \end{cases}$$

and hence

$$\Phi_1 \circ (- \otimes \mathcal{O}(-iH)) \circ q_{N-1}^* \simeq (- \otimes \mathcal{O}(-iH)) \circ p_* \circ \Phi_{\mathcal{E}}.$$

Since $\Phi_{\mathcal{E}} = \gamma \circ \phi$ it follows from Setup 8.1(3c) that $\mathrm{im}\,\Phi_1$ contains the categories $\mathcal{A}_0(-iH)$ for $0 \le i \le N-2$. The stable subcategory of $\mathcal{C}$ generated by these categories is all of $\mathcal{C}$

by the assumption that $\mathcal{C}$ is a nondegenerate Lefschetz category. But $\operatorname{im}\Phi_1 \subset \mathcal{C}$ is a stable subcategory since Φ_1 is left splitting, so we conclude $\operatorname{im}\Phi_1 = \mathcal{C}$. □

Proposition 8.31. *We have*

$$\mathbf{L}_r(\mathcal{C}) = \left\langle \operatorname{im}\Phi_r, \mathcal{A}_s(H) \otimes_{\operatorname{Perf}(S)} \operatorname{Perf}(\mathbf{G}_r), \dots, \mathcal{A}_{m-1}((m-s)H) \otimes_{\operatorname{Perf}(S)} \operatorname{Perf}(\mathbf{G}_r) \right\rangle .$$

Proof. Analogous to Proposition 8.29 and left to the reader. □

References

1. Benjamin Antieau and David Gepner, *Brauer groups and étale cohomology in derived algebraic geometry*, Geom. Topol. **18** (2014), no. 2, 1149–1244.
2. Matthew Ballard, Dragos Deliu, David Favero, M. Umut Isik, and Ludmil Katzarkov, *Homological projective duality via variation of geometric invariant theory quotients*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 4, 1127–1158.
3. David Ben-Zvi, John Francis, and David Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc. **23** (2010), no. 4, 909–966.
4. David Ben-Zvi, David Nadler, and Anatoly Preygel, *Integral transforms for coherent sheaves*, arXiv:1312.7164, to appear in J. Eur. Math. Soc. (JEMS) (2013).
5. Alexei Bondal, *Representations of associative algebras and coherent sheaves*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 25–44.
6. Alexei Bondal and Mikhail Kapranov, *Representable functors, Serre functors, and mutations*, Mathematics of the USSR-Izvestiya **35** (1990), no. 3, 519.
7. Dennis Gaitsgory and Nick Rozenblyum, *A study in derived algebraic geometry*, available at `http://www.math.harvard.edu/~gaitsgde/GL/`, 2016.
8. Qingyuan Jiang, Naichung Conan Leung, and Ying Xie, *Categorical Plücker Formula and Homological Projective Duality*, arXiv:1704.01050 (2017).
9. Alexander Kuznetsov, *Homological projective duality*, Publ. Math. Inst. Hautes Études Sci. (2007), no. 105, 157–220.
10. ______, *Homological projective duality*, habilitation thesis (in Russian), unpublished (2008).
11. ______, *Lefschetz decompositions and categorical resolutions of singularities*, Selecta Math. (N.S.) **13** (2008), no. 4, 661–696.
12. ______, *Semiorthogonal decompositions in algebraic geometry*, Proceedings of the International Congress of Mathematicians, Vol. II (Seoul, 2014), 2014, pp. 635–660.
13. Alexander Kuznetsov and Valery Lunts, *Categorical resolutions of irrational singularities*, Int. Math. Res. Not. IMRN (2015), no. 13, 4536–4625.
14. Alexander Kuznetsov and Alexander Perry, *Categorical joins*, in preparation.
15. Joseph Lipman and Amnon Neeman, *Quasi-perfect scheme-maps and boundedness of the twisted inverse image functor*, Illinois J. Math. **51** (2007), no. 1, 209–236.
16. Valery Lunts and Olaf Schnürer, *New enhancements of derived categories of coherent sheaves and applications*, J. Algebra **446** (2016), 203–274.
17. Jacob Lurie, *Higher algebra*, available at `http://www.math.harvard.edu/~lurie/`.
18. ______, *Spectral algebraic geometry*, available at `http://www.math.harvard.edu/~lurie/`.
19. ______, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.
20. Amnon Neeman, *Strong generators in $D^{perf}(X)$ and $D^b_{coh}(X)$*, arXiv:1703.04484 (2017).
21. Dmitri Orlov, *Smooth and proper noncommutative schemes and gluing of DG categories*, Adv. Math. **302** (2016), 59–105.
22. Jørgen Rennemo, *The fundamental theorem of homological projective duality via variation of GIT stability*, arXiv:1705.01437 (2017).
23. The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, 2017.
24. Richard Thomas, *Notes on HPD*, arXiv:1512.08985, to appear in Proceedings of the 2015 AMS Summer Institute, Salt Lake City (2017).
25. Bertrand Toën and Michel Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387–444.

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