SUPPLEMENTARY MATERIAL
to "Stochastic growth in time dependent environments"

We give here the details of the calculations and their applications, as described in the main text of the Letter.

CONTENTS

I SPATIALLY LINEAR CHANGE OF VARIABLES ON THE KPZ EQUATION

1) General time-inhomogeneous KPZ equation

Let us consider the general time inhomogeneous KPZ equation

$$\partial_t h = \nu(t)\partial_x^2 h + \frac{\lambda(t)}{2}(\partial_x h)^2 + V(x,t) + \sqrt{2c(t)}\,\xi(x,t) \tag{38}$$

in presence of an external potential $V(x,t)$. The change of variable method considered here works only for the case $\lambda(t) \propto \nu(t)$, which we will assume from now on (see [60] for a study of the general $\lambda(t)$). Since an additional rescaling by space-time independent coefficients $x, t, h \to x/x^*, t/t^*, h/h^*$ is always possible, akin to a choice of units, we assume from now $\lambda(t) = 2\nu(t)$. If this condition is violated, a new term $\propto h$ appears in the transformed equation, not studied here.

To treat also cases including linear potentials, we consider the following change of variable from $h(x,t)$ to $H(y,\tau)$

$$h(x,t) = H(c(t)x + y_0(t), \tau(t)) + \frac{c'(t)}{4c(t)\nu(t)} x^2 + \frac{y_0'(t)}{2c(t)\nu(t)} x + \frac{1}{2} \log \frac{c(t)}{c(0)} + J(t) \quad , \quad \tau(t) = \int_0^t ds\, c(s)^2 \nu(s) \tag{39}$$

Then we find that if $h(x,t)$ satisfies the time inhomogeneous equation (38) with white noise, then $H(y,\tau)$ satisfies the time-homogeneous equation

$$\partial_\tau H = \partial_y^2 H + (\partial_y H)^2 + W(y,\tau) + \sqrt{2}\, \tilde{\xi}(y,\tau) \tag{40}$$

in the external potential

$$W(y,\tau) = \frac{V(x = \frac{y - y_0(t)}{c(t)}, t)}{c(t)^2 \nu(t)} - A(\tau)\frac{y^2}{2} - B(\tau) y + W_0(\tau) \tag{41}$$

$$A(\tau(t)) = \frac{1}{2c(t)^6 \nu(t)^2} \left(c(t)c''(t) - 2c'(t)^2 - c(t)c'(t)\frac{\nu'(t)}{\nu(t)} \right) = \frac{-1}{2c(t)^3 \nu(t)} \frac{d}{dt} \left(\frac{1}{\nu(t)} \frac{d}{dt} \frac{1}{c(t)} \right) \tag{42}$$

$$B(\tau(t)) = \frac{1}{2c(t)^3 \nu(t)} \frac{d}{dt} \left(\frac{1}{\nu(t)} \frac{d}{dt} \frac{y_0(t)}{c(t)} \right) \tag{43}$$

The coefficients in (39) have been determined so that no term linear in $\partial_y H$ appears in the equation (40). In (39) the last term reads

$$J(t) = \int_0^t ds \left[\frac{1}{4c(s)} \frac{d}{ds} \left(\frac{1}{\nu(s)} \frac{d}{ds} \frac{y_0(s)^2}{c(s)} \right) - \frac{y_0'(s)^2}{4\nu(s)c(s)^2} - c(s)^2 \nu(s) W_0(\tau(s)) \right] \tag{44}$$

Note that the function W_0 can be chosen arbitrarily, for convenience.

A case of particular interest is when the initial equation (38) contains no external potential, i.e. $V(x,t) = 0$. Then, in the subcase such that

$$c(t) = \frac{c(0)}{1 + \int_0^t \nu(t')dt'} \tag{45}$$

the transform (39) with the choice $y_0(t) = 0$, $W_0(\tau) = 0$, maps the problem to the standard KPZ equation with $W(y,\tau) = 0$.

In the case $\nu(t) = 1$ and $V(x,t) = a(t)\frac{x^2}{2}$, choosing $y_0(t) = 0$ and $W_0(\tau) = 0$, one recovers the formula (4), (5) and (8) given in the text.

KPZ equation in presence of a linear potential. Consider now the usual KPZ equation in presence of a linear time-dependent potential $V(x,t) = b(t)x$

$$\partial_t h(x,t) = \partial_x^2 h(x,t) + (\partial_x h(x,t))^2 + b(t)x + \sqrt{2}\xi(x,t) \tag{46}$$

With $\nu(t) = 1$, $c(t) = 1$, choosing

$$y_0(t) = 2\int_0^t ds \int_0^s du\, b(u) \tag{47}$$

and $W_0(\tau(t)) = b(t)y_0(t)$, it is mapped under the shift

$$h(x,t) = H(x + y_0(t), t) + \frac{y_0'(t)}{2} x + \frac{1}{4}\int_0^t ds\, y_0'(s)^2 \tag{48}$$

to the standard KPZ equation for $H(y,\tau)$ without external potential $W(y,\tau) = 0$ and the same initial condition $H(y,\tau = 0) = h(y, t = 0)$ (since $y_0(0) = y_0'(0) = 0$).

2) Mapping and solution in the absence of noise: time-dependent harmonic oscillator and blow up

In the absence of noise, the equation $\partial_t h = \partial_x^2 h + (\partial_x^2 h)^2 + a(t)\frac{x^2}{2}$ (related to the quantum time-dependent harmonic oscillator – in imaginary time) can be solved by the rescaling method [14] as $h(x,t) = H(\frac{x}{L(t)}, \tau(t)) - \frac{L'(t)}{4L(t)}x^2 - \frac{1}{2}\log(L(t)/L(0))$, where $\partial_\tau H = \partial_y^2 H + (\partial_y^2 H)^2 - \frac{A}{2}y^2$, and $L(t)$ satisfies the Ermakov equation [30–33], i.e. $L''(t) + 2a(t)L(t) + \frac{2A}{L(t)^3} = 0$ and $\tau'(t) = 1/L(t)^2$. Here A is an arbitrary constant. Since it is a second order differential equation, for any given $a(t)$ there is in addition a two-parameter family of solutions $L(t)$ indexed by $(L(0), L'(0))$. The solution for $h(x,t)$ should be invariant under the possible choices of these three parameters, provided the initial condition is modified correspondingly.

For instance, choosing $A = 0$ and using the solution of the standard heat equation, one obtains $h(x,t)$ as

$$e^{h(x,t)} = \sqrt{\frac{L(0)}{L(t)}}\frac{e^{-\frac{L'(t)}{4L(t)}x^2}}{\sqrt{4\pi\int_0^t \frac{du}{L(u)^2}}}\int_{-\infty}^{+\infty} dy\, e^{-\frac{\left(\frac{x}{L(t)}-y\right)^2}{4\int_0^t \frac{du}{L(u)^2}} + h(L(0)y,0) + \frac{L'(0)L(0)}{4}y^2} \tag{49}$$

where $L(t)$ satisfies $L''(t) = -2a(t)L(t)$. Let us check that (49) is indeed independent of the choice $(L(0), L'(0))$, which is not immediately obvious. First note that the r.h.s. of (49) is invariant by the rescaling $L(t) \to \lambda L(t)$, $y \to y/\lambda$, hence one can always choose $L(0) = 1$, which we do from now on (we will not consider the case $L(0) = 0$). To see that (49) does not depend on the choice of $L'(0)$, let us consider the Wronskian of two solutions $L_1(t)$, $L_2(t)$ (with $L_1(0) = L_2(0) = 1$) for the same $a(t)$. One has $L_1'(t)L_2(t) - L_2'(t)L_1(t) = L_1'(0) - L_2'(0)$. One can solve this equation for $L_2(t)$ as a function of $L_1(t)$ and one obtains

$$L_2(t) = L_1(t)(1 + (L_2'(0) - L_1'(0))\int_0^t \frac{du}{L_1(u)^2}) \tag{50}$$

Solving instead for $L_1(t)$ as a function of $L_2(t)$ leads to the same equation with L_1 and L_2 exchanged. Combining both equations we obtain

$$L_1(t)\int_0^t \frac{du}{L_1(u)^2} = L_2(t)\int_0^t \frac{du}{L_2(u)^2} = \frac{L_2(t) - L_1(t)}{L_2'(0) - L_1'(0)} \tag{51}$$

Hence the pre-exponential factor in (49), as well as the term proportional to xy in the exponential, take the same value for both solutions. Now, (51) implies

$$L_2'(0) - L_1'(0) = \frac{L_2(t) - L_1(t)}{L_1(t)\int_0^t \frac{du}{L_1(u)^2}} = \frac{L_2(t)}{L_1(t)\int_0^t \frac{du}{L_1(u)^2}} - \frac{1}{\int_0^t \frac{du}{L_1(u)^2}} = \frac{1}{\int_0^t \frac{du}{L_2(u)^2}} - \frac{1}{\int_0^t \frac{du}{L_1(u)^2}} \tag{52}$$

hence

$$L_2'(0) - \frac{1}{\int_0^t \frac{du}{L_2(u)^2}} = L_1'(0) - \frac{1}{\int_0^t \frac{du}{L_1(u)^2}} \tag{53}$$

which is precisely the coefficient of $y^2/4$ in the exponential in (49). Finally, dividing the Wronskian by $L_1(t)L_2(t)$ one obtains, using (51)

$$\frac{L_1'(t)}{L_1(t)} - \frac{L_2'(t)}{L_2(t)} = \frac{L_1'(0) - L_2'(0)}{L_1(t)L_2(t)} = \frac{L_1(t) - L_2(t)}{L_1(t)L_2(t)\int_0^t \frac{du}{L_1(u)^2}} = \frac{1}{L_2(t)L_1(t)\int_0^t \frac{du}{L_1(u)^2}} - \frac{1}{L_1(t)^2\int_0^t \frac{du}{L_1(u)^2}} \tag{54}$$

Using (51) once more we finally obtain

$$\frac{L_1'(t)}{L_1(t)} + \frac{1}{L_1(t)^2\int_0^t \frac{du}{L_1(u)^2}} = \frac{L_2'(t)}{L_2(t)} + \frac{1}{L_2(t)^2\int_0^t \frac{du}{L_2(u)^2}} \tag{55}$$

which is the coefficient of $-x^2/4$ in the exponential in (49). Hence all combinations of $L(t)$ appearing in (49) are indeed independent of the choice of $L'(0)$, via some elementary identities.

