Problems

1) Let $V$ be a vector space over $\mathbb{C}$ with a positive definite Hermitian inner product. Prove that if $v_1, v_2, \ldots, v_r$ are mutually orthogonal vectors in $V$ then they are linearly independent.

2) Let $V$ be a representation of $G$ with character $\chi$.
   (a) Prove that if $(\chi, \chi) = 1$ then $V$ is irreducible.
   (b) Prove that if $(\chi, \chi) = 2$ then $V$ decomposes into two distinct irreducible representations.

3) Let $V$ be vector space of homogeneous polynomials of degree 4 in variables $x, y, z$. Let $S_3$ act on $V$ by permuting the variables.
   (a) Find character $\chi_V$ of the representation.
   (b) How does $V$ decompose into irreducible representations?

4) Let group $G$ act on a set $X$.
   (a) Prove that the character of the associated permutation representation is given by $\chi(g) = |X^g|$. ($X^g$ denotes the set of elements of fixed by $g$).
   (b) Suppose the action of $G$ is transitive on $X$. Prove that
   $$\frac{1}{|G|} \sum_{g \in G} \chi(g) = 1$$
   More generally,
   $$\frac{1}{|G|} \sum_{g \in G} \chi(g) = |X/G|$$
   where $X/G$ denotes the set of orbits.
   (c) We say that the action of $G$ on $X$ is doubly transitive if for all $a, b, \alpha, \beta \in X$ such that $a \neq b$ and $\alpha \neq \beta$ there exists $g \in G$ such that $g(a) = \alpha$ and $g(b) = \beta$. Prove that if the action of $G$ is doubly transitive we have
   $$\frac{1}{|G|} \sum_{g \in G} \chi(g)^2 = 2$$
   (Hint: Consider the diagonal action of $G$ on $X \times X$).
   (d) Use this to prove that the standard representation of $S_n$ is irreducible.
5) Let \( \rho_V : G \to GL(V) \) be a representation of a group \( G \). Prove that \( \det \circ \rho_V \) is a one dimensional representation of \( G \).

6) Let \( \rho \) be a one dimensional representation of \( G \) with character \( \chi \). Suppose \( \rho_V : G \to GL(V) \) is any other representation with character \( \chi_V \). Show that

\[
\rho \otimes \rho_V = \rho \cdot \rho_V
\]
is another representation of \( G \) with character \( \chi \cdot \chi_V \). Show that \( \rho_V \) is irreducible iff \( \rho_V \otimes \rho \) is irreducible.

7) Prove that every group \( G \) has a faithful representation. i.e. There exists \( \rho_V : G \to GL(V) \) which is injective.

8) Let \( G \) be a finite group. Prove that all irreducible representations of \( G \) are one dimensional if and only if \( G \) is abelian.

9) Let \( \mathbb{C}\{S_3\} \) denote the group ring. Let \( p = \frac{1}{6} \sum_{g \in S_3} \text{sign}(g)g \). Compute \( p^2 \).

10) Let \( V_i \) for \( 1 \leq i \leq n \) be the set of irreducible characters of \( G \). Let \( d_i \) be the dimension of \( V_i \). Prove that the regular representation \( V_{\text{reg}} \) decomposes as \( V_{\text{reg}} = \sum_{i=1}^{n} d_i \cdot V_i \).

Conclude that \( d_1^2 + d_2^2 + \ldots + d_n^2 = |G| \).