Blow-up. For each choice of $a(t)$ there is a class of initial conditions which lead to blow-up. Blow-up occurs when the integral over y in (49) diverges. For simplicity let us consider IC of the type $h(x, t = 0) = -Bx^2$, $B = 0$ being the flat IC. For each $a(t)$ there is a B_c such that blow-up occurs for $B < B_c$ and no blow-up for $B \geq B_c$. For the standard noiseless KPZ equation $a(t) = 0$, a blow-up occurs when $B < 0$ and no blow up for $B \geq 0$ (as seen choosing $L(t) = 1$ in (49)). Hence $B_c = 0$ in that case. More generally the condition for absence of blow up is that the coefficient of $y^2/4$ in (49) remains negative, i.e. $L'(0) - \frac{1}{\int_0^t \frac{du}{L(u)^2}} - 4B < 0$ for all t. It this quantity changes sign for the first time at some t^*, there is a blow-up at $t = t^*$ which satisfies

$$\tau(t^*) = \int_0^{t^*} \frac{du}{L(u)^2} = \frac{1}{L'(0) - B} \tag{56}$$

leading to a blow-up of the solution $h(x,t)$ towards $+\infty$ at $t = t^*$. From (53) we see that t^* is independent of the choice of $L'(0)$, as expected, since the blow-up is an intrinsic property which depends only on the choice of $a(t)$ and B. The critical value is thus given $4B_c = \max_t [L'(0) - \frac{1}{\int_0^t \frac{du}{L(u)^2}}]$.

Let us give an example. Consider $L_1(t) = \sqrt{1+t}$ and flat IC $B = 0$. It corresponds to $a(t) = -\frac{L_1''(t)}{2L_1(t)} = \frac{1}{8(1+t)^2}$. The coefficient of $y^2/4$ in the exponential in (49) is given by (53) and equal to $\frac{1}{2} - (1/\log(1+t))$. There is thus a blow-up of $h(x,t)$ at time $t^* = e^2 - 1 = 6.389..$ where this coefficient vanishes. One could naively argue that the blow-up arises from choosing a function $L(t)$ such that $L'(0) > 0$, which gives a positive contribution to the coefficient of $y^2/4$. Choosing $L'(0) = 0$ would naively seem as a way to push the root for t^* in (56) to infinity. This is not the case however. Indeed let us make the equivalent choice $L_2(t)$ such that $L_2'(0) = 0$. One finds from (50), that it is given by $L_2(t) = \sqrt{1+t}(1 - \frac{1}{2}\log(1+t))$. The associated $a(t) = \frac{1}{8(1+t)^2}$ is the same as for $L_1(t)$, and the blowup still occurs at the same t^* but it is because $L_2(t)$ changes sign at $t = t^*$, and the integral on the l.h.s. of (56) diverges at $t = t^*$.

It is interesting to ask which choices of $a(t)$ and B lead to blow-up and which do not. The equation $L''(t) + 2a(t)L(t) = 0$ can be interpreted as a Schrodinger eigenvalue equation for a wave-function $\psi(t) \equiv L(t)$ in one space dimension, where the space variable is t, i.e. $-\psi''(t) + V(t)\psi(t) = E\psi(t)$. One can choose the potential $V(t) = -2a(t)$ and $L(t)$ corresponds to any zero energy solution at $E = 0$. Let us consider the flat IC, $B = 0$, in which case it is natural to choose $L'(0) = 0$ (i.e. $\psi'(0) = 0$) as the boundary condition for the Schrodinger operator on $t \in [0, +\infty[$. Generally, let us denote E_0 the lowest energy of the spectrum of the quantum potential $V(t)$ with this boundary condition. If $E_0 < 0$ one expects that the solutions for $L(t)$ at zero energy $E = 0$ will oscillate and change sign, leading to a blow-up (see the example of the previous paragraph). At the contrary, if $E_0 > 0$ one expects exponentially decaying or exploding solutions for $L(t)$, with no sign change, hence no blow-up.

Conversely one can pick any explicit function $L(t)$ and obtain a function $a(t)$ for which the solution $h(x,t)$ can be written explicitely. A case of interest in this paper is $L(t) = (1+t)^\alpha$ in which case $a(t) = \frac{\alpha(1-\alpha)}{2(1+t)^2}$. For flat IC, $B = 0$, there is a blowup for $\alpha < 1$ at time t^* given by $t^* = (\frac{1-\alpha}{\alpha})^{\frac{1}{1-2\alpha}} - 1$. For $\alpha \geq 1$ there is no blow up. This can be understood since $a(t) > 0$ for $\alpha < 1$, while $a(t) < 0$ for $\alpha > 1$, hence the potential $V(t) = -2a(t) < 0$ for $\alpha < 1$ with $E_0 < 0$ and positive for $\alpha > 1$ with $E_0 > 0$.

It is interesting to consider the case $L(t) = (1+t^2)^{\alpha/2}$, which has the same large time behavior as the previous example, but has $L'(0) = 0$. It corresponds to $a(t) = -\frac{\alpha\left(1-(1-\alpha)t^2\right)}{2(t^2+1)^2}$. It is easy to see from the condition (56) that for flat IC $B = 0$ there is never a blow-up for any $\alpha > 0$. The function $a(t)$ starts negative at small times with $a(0) = -\alpha/2$, hence $V(0) = \alpha > 0$, which seems to be sufficient to avoid the blow up (and leads to $E_0 > 0$).

Connection to the noisy KPZ equation. The main point of the present paper is to note that the above mapping for the noiseless KPZ equation, which we can denote $(a(t), 0) \to (-A, 0)$, extends in presence of white noise, to the mapping $(a(t), c(t)) \to (-A, 1)$, if one chooses $c(t) = 1/L(t)$ (where $L(t)$ is associated to $a(t)$ as described above). One can check that indeed Eqs. (8) and (7) are equivalent in that sense to Ermakov's equation. Although the noise generates fluctuations in $h(x,t)$, the question of the blowup for a given $(a(t), c(t))$ model can be discussed already in the absence of noise, as we have shown (see also below).

It is interesting to note that the solution of the noiseless KPZ problem, i.e. $(a(t), 0)$, with the droplet initial condition $e^{h(y,0)} \to \delta(y)$ is a simple gaussian

$$e^{h(x,t)} = \frac{\sqrt{c(t)}}{\sqrt{4\pi c(0)\tau(t)}} e^{-\frac{x^2}{4}\left[\frac{c(t)^2}{\tau(t)} - \frac{c'(t)}{c(t)}\right]}\Big|_{c(t)=1/L(t)} \tag{57}$$

This solution will provide the "mean profile" for the droplet IC in presence of noise. Indeed consider Eq. (4) of the text for the problem $(a(t) = a_c(t), c(t))$. Since $A(\tau) = 0$ in (6), we know that one has equivalence in one-point PDF law $H(y,\tau) \equiv H(0,\tau) - \frac{y^2}{4\tau}$. This implies the equality in law

$$h(x,t) \equiv h(0,t) - \frac{x^2}{4}\left[\frac{c(t)^2}{\tau(t)} - \frac{c'(t)}{c(t)}\right] \tag{58}$$

This can be obtained, more generally, by studying the symmetries of the inhomogeneous KPZ equation as we now discuss

3) Statistical tilt symmetry (STS), Galilean invariance

Consider the case $\nu(t) = 1$, i.e. the model $(a(t), b(t), c(t))$

$$\partial_t h(x,t) = \partial_x^2 h(x,t) + (\partial_x h(x,t))^2 + \frac{1}{2}a(t)x^2 + b(t)x + \sqrt{2c(t)}\xi(x,t) \tag{59}$$

Let us define $\tilde{h}(y,t)$ via the relation $h(x,t) = \tilde{h}(x+f(t),t) + \frac{f'(t)}{2}x + F(t)$. If we choose $f(t)$ and $F'(t)$ such that

$$f''(t) + 2a(t)f(t) = 0 \quad , \quad F'(t) = \frac{1}{4}f'(t)^2 - \frac{1}{2}a(t)f(t)^2 - b(t)f(t) \tag{60}$$

then $\tilde{h}(y,t)$ satisfies exactly the same equation (59) with $x \to y$, and $\sqrt{2c(t)}\xi(x,t) \to \sqrt{2c(t)}\tilde{\xi}(y,t)$, with $\tilde{\xi}$ an equivalent white noise in y. If we further choose $f(0) = F(0) = 0$, and the droplet IC for h, i.e. $e^{h(x,0)} = \delta(x)$, this also corresponds also to droplet IC for $\tilde{h}$, $e^{\tilde{h}(y,0)} = \delta(y)$. Hence the two fields $h(x,t)$ and $\tilde{h}(x,t)$ have the same statistics. Note that this is independent of the choice of $c(t)$.

In the case $a(t) = b(t) = 0$, $f(t) = vt$, $F(t) = \frac{v^2}{4}t$, and this is the usual Galilean invariance of the KPZ equation (and its derivative, Burger's equation). It states that for droplet IC

$$h(x,t) \equiv h(x+vt,t) + \frac{v}{2}x + \frac{v^2}{4} \tag{61}$$

where $\equiv$ means the same statistics. For a fixed x, t, choosing $v = -x/t$ we obtain the celebrated identity of the one point PDF's, which we note here extends to any choice of $c(t)$

$$h(x,t) \equiv h(0,t) - \frac{x^2}{4t} \tag{62}$$

Let us take $b(t) = 0$. Then from (60) one finds $F(t) = \frac{1}{4}f(t)f'(t)$. We can write the solution $f(t) = vf_0(t)$ where $f_0'(0) = 1$. Then we can choose $v = -x/f_0(t)$ and we obtain the general STS relation (equivalence in law)

$$h(x,t) \equiv h(x+vf_0(t),t) + v\frac{f_0'(t)}{2}x + \frac{v^2}{4}f_0(t)f_0'(t) = h(0,t) - \frac{x^2}{4}\frac{f_0'(t)}{f_0(t)} \tag{63}$$

where we recall that $f_0(t)$ is the unique solution of $f_0''(t) + 2a(t)f_0(t) = 0$ with $f_0(0) = 0$ and $f_0'(0) = 1$. Again we stress that this is valid for any $(a(t), b(t) = 0, c(t))$.

Suppose now that we choose $a(t) = a_c(t) := -\frac{1}{2}c(t)(\frac{1}{c(t)})''$. One can write the Wronskian of the two solutions $f_0(t)$ and $1/c(t)$ (using $c(0) = 1$), as $\frac{f_0'(t)}{c(t)} - f_0(t)(\frac{1}{c(t)})' = 1$. This leads to $f_0(t) = \tau(t)/c(t)$ with $\tau(t) = \int_0^t c(t)^2$. Then (63) leads to

$$h(x,t) \equiv h(0,t) - \frac{x^2}{4}\left[\frac{c(t)^2}{\tau(t)} - \frac{c'(t)}{c(t)}\right] \tag{64}$$

as anticipated in (58).

One can consider other choices, e.g. $a(t) = a$. One has $f_0(t) = \frac{\sin\sqrt{2a}t}{\sqrt{2a}}$ for $a > 0$ and $f_0(t) = \frac{\sinh\sqrt{2|a|}t}{\sqrt{2|a|}}$ for $a < 0$, hence

$$h(x,t) \equiv h(0,t) - \frac{x^2}{4}\sqrt{2a}\cot(\sqrt{2a}t) \quad , \quad a > 0 \tag{65}$$

$$\equiv h(0,t) - \frac{x^2}{4}\sqrt{2|a|}\coth(\sqrt{2|a|}t) \quad , \quad a < 0 \tag{66}$$

which will be used below.

4) IC classes: known results for the standard KPZ equation

For later use below, we recall here some known results for the standard KPZ equation, satisfied by $H(y,\tau)$ i.e. (40) with $W(y,\tau)=0$. In particular about the one-time statistics of the KPZ field in the limit of large τ. From the Cole-Hopf mapping one has

$$\hat{Z}(y,\tau) = e^{H(y,\tau)} = \int dz \hat{Z}(y,\tau|z,0) e^{H(z,0)} \tag{67}$$

where $\hat{Z}(y,\tau|z,0)$ is the partition function of the continuous directed polymer from space-time point $z,0$ to y,τ. For large $\tau \gg 1$ the integral on the r.h.s. is dominated by its maximum. The scaled droplet solution $\tau^{-1/3}\log \hat{Z}(y,\tau|0,0)$ is conjectured to converge, in rescaled coordinates $\hat{y} = \frac{y}{2\tau^{2/3}}$ to the so-called Airy$_2$ process minus a parabola, $\mathcal{A}_2(\hat{y}) - \hat{y}^2$ [17, 18]. The height field is then determined [61], as a process in $\hat{y}$, by a variational problem

$$H(y,\tau) + \frac{\tau}{12} \simeq \tau^{1/3} \max_{\hat{z}} \left\{ \mathcal{A}_2(\hat{z}-\hat{y}) - (\hat{z}-\hat{y})^2 + \mathsf{H}_0(\hat{z}) \right\}. \tag{68}$$

Here

$$\mathsf{H}_0(\hat{z}) = \tau^{-1/3} H(2\hat{z}\tau^{2/3}, 0) \tag{69}$$

is the so-called rescaled IC (in (68) and (69) the limit of large $\tau \gg 1$ is understood). All IC which share the same H_0 lead to the same universal height PDF at large τ. The droplet IC class corresponds to $\mathsf{H}_0(\hat{z}) = -\infty$ for $\hat{z} \neq 0$ and $\mathsf{H}_0(0) = 1$, shared e.g. by any wedge, $H(z,0) = -w|z|$, of a large class of IC where $e^{H(x,0)}$ is localized in space. In that case the maximum in (68) is attained at $\hat{z}=0$, and the one point PDF of H is related to the one of $\mathcal{A}_2(0)$ which is the GUE-TW distribution. The flat IC corresponds to $\mathsf{H}_0(\hat{z}) = 0$ and includes a class of IC extended over the whole axis. It leads to the GOE TW one point distribution. Eq. (68) expresses the solution for arbitrary IC, and its one point distribution can be expressed in terms of a Fredholm determinant in terms of a kernel depending on H_0, in general quite complicated [19, 34].

Let us recall also that for small $\tau \ll 1$, the KPZ field has Gaussian statistics: this is the so-called Edwards-Wilkinson (EW) regime, with fluctuations growing as $\delta H \sim \tau^{1/4}$. The spatial correlation scale of the standard KPZ field, denoted here $y(\tau)$, changes from $y(\tau) \propto \tau^{1/2}$ for $\tau \ll 1$ to $y(\tau) = 2\tau^{2/3}$ at large $\tau \gg 1$.

5) Large time asymptotics for the time-inhomogeneous KPZ equation

In this section we provide more details to the study in the main text of the statistics of the height field (2) for various initial conditions. The questions are (i) for a given IC for h, what is the effective initial condition to use which determine χ (ii) what are the statistics of the full height profile $h(x,t)$ at large time.

We will center the discussion on the model $(a(t) = a_c(t), c(t))$ with noise variance $c(t) = \left(\frac{t_0}{t+t_0}\right)^\alpha$ and external potential $V(x,t) = a(t)\frac{x^2}{2}$ and $a(t) = a_c(t) = \frac{\alpha(1-\alpha)}{2}(t+t_0)^{-2}$, although we will consider a few other cases below (we always assume $c(0)=1$). The variations of $c(t)$ thus occur on a time scale t_0. There are two distinct cases to be studied. One is $t_0 = O(1)$ fixed and large $t \gg 1$, which leads to non-universal results for $\alpha > 1/2$. The other is $t_0 \gg 1$ large. In the latter case one can study the regime where both t and t_0 large with a fixed ratio, and the results are always universal and can be quantified more precisely. Under the transformation (4), (5),

$$h(x,t) = H(y,\tau(t)) - \frac{\alpha x^2}{4(t+t_0)} + \frac{\alpha}{2}\log\frac{t_0}{t+t_0}, \quad y = \left(\frac{t_0}{t+t_0}\right)^\alpha x, \quad \tau(t) = \frac{t_0}{1-2\alpha}\left(\left(1+\frac{t}{t_0}\right)^{1-2\alpha} - 1\right), \tag{70}$$

the inhomogeneous KPZ equation (2) is mapped to the standard KPZ equation for $H(y,\tau)$, i.e. (40) with $A(\tau)=0$, with the new time $\tau(t)$, for which we can use the results of Section I 4). In particular, the spatial correlation scale of the growth can be defined as $x(t) = \frac{y(\tau(t))}{c(t)}$, where $y(\tau)$ is the spatial correlation scale associated to the standard KPZ equation (40). We also recall the relation between the initial conditions, i.e. (70) at $t=0$

$$H(y,0) = h(y,0) - \frac{c'(0)}{4}y^2 = h(y,0) + \frac{\alpha}{4t_0}y^2 \tag{71}$$

Asymptotics for $t_0 = O(1)$. Let us first consider the case where $t_0 = O(1)$ is fixed. The new time $\tau(t)$ has a different behavior depending on whether $\alpha < 1/2$ or $\alpha > 1/2$. In the limit $t/t_0 \gg 1$, it diverges for $\alpha < 1/2$ as $\tau(t) \simeq \frac{t_0}{1-2\alpha}(\frac{t}{t_0})^{1-2\alpha}$, while it saturates to a finite value for $\alpha > 1/2$, as $\tau(t) \to \frac{t_0}{2\alpha-1}$.

- For $\alpha < 1/2$ we thus predict that the one point PDF of $h(x,t)$, e.g. at $x = 0$, behaves as $t \to +\infty$ as

$$h(0,t) \simeq -\frac{1}{12} c_1 t_0 \left(\frac{t}{t_0}\right)^{1-2\alpha} + (c_1 t_0)^{1/3} \left(\frac{t}{t_0}\right)^{\beta(\alpha)} \chi, \quad \beta(\alpha) = \frac{1-2\alpha}{3} \tag{72}$$

with $c_1 = \frac{1}{1-2\alpha}$, and where the random variable χ is the TW type distribution associated to the KPZ fixed point with the initial condition given by (71). Because of the positive quadratic part in (71) there is an important restriction on the class of IC for h which lead to (72). Consider the subset of IC such that $h(x, t = 0) \simeq -Bx^2$ for $x \gg 1$. Let us use the results of Section I 4). It is easy to see from (68), (71) and (69) that if $B > B_c = \frac{c'(0)}{4} = \frac{\alpha}{4t_0}$ then H_0 belongs to the droplet IC and χ is GUE-TW distributed. If $B = B_c$ and if at large x, $|h(x, t = 0) + B_c x^2| < |x|^r$ with $r < 1/2$, then (72) holds, with $\mathsf{H}_0(\hat{z}) = 0$ implying that χ is GOE-TW distributed. The case $B < B_c$ maps to the standard KPZ equation for H with a convex parabolic initial condition. It is well known that this leads to a finite-time blow-up, i.e. the solution for $H(y,\tau)$, hence also for $h(x,t)$, blows up to $+\infty$ at a finite time t^*, hence (72) does not hold. The existence of a blow up for IC with $B < B_c$ is in fact already a property of the equation without noise, i.e. of the model $(a(t), 0)$ studied in Section I 2). It is related to the special form of $a(t) = a_c(t)$, dictated by the choice of $c(t)$, here $c(t) = (1 + \frac{t}{t_0})^{-\alpha}$. For $\alpha < 1$, $a(t) < 0$ which, for a flat IC, $B = 0$, leads to a blow up, while for $\alpha \geq 1$, $a(t) > 0$ and there is no blow up. This is discussed in detail in Section I 2). In particular, choosing e.g. $c(t) = (1 + t^2)^{-\alpha/2}$, does not change the large time behavior, but leads to $a(t)$ which is negative at small time and avoids the blow up (in that case $B_c = 0$).

In the cases where (72) holds ($B \geq B_c$, no blow up) the spatial correlation scale behaves as $t \to +\infty$ as

$$x(t) = \frac{2\tau(t)^{2/3}}{c(t)} \simeq \frac{2t_0^{\frac{\alpha}{3}}}{(1-2\alpha)^{2/3}} t^{\zeta(\alpha)}, \quad \zeta(\alpha) = \frac{2-\alpha}{3} \quad , \quad \alpha < \frac{1}{2} \tag{73}$$

For $B > B_c$, one obtains from (68) the one time statistics of the field $h(x,t)$ in the large t limit as [62]

$$h(x,t) + \frac{\tau(t)}{12} \simeq \tau(t)^{1/3} \left[\mathcal{A}_2(\tilde{x}) - \tilde{x}^2\right] - \frac{\alpha x^2}{4t} = \tau(t)^{1/3} \mathcal{A}_2(\tilde{x}) - \frac{(1-\alpha)x^2}{4t} \tag{74}$$

$$= \frac{t_0^{\frac{2\alpha}{3}}}{(1-2\alpha)^{1/3}} t^{\frac{1-2\alpha}{3}} \left[\mathcal{A}_2(\tilde{x}) - \frac{1-\alpha}{1-2\alpha}\tilde{x}^2\right] \quad , \quad \tilde{x} = \frac{x}{x(t)} \tag{75}$$

i.e. GUE-TW for the one point PDF. Note however that the deterministic quadratic dependence in $\tilde{x}$ is different from the one for the standard KPZ equation (recovered for $\alpha = 0$). It will be of importance when analyzing the endpoint PDF of the directed polymer, see below. Finally, for $B = B_c$ one finds from (68) for $t \to +\infty$

$$h(x,t) + \frac{\tau(t)}{12} \simeq \tau(t)^{1/3} 2^{-2/3} \chi_1 - \frac{\alpha x^2}{4t} \tag{76}$$

where $\chi_1 = 2^{2/3} \max_z [\mathcal{A}_2(\hat{z}) - \hat{z}^2]$ is distributed according to GOE-TW. The spatial dependence of the field is trivial in that case.

- For $\alpha = 1/2$ one has $\tau(t) = t_0 \log(1 + \frac{t}{t_0})$ and $x(t) = \frac{2\tau(t)^{2/3}}{c(t)} \simeq 2t_0^{1/6} t^{1/2} [\log(t/t_0)]^{2/3}$ at large t. For $B > B_c = \frac{1}{8t_0}$ the first equation in (75) is still valid, but not the second, because the quadratic terms are now, at large t, equivalent to $-\frac{x^2}{4t\log(t/t_0)} - \frac{x^2}{8t}$. The first one is now negligible compared to the second. Hence we obtain

$$h(x,t) + \frac{\tau(t)}{12} \simeq t_0^{1/3} [\log(t/t_0)]^{1/3} \mathcal{A}_2(\tilde{x}) - \frac{x^2}{8t} \quad , \quad \tilde{x} = \frac{x}{x(t)} \tag{77}$$

- For $\alpha > 1/2$ the fluctuations of the field $h(x,t)$ saturate at $t \to +\infty$, and the limit statistics, including the one-point PDF, is related via (70) to the one of the standard KPZ equation at finite time $\tau(+\infty) = \frac{t_0}{2\alpha-1}$, with the modified initial condition (71). These asymptotic distributions are specific to the standard KPZ equation, and are not universal across the KPZ class. They are not known analytically except for the one-point PDF and only for a few special IC. For $h(x,t)$ these are (i) the droplet IC, which maps to the droplet IC for H, for which one can use the finite time results of [2–5] (ii) $h(x,t=0)=e^{-\frac{\alpha}{4t_0}y^2}$, which corresponds to the flat IC for H, for which one can use the finite time results of [35]. More generally, one can define a correlation scale, as above, which grows at large time t as

$$x(t)=\frac{y(\tau(+\infty))}{c(t)}\propto t^{\alpha},\quad \alpha>1/2. \tag{78}$$

The interpretation of this scale is discussed in Fig. (4). Again these results at large time hold only when there is no blow-up. From Section I 2), we can surmise that this is the case for $B\geq B_c$ with $4B_c=-c'(0)-\frac{1}{\tau(+\infty)}=\frac{1-\alpha}{t_0}$.

Thus for $\alpha<1/2$ and $t_0=O(1)$, in particular for $\alpha=1$ a case of special interest, the limiting height distribution at $t=+\infty$ is non-universal. There are two cases however where it can be characterized more precisely. Small $t_0\ll 1$, in which case it becomes Gaussian and described by the EW fixed point (we will not study that case). And large $t_0\gg 1$, in which case it becomes again universal and described by the KPZ fixed point, as we now discuss.

Asymptotics for large $t_0\gg 1$. We now study the case where t_0, the time scale over which $c(t)$ varies, is chosen large, $t_0\gg 1$. Note that this situation is natural in disordered systems undergoing aging or coarsening dynamics (in that case t_0 is the waiting time, which maybe large). In that case it is natural to study the regime where both times are large $t,t_0\gg 1$, with t/t_0 fixed. As a result $\tau(t)$ is also large, i.e. $\tau(t)\gg 1$ and $\tau(t)/t_0$ is a function of t/t_0. All asymptotics below are thus controled by $t_0\gg 1$, at fixed ratio t/t_0. The most general such model is defined by a shape function $\hat{c}(s)$

$$c(t)=\hat{c}(s)\quad,\quad s=\frac{t}{t_0}\quad,\quad \tau(t)/t_0=\hat{\tau}(s)=\int_0^s\hat{c}(u)^2du \tag{79}$$

where $\hat{\tau}(s)$ is the shape function of the new time (we impose $\hat{c}(0)=1$ for simplicity). We recall that the equation for $h(x,t)$ contains also a quadratic external potential $V(x,t)=\hat{a}_c(s)\frac{x^2}{2t_0}$ with $\hat{a}_c(s)=\frac{-\hat{c}(s)}{2}(\frac{1}{\hat{c}(s)})''$. Our main example here is a shape function chosen as $\hat{c}(u)=(1+u)^{-\alpha}$. Hence in this case, for any α we may use the known asymptotics of H in (68) and the limit is universal. We obtain the one point statistics as

$$h(0,t)\simeq-\frac{1}{12}\hat{\tau}(s)\,t_0+(\hat{\tau}(s)\,t_0)^{1/3}\chi_s\quad,\quad s=\frac{t}{t_0}=O(1)\quad,\quad s\text{ fixed} \tag{80}$$

where $\hat{\tau}(s)$ is given in (79). Here χ_s is a TW type distribution depending on the initial data *and* on (i) the shape function $\hat{c}$, (ii) the parameter s: the dependence of χ_s in s is non trivial, and the IC classes of H and h are not identical anymore (see the discussion below).

When $\alpha<1/2$ one has, from (70), $\hat{\tau}(s)\simeq\frac{1}{1-2\alpha}s^{1-2\alpha}$ at large $s\gg 1$, and one recovers, for $t/t_0\gg 1$, the same behavior as in Eq. (72) with $\chi=\chi_{+\infty}$. The additional information in (80) is the complete dependence in the parameter s. When $\alpha<1/2$, the prefactor $\hat{\tau}(s)$ saturates at large $s\gg 1$, $\hat{\tau}(s)\simeq\frac{1}{2\alpha-1}$, and the limit $t\to+\infty$, that is $t\gg t_0$, leads to interesting new results (see below).

Mapping initial conditions. Let us address the one-time, full space statistics of the field $h(x,t)$, and identify the IC classes. We use (70) together with the asymptotic large τ result for $H(y,\tau)$, as given by (68). From (71) one now finds that $\mathsf{H}_0(\hat{z})=\mathsf{h}_0(\hat{z})+\frac{c_2\tau(t)}{t_0}\hat{z}^2$, with $c_2=-\hat{c}'(0)/\hat{c}(0)=\alpha$. Here, for fixed $t/t_0=s=O(1)$, $\frac{\tau(t)}{t_0}=\hat{\tau}(s)=\int_0^s\hat{c}(u)^2du$ is a fixed number. Hence the shift between IC of h and H (an additional parabola) remains important in this regime. We then find

$$h(x,t)+\frac{\hat{\tau}(s)t_0}{12}\simeq(\hat{\tau}(s)t_0)^{1/3}\left[\max_{\hat{z}}\left(\mathcal{A}_2(\hat{z}-\tilde{x})-(\hat{z}-\tilde{x})^2+\mathsf{h}_0(\hat{z})+c_2\hat{\tau}(s)\hat{z}^2\right)+\frac{\hat{c}'(s)\hat{\tau}(s)}{\hat{c}(s)^3}\tilde{x}^2\right],\quad \tilde{x}=\frac{x}{x(t)} \tag{81}$$

where we recall that $x(t)=\frac{2\hat{\tau}(s)^{2/3}}{\hat{c}(s)}t_0^{2/3}$ is the spatial correlation scale defined above, and $\mathsf{h}_0(\hat{z})=(t_0\hat{\tau}(s))^{1/3}h(2(t_0\hat{\tau}(s))^{2/3},0)$ (with $t_0\gg 1$). The variable χ_s introduced in (80) is thus equal to the square bracket in (81). The variational equation (81) characterizes completely the scaled height field as a process in the variable

$\tilde{x} = x/x(t)$. One can see that is *does not depend only* on the final value $\hat{c}(s)$ of the noise, but on integrated information on the full shape function, $\hat{c}(u)$, e.g. via $\hat{\tau}(s)$, as well as $c_2 = -\hat{c}'(0)/\hat{c}(0)$.

Consider initial conditions such that $h_0(\hat{z}) \simeq -\hat{B}\hat{z}^2$ at large $\hat{z}$. We see that the result in (81), for a given value of s, is finite if and only if $\hat{B} \geq c_2\hat{\tau}(s) - 1$. If not, there is a blow-up. This condition corresponds, upon rescaling, to the one given in Section I 2) and in the discussion above. For $\alpha < 1/2$ since $\hat{\tau}(s)$ diverges at large s there is always a blow up for some $s = s^*$. Only the droplet IC, with $\hat{B} = +\infty$, has no blow up. For $\alpha > 1/2$ the IC which have no blow up are such that $\hat{B} \geq \hat{B}_c = c_2\hat{\tau}(+\infty) - 1 = \frac{1-\alpha}{2\alpha-1}$. The flat IC thus has no blow-up for $\alpha \geq 1$. Interestingly, the absence of a blow-up for the flat IC is also guaranteed if $c_2 = 0$, i.e. $\hat{c}'(0) = 0$, that is if $\hat{c}(s)$ is sufficiently smooth around the origin. In that case $\hat{B}_c = -1$. This is the case for instance for $\hat{c}(s) = (1+s^2)^{\alpha/2}$, with $\hat{a}(s) = -\frac{\alpha(1+s^2(\alpha-1))}{2(1+s^2)^2}$ (see discussion in Section I 2)).

Let us rewrite (81) in the special case $\alpha = 1$, for which $a(t) = a_c(t) = 0$, in the more explicit form

$$h(x,t) + \frac{t}{12(t+t_0)} \simeq \left(\frac{t_0 t}{t+t_0}\right)^{1/3} \left[\max_{\hat{z}} \left(\mathcal{A}_2(\hat{z}-\tilde{x}) - (\hat{z}-\tilde{x})^2 + \mathsf{h}_0(\hat{z}) + \frac{t}{t+t_0}\hat{z}^2\right) - \frac{t}{t_0}\tilde{x}^2\right], \ \tilde{x} = \frac{x}{2t^{2/3}(1+\frac{t}{t_0})^{1/3}}. \tag{82}$$

Droplet IC. The droplet IC formally corresponds to $\mathsf{h}_0(0) = 0$ and $\mathsf{h}_0(\hat{z} \neq 0) = -\infty$. In that case the maximum in (81) is attained for $\hat{z} = 0$ and [62]

$$h(x,t) + \frac{\hat{\tau}(s)t_0}{12} \simeq (\hat{\tau}(s)t_0)^{1/3}\left[\mathcal{A}_2(\tilde{x}) - \omega(s)\tilde{x}^2\right], \quad \omega(s) = (1 - \frac{\hat{c}'(s)\hat{\tau}(s)}{\hat{c}(s)^3}) \tag{83}$$

Hence the one-point statistics is GUE-TW and the height field statistics is the Airy$_2$ process plus, however, a parabola with amplitude depending continuously on $s = t/t_0$. In the units of the correlation scale $x(t)$, the amplitude of the parabola saturates at large $s = t/t_0$ for $\alpha < 1/2$ as $\omega(+\infty) = \frac{1-\alpha}{1-2\alpha}$. This limit is consistent with the result (75) obtained there for $t/t_0 \gg 1$ at fixed $t_0 = O(1)$.

On the contrary, for $\alpha = 1/2$ one has $\hat{\tau}(s) = \log(1+s)$ and $\omega(s) = 1 + \frac{1}{2}\log(1+s)$, which diverges as $\omega(s) \simeq \frac{1}{2}\log(s)$ at large s. For $\alpha > 1/2$, the large s divergence is $\omega(s) \simeq \frac{\alpha}{2\alpha-1}s^{2\alpha-1}$ for $s = t/t_0 \gg 1$. In the case $\alpha = 1$

$$h(x,t) + \frac{t}{12(t+t_0)} \simeq \left(\frac{t_0 t}{t+t_0}\right)^{1/3}\left[\mathcal{A}_2(\tilde{x}) - (1+\tfrac{t}{t_0})\tilde{x}^2\right] = \left(\frac{t_0 t}{t+t_0}\right)^{1/3}\mathcal{A}_2(\tilde{x}) - \frac{x^2}{4t} \tag{84}$$

where we recall that the Airy$_2$ process is statistically invariant by translation and reflection.

Flat IC. Let us consider now the flat initial condition, $h(x,0) = 0$, i.e. $\mathsf{h}_0(\hat{z}) = 0$, and focus on $\alpha = 1$ for simplicity. In Eq. (82) we can redefine $\hat{z} \to \hat{z} + \tilde{x}$. Then we observe that the remaining deterministic terms form a perfect square. Hence we obtain

$$h(x,t) + \frac{t}{12(t+t_0)} \simeq \left(\frac{t_0 t}{t+t_0}\right)^{1/3} \max_{\hat{z}}\left(\mathcal{A}_2(\hat{z}) - \frac{t_0}{t+t_0}\left(\hat{z} - \frac{t}{t_0}\tilde{x}\right)^2\right), \ \tilde{x} = \frac{x}{2t^{2/3}(1+\frac{t}{t_0})^{1/3}}. \tag{85}$$

For $\tilde{x} = 0$ we find the result mentionned in the main text, namely that the CDF of the scaled fluctuating part $(\frac{t_0 t}{t+t_0})^{-1/3}\delta h(0,t)$ is given by $F^{\beta,\beta}_{\rm parbl}(s) = \mathrm{Prob}\left(\max_{\hat{z}}(\mathcal{A}_2(\hat{z}) - (1+\beta)\hat{z}^2) \leq s\right)$, with $\beta = -\frac{\tau(t)}{t_0} = -\frac{t}{t+t_0}$, a distribution for which a formula was obtained in [19, Example 1.25]. It interpolates between the GOE TW for small t/t_0 (small negative β) and the Gumbel distribution for $t/t_0 \to +\infty$ ($\beta \to -1$) [36]. The latter can be seen from the following heuristics. As $\beta \to -1$, the parabola weakens and $\hat{z}$ explores a larger region $|\hat{z}| \propto (1+\beta)^{-1/2}$. Since correlations of $\mathcal{A}_2(\hat{z})$ decay fast enough (as $1/\hat{z}^2$) on scales $\hat{z} = O(1)$ the problem becomes similar to the maximum of $M \propto (1+\beta)^{-1/2}$ i.i.d. random variables. One obtains the estimate

$$F^{\beta,\beta}_{\rm parbl}\left(A_\beta + \frac{s}{2\sqrt{A_\beta}}\right) \to e^{-e^{-s}} \tag{86}$$

as β goes to -1 with $A_\beta \simeq (-\frac{3}{8}\log(1+\beta))^{2/3}$, where we used that the CDF of $v = \mathcal{A}_2(0)$ decays as $\propto v^{-3/2}e^{-\frac{4}{3}v^{3/2}}$ for large positive v.

Note that in [20] the above optimisation problem was simulated using the Dyson Brownian motion and compared to inward KPZ growth experiments. In particular the few lowest cumulants of the distribution $F^{\beta,\beta}_{\rm parbl}$ have been computed numerically. This thus provides another nice example where this "parabolic" KPZ fixed point distribution appears.

Note also that in (85), since the Airy$_2$ process is statistically translationally invariant, the one point PDF of $h(x,t)$ is independent of x, as is expected for a flat initial condition.

Remark. The fact that the case $c(t) = 1/t$ has special properties can also be seen from the invariance of the Brownian motion under the transformation $t \to 1/t$, i.e. $\hat{B}(1/t) = B(t)/t$, where B and $\hat{B}$ are two unit Brownians. For the point to point DP partition sum with $V(x,t) = 0$ and noise $c(t) = 1/t$, one can write the solution of the SHE in the time interval $[t_1, t_2]$ as an expectation over a Brownian

$$Z(x_2, t_2|x_1, t_1) = \mathbb{E}\left[\exp\left(\int_{t_1}^{t_2} dt\sqrt{\frac{2}{t}}\xi(B(t), t)\right)|B(t_1) = x_1, B(t_2) = x_2\right] \tag{87}$$

$$= \mathbb{E}\left[\exp\left(\int_{t_1}^{t_2} \frac{dt}{t^2}\sqrt{2}\hat{\xi}(\frac{B(t)}{t}, \frac{1}{t})\right)|B(t_1) = x_1, B(t_2) = x_2\right] \tag{88}$$

$$= \mathbb{E}\left[\exp\left(\int_{u_2}^{u_1} du\sqrt{2}\hat{\xi}(\hat{B}(u), u)\right)|\hat{B}(u_1) = \frac{x_1}{t_1}, \hat{B}(u_2) = \frac{x_2}{t_2}\right] = \hat{Z}(y_1, u_1 = 1/t_1|y_2, u_2 = 1/t_2) \tag{89}$$

with $y_i = x_i/t_i$. In the second identity we only used the scale invariance of the space-time white noise ($\hat{\xi}$ being here another unit space-time white noise) and in the third we used the change of variable $t_i = 1/u_i$ and the above property of the Brownian motion. Note that here the time change $t \to 1/u$ has reversed time order, and to connect to our result for $\alpha = 1$ we can use the reversibility symmetry $\hat{Z}_{\hat{\xi}(y,u)}(y_1, u_1|y_2, u_2) = \hat{Z}_{\hat{\xi}(y,u_1+u_2-u)}(y_2, u_1|y_1, u_2)$.

Case of a linear potential. We now discuss the case of the KPZ equation with linear potential (46). For droplet initial conditions we know that for the one point PDF, $H(y,t) \equiv H(0,t) - \frac{y^2}{4t}$, where $\equiv$ means equality in distribution. Hence we have using (48)

$$h(x,t) \equiv H(0,t) - \frac{(x + y_0(t))^2}{4t} + \frac{y_0'(t)}{2}x + \frac{1}{4}\int_0^t ds\, y_0'(s)^2. \tag{90}$$

The height is thus a parabola centered at $x = x_m(t)$

$$x_m(t) = ty_0'(t) - y_0(t) \tag{91}$$

plus droplet KPZ fluctuations. In the case of the KPZ equation (2) with $b(t) = b$ (90) corresponds to the result discussed in the text with $y_0(t) = bt^2$. Note that the droplet result (GUE-TW at large time) requires an initial condition such that $Z(x,t=0) = e^{h(x,t=0)}$ decays sufficiently fast. Indeed, in (48) the field H is probed at space point $x + bt^2$ very far from the origin at large time. Let us consider an initial condition $h(x,t=0) = -\phi(x)$ with $\phi(x) > 0$. If the initial condition is e.g. a wedge, $\phi(y) = B|y|$, one expects that fluctuations of $h(x,t)$ for x fixed (e.g. at $x = 0$) at large t will be given instead by the flat IC class, i.e. the GOE-TW distribution.

Let us write the solution $h_0(x,t)$ in presence of the linear potential, but in the absence of the noise. From (90) it gives a heuristic description of the "mean profile" in the presence of noise, replacing $h(x,t)$ by its average. It reads

$$e^{h_0(x,t)} = \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{4\pi t}} e^{-\frac{(x+bt^2-z)^2}{4t} + xbt - \phi(z)} \tag{92}$$

Consider first the wedge, $\phi(z) = -|z|$. At large time one can use the saddle point method to estimate the integral. The argument of the exponential is maximum at $z = z(t) = bt^2 - 2t + x$: Indeed, $|z|$ can be replaced by z and this argument can be approximated as $\simeq -(z - z(t))^2/(4t) + (bt - 1)(x - t)$. The problem thus looks like the standard KPZ problem with flat IC. Hence we expect the GOE-TW distribution. One sees that the profile $h_0(x,t) \simeq bt(x - t)$ is linear, consistent with this expectation.

One can ask about the general form of the profile $h_0(x,t)$ for a larger class of IC. At large time, we may approximate $h_0(x,t)$ by the maximum of the argument of the exponential in (92). The maximal argument is reached at $z = z_m(x)$ solution of $2t\phi'(z_m(x)) = bt^2 + x - z_m(x)$. The profile $h_0(x,t)$ has a maximum at some $x = x_m$. Denoting $z_m = z_m(x_m)$ one finds that the maximum is reached at $x_m = bt^2 + z_m$ with $\phi'(z_m) = bt$. For the class $\phi(z) = \frac{1}{1+\delta}|z|^{1+\delta}$ with $\delta > 0$, one thus finds $x_m = bt^2 + (bt)^{1/\delta}$. Further, one may check that the curvature of the profile $h_0(x,t)$ (in the large time limit) is given by $\partial_x^2 h_0(x,t) \simeq -\phi''(z_m)/(1+2t\phi''(z_m))$. It is negative for $\delta > 0$, which is a sign that we are in the droplet IC class. For $\delta \to 0^+$, this curvature vanishes, and one recovers the above result about the wedge: the profile is linear at large time and does not exhibit a maximum.

A solvable case of the KPZ equation in presence of a quadratic potential. It is interesting to note that, although no exact solutions exist for the usual KPZ equation in presence of a time-independent quadratic potential $V(x,t) = a\frac{x^2}{2}$, it can be solved for certain time-dependent disorders $c(t)$. It is easy to solve for $a = a_c(t)$, using that $a_c(t) = -\frac{c(t)}{2}(1/c(t))''$.

Consider $a < 0$ i.e. a confining potential for the DP. The general solution is $c(t) = 1/(c_1 e^{\sqrt{2|a|}t} + c_2 e^{-\sqrt{2|a|}t})$. The simplest example is $V(x,t) = -\frac{x^2}{4t_0^2}$ and $c(t) = e^{-t/t_0}$, the solution is then $h(x,t) = H(xe^{-t/t_0}, \frac{t_0}{2}(1-e^{-2t/t_0})) - \frac{x^2}{4t_0} - \frac{t}{2t_0}$ with $H(y,\tau)$ solution of the standard KPZ equation with $H(x,0) = h(x,0) + \frac{x^2}{4t_0}$. At large t is leads to standard finite time KPZ fluctuations. The asymptotic PDF of $h(x,t=+\infty)$ scaled by $(t_0/2)^{1/3}$ becomes the GUE-TW distribution as $t_0 \gg 1$ (for droplet IC).

Interestingly, the opposite case, $c(t) = e^{t/t_0}$, leads to the solution $h(x,t) = H(xe^{t/t_0}, \frac{t_0}{2}(e^{2t/t_0} - 1)) + \frac{x^2}{4t_0} + \frac{t}{2t_0}$ with $H(y,\tau)$ solution of the standard KPZ equation with $H(x,0) = h(x,0) - \frac{x^2}{4t_0}$. At large time it leads to TW type (and KPZ fixed point) type fluctuations for any value of t_0.

Finally for $a < 0$, the solvable cases are $c(t) = c_1/\cos(\sqrt{2a}(t+t_0))$, which, however, lead to diverging $c(t)$ for some periodic times.

6) Directed polymer and its wandering exponent

We recall that the partition sum $Z(x,t)$ of the $d = 1{+}1$ continuum directed polymer (DP) in a (time inhomogeneous) random potential $\sqrt{2c(t)}\xi(x,t)$ with one fixed endpoint at (x,t) and in presence of an external potential $V(x,t)$ is solution of the stochastic heat equation (SHE), Equation (3) in the text. The special solution, denoted $Z(x,t|x_0,0)$, with initial condition $Z(x,t=0|x_0,0) = \delta(x-x_0)$ is called the droplet IC solution and corresponds to a DP with both endpoints fixed (point to point DP). More general IC conditions correspond to other DP geometries, for instance the flat IC $Z(x,t=0) = 1$ corresponds to the point to line DP.

Similarly one defines the second DP problem, in the random potential $\sqrt{2}\hat{\xi}(y,\tau)$ and in presence of a quadratic external potential whose partition sum, $\hat{Z}(y,\tau)$, is solution of

$$\partial_\tau \hat{Z} = \partial_y^2 \hat{Z} + (-A(\tau(t))\frac{y^2}{2} + \sqrt{2}\hat{\xi}(y,\tau))\hat{Z} \tag{93}$$

and one denotes the droplet solution, $\hat{Z}(y,\tau|y_0,0)$, with initial condition $\hat{Z}(y,\tau=0|y_0 0) = \delta(y-y_0)$.

When both ξ and $\hat{\xi}$ are unit space-time white noises, the relation between the two partition sums is then, from (4), (5) in the text

$$Z(x,t) = \sqrt{c(t)}\hat{Z}(c(t)x,\tau(t))e^{\frac{c'(t)}{4c(t)}x^2} \quad , \quad Z(x,t|0,0) = \sqrt{c(t)}\hat{Z}(c(t)x,\tau(t)|0,0)e^{\frac{c'(t)}{4c(t)}x^2}, \tag{94}$$

(up to an immaterial multiplicative constant). The first identity leads to some correspondence between the DP geometries, as discussed in the text, see (9), and we see that the point to point DP are in correspondence in both problems. These relations can also be checked directly from (93).

An important observable is the PDF of the endpoint of the DP. It is defined as the average

$$P(x,t) = \overline{P_\xi(x,t)} = \overline{\frac{Z(x,t)}{\int dx' Z(x',t)}} \tag{95}$$

where $P_\xi(x,t)$ is the endpoint PDF in a given sample. From it one defines the moments of the endpoint PDF $\langle x^n \rangle = \int dx x^n P(x,t)$, and the transverse wandering $\delta x(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. Using the mapping (94) we obtain

$$P(x,t) = \overline{\frac{\hat{Z}(c(t)x,\tau(t))e^{\frac{c'(t)x^2}{4c(t)}}}{\int dx' \hat{Z}(c(t)x',\tau(t))e^{\frac{c'(t)(x')^2}{4c(t)}}}} \quad , \quad \hat{Z}(y,\tau) = e^{H(y,\tau)} \tag{96}$$

Let us restrict to the droplet solution, i.e. the point to point DP. Consider the case $c(t) = \left(\frac{t_0}{t+t_0}\right)^\alpha$, and $a(t) = a_c(t)$, as above.

Let us start with $\alpha < 1/2$ and $t_0 = O(1)$. The denominator in (96) is $e^{h(x,t)}$, and in the limit $t \gg 1$ the statistics of $h(x,t)$ is described by Eq. (75). Given that the prefactor in e.g. (75) is large, the PDF in any given sample is concentrated around $\tilde{x}^*$ which realizes the maximum of the bracket in (75). Let us define for $\omega > 0$, $\mathcal{P}_\omega(\tilde{x}^*)$ the PDF of $\tilde{x}^* = \mathrm{argmax}_{z\in\mathbb{R}}\left[\mathcal{A}_2(z) - \omega z^2\right]$. It is a one-parameter universal distribution: for $\omega = 1$ it is the known endpoint PDF for the standard DP [37–39], and it was calculated recently [40] for other values of ω. Our conclusion is that for large t, and fixed $t_0 = O(1)$, the endpoint distribution takes the form

$$P(x,t) \simeq \frac{1}{x(t)}\mathcal{P}_{\frac{1-\alpha}{1-2\alpha}}\left(\frac{x}{x(t)}\right) \quad , \quad \alpha < 1/2 \tag{97}$$

where $x(t) \sim t^{\zeta(\alpha)}$ is the correlation length scale given in Eq. (73). For $\alpha < 1/2$ the transverse wandering length of the DP is thus proportional to the correlation length scale, i.e. $\delta x(t) \propto x(t)$.

Let us study the limit of $\mathcal{P}_\omega$ for $\omega \to +\infty$, which is relevant for $\alpha \to 1/2^-$. In that limit the quadratic well is strong and the position of the maximum $\tilde{x}^*$ is close to zero. Hence one can rescale $z = \omega^{-2/3}u$ and one has

$$\max_{z\in\mathbb{R}}\left[\mathcal{A}_2(z) - \omega z^2\right] - \mathcal{A}_2(0) \simeq \omega^{-1/3}\max_{u\in\mathbb{R}}\left[\sqrt{2}B(u) - u^2\right] \tag{98}$$

where $B(u)$ is the two sided Brownian motion, and we have used that the Airy process is locally Brownian, i.e. that as $\epsilon \to 0$, $\mathcal{A}_2(\epsilon u) - \mathcal{A}_2(0) = \sqrt{2\epsilon}B(u) + o(\epsilon)$ [41]. We also use here and below the Brownian scaling, $B(au) = \sqrt{a}B(u)$. Hence one has $\tilde{x}^* = \omega^{-2/3}u^*$ where $u^* = \mathrm{argmax}[\sqrt{2}B(u) - u^2]$. The PDF of u^*, which we denote $P_0(u^*)$, is well known [42, 43]

$$P_0(u) = g(u)g(-u) \quad , \quad g(u) = \int_{-\infty}^{+\infty}\frac{dw}{2\pi}\frac{e^{-iwu}}{\mathrm{Ai}(iw)} \tag{99}$$

with $P_0(u) \sim |u|e^{-|u|^3/3}$ at large $|u|$. Hence one has, in the large $\omega \to +\infty$ limit

$$\omega^{-2/3}\mathcal{P}_\omega(\omega^{-2/3}u) \to P_0(u) \tag{100}$$

Let us consider now $\alpha = 1/2$, the marginal case. We already surmise that as $\alpha \to 1/2^-$, since $\omega = \frac{1-\alpha}{1-2\alpha} \to +\infty$, the distribution P_0 should arise. Let us show now how it works. Let us start from Eq. (77). Anticipating the result, let us write $x = 2^{5/3}t_0^{1/6}t^{1/2}z$. One can rewrite the r.h.s. in (77) as $(2t_0)^{1/3}[\sqrt{2}B(z) - z^2]$. This means that in a given sample

$$P_\xi(x,t) \propto e^{(2t_0)^{1/3}[\sqrt{2}B(z)-z^2]} \quad , \quad z = \frac{x}{2^{5/3}t_0^{1/6}t^{1/2}} \tag{101}$$

up to a normalization constant. This identifies with the random Gibbs measure at "temperature" $(2t_0)^{-1/3}$ of the so-called "toy model", much studied in the disordered system literature [43–45]. If $t_0 \gg 1$, it is dominated by the maximum in the exponential and the PDF of $x/x_1(t)$ is given by P_0 as $t/t_0 \gg 1$

$$P(x,t) \simeq \frac{1}{x_1(t)}P_0(\frac{x}{x_1(t)}) \quad , \quad x_1(t) = 2^{5/3}t_0^{1/6}t^{1/2} \tag{102}$$

Some results are also known for the "finite temperature" regime $t_0 = O(1)$. Note that the transverse wandering length of the DP is now set by the new length scale $\delta x(t) \propto x_1(t) \sim t^{1/2}$, i.e. leading to diffusion, different for the super-diffusive correlation length $x(t) \sim t^{1/2}[\log(t/t_0)]^{2/3}$.

For $\alpha > 1/2$, and $t_0 = O(1)$, the quadratic part $-\alpha x^2/(4t)$ dominates the endpoint PDF, and the random part becomes negligible. Let us write $x = \hat{x}t^{1/2}$ with fixed $\hat{x}$ and large t. Then we can neglect the dependence in $\hat{x}$ of the factor $\hat{Z}$ in (96), i.e. $\hat{Z}(c(t)x, \tau(t))$ is asymptotically constant as x varies in the $t^{1/2}$ scale. Conversely, this factor is expected to vary on scales $x = x(t) \sim t^\alpha \gg t^{1/2}$. More precisely, because $\alpha > 1/2$, $c(t)t^{1/2} \to 0$ as t goes to infinity, so that for fixed $\hat{x}$, the numerator of (96) can be approximated by

$$\hat{Z}(c(t)x, \tau(t))e^{\frac{c'(t)x^2}{4c(t)}} \simeq \hat{Z}\left(0, \frac{t_0}{2\alpha - 1}\right)e^{-\frac{\alpha}{4}\tilde{x}^2}. \tag{103}$$

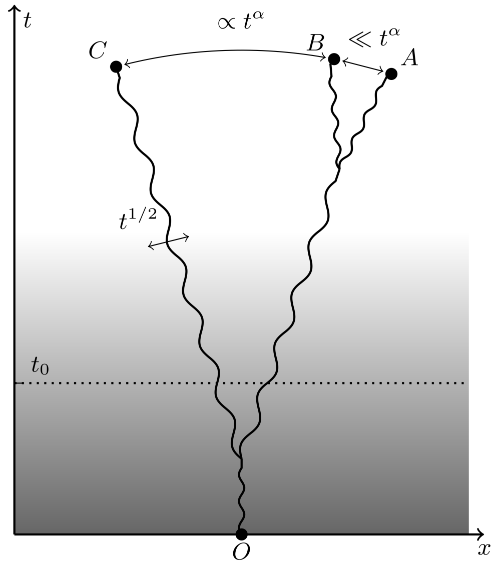


FIG. 4. The gray shading indicates the amplitude of the noise which decreases with time. The directed polymer partition function from O to A or B is dominated by paths which will branch to A or B after exiting the area with large weights, thus the partition functions are highly correlated. However, the paths going to B or C branch much earlier and thus the partition functions are different. If the distance from B to C was much larger than t^α, the partition function would fully decorrelate.

Hence we obtain that at large t, $P(x,t) \simeq (\frac{\alpha}{4\pi})^{1/2} e^{-\frac{\alpha}{4}\hat{x}^2}$. Thus, as stated in the main text, the polymer wandering scale is $\delta x(t) \sim t^{1/2}$, i.e. diffusive, and is typically different from the spatial correlation scale which is $x(t) \propto t^\alpha$. Figure 4 provides a heuristic explanation for the relation between these spatial scales and the geometry of polymer paths.

Finally let us discuss the case $t_0 \gg 1$. In the large t limit with $s = t/t_0$ fixed we can now use Eq. (83), and we find $P(x,t) \simeq \frac{1}{x(t)} \mathcal{P}_{\omega(s)}(\frac{x}{x(t)})$, where $x(t) = t_0^{2/3} \frac{2\hat{\tau}(s)}{\hat{c}(s)}$ and $\omega(s)$ is defined in (83). For $s = O(1)$ this result is valid for any α. For $\alpha < 1/2$ the result is thus qualitatively similar to the one above in (97) for $t_0 = O(1)$, the two results matching perfectly when $s = t/t_0 \to +\infty$, since $\omega(+\infty) = \frac{1-\alpha}{1-2\alpha}$. For $\alpha \geq 1/2$ and $s = O(1)$ the result is quite different from the result obtained above for $O(t_0) = 1$. They can still be matched as $s = t/t_0 \to \infty$, but the matching is more complicated since $\omega(s)$ diverges at large s. Let us focus on $\alpha = 1$ for simplicity, and consider the Eq. (84) where we recall $\tilde{x} = \frac{x}{x(t)} = \frac{x}{2t^{2/3}(1+\frac{t}{t_0})^{1/3}}$. Let us rescale $\tilde{x} = (1 + \frac{t}{t_0})^{-2/3} z$, and study the regime $t/t_0 \gg 1$. Using that the Airy process is locally Brownian we have

$$P_\xi(x,t) \propto e^{t_0^{2/3} t^{-1/3} [\sqrt{2} B(z) - z^2]} \quad , \quad z = \frac{x}{2(t_0 t)^{1/3}} \tag{104}$$

thus there is an intermediate regime $t_0 \ll t \ll t_0^2$ where the endpoint PDF behaves as $P(x,t) \sim \frac{1}{x_2(t)} P_0(x/x_2(t))$ where $x_2(t) = 2(t_0 t)^{1/3}$ is a new length scale, intermediate between the correlation scale $x(t) \sim t$ and the diffusive scale $x_1(t) \sim t^{1/2}$. Finally for $t/t_0 \gg t_0 \gg 1$ one recovers the diffusive result obtained in (103).

7) The case $a(t) = 0$: absence of external potential

When $a(t) = 0$, i.e. $V(x,t) = 0$, the KPZ equation with a general time-dependent noise $c(t)$ maps to the KPZ equation with unit noise in presence of a quadratic potential $-\frac{A(\tau)}{2} y^2$, with $A(\tau(t)) = a_c(t)/c(t)^4 = -\frac{1}{2} c(t)^{-3} (1/c(t))''$.

One can ask which $c(t)$ lead to $A(\tau) = A$ a positive constant. The general solution with $c(0) = 1$ is the two parameter family (where one can choose $t_1 \geq t_0$)

$$c(t) = \frac{1}{\sqrt{(1 + \frac{t}{t_0})(1 + \frac{t}{t_1})}} \quad , \quad 8A = \left(\frac{1}{t_0} - \frac{1}{t_1}\right)^2 \quad , \quad \tau(t) = \frac{1}{\frac{1}{t_0} - \frac{1}{t_1}} \left(\log\left(1 + \frac{t}{t_0}\right) - \log\left(1 + \frac{t}{t_1}\right) \right) \tag{105}$$

with $\tau(+\infty) = \frac{1}{\frac{1}{t_0} - \frac{1}{t_1}} \log(t_1/t_0)$. The case $t_1 = +\infty$, i.e. $c(t) = 1/\sqrt{1 + t/t_0}$, with $A = 1/(8t_0^2)$, is the "critical case" studied in the main text. The case $t_1 = t_0$, i.e. $c(t) = 1/(1 + t/t_0)$, with $A = 0$, is the case $\alpha = 1$ studied above and in the main text. In the general case, and especially for $t_1 \gg t_0$, $c(t)$ exhibits a crossover between the two behaviors $c(t) \sim 1/\sqrt{t}$ (on scale t_0) and $c(t) \sim 1/t$ (on scale t_1).

Assume $t_1 \geq t_0 \gg 1$, hence the curvature of the quadratic well is small $A \ll 1$. Let us examine qualitatively the problem of the point to line DP of length τ in a quadratic well $-\frac{A}{2}y^2$ with a unit white noise random potential (hence we restrict to the droplet IC). In the limit $A \ll 1$ the confinement due to the quadratic well acts only at large scale: by scaling one sees that it cuts the growth of the variance of the DP endpoint fluctuations, noted $\langle y^2 \rangle$, at a crossover time $\tau \simeq \tau_A = 1/\sqrt{8A} \gg 1$. This is obtained by considering [63] a segment of length τ of the DP, wandering over a distance y, and balancing the elastic energy, $\propto y^2/\tau$, with the potential energy, $\propto Ay^2\tau$. Hence one has

$$\langle y^2 \rangle \simeq \begin{cases} Y\tau^{4/3} & , \quad 1 \ll \tau \ll \tau_A \\ y_2(8A)^{-2/3} & , \quad \tau \gg \tau_A \end{cases} \tag{106}$$

where Y and y_2 are numbers of order unity. Similarly, since segments of length τ_A become essentially uncorrelated, one expects that the free energy fluctuations of the point to point DP scale as

$$\delta H(0,\tau) \simeq \begin{cases} \tau^{1/3}\chi_2 & , \quad 1 \ll \tau \ll \tau_A \\ \tau_A^{1/3}(\frac{\tau}{\tau_A})^{1/2}\omega & , \quad \tau \gg \tau_A \end{cases} \tag{107}$$

where χ_2 is GUE-TW distributed, and ω a unit Gaussian.

Transporting these results to the original problem one finds for the free energy fluctuations, in the case where $t_1 \geq t_0 \gg 1$

$$\delta h(0,t) \simeq \begin{cases} \left(\frac{t_0 t_1}{t_1 - t_0} \log(\frac{1+\frac{t}{t_0}}{1+\frac{t}{t_1}}) \right)^{1/3} \chi_2 & , \quad \frac{t_1 - t_0}{t_1 t_0} \ll \log(\frac{1+\frac{t}{t_0}}{1+\frac{t}{t_1}}) \ll 1 \\ \left(\frac{t_0 t_1}{t_1 - t_0} \right)^{1/3} \left(\log(\frac{1+\frac{t}{t_0}}{1+\frac{t}{t_1}}) \right)^{1/2} \omega & , \quad \log(\frac{1+\frac{t}{t_0}}{1+\frac{t}{t_1}}) \gg 1 \end{cases} \tag{108}$$

For the case $t_1 = +\infty$ one recovers the result given in the main text (Eq. 19)

$$\delta h(0,t) \simeq \begin{cases} t^{1/3}\chi_2 & , \quad 1 \ll t \ll t_0 \\ t_0^{1/3}(\log \frac{t}{t_0})^{1/2}\omega & , \quad t \gg t_0 \end{cases} \tag{109}$$

In the case $t_1 = t_0$, i.e. $\alpha = 1$ and droplet IC, only the first regime exists (since $A = 0$ and $\tau_A = +\infty$) and

$$\delta h(0,t) \simeq \left(\frac{t t_0}{t + t_0} \right)^{1/3} \chi_2 \quad , \quad 1 \ll t \tag{110}$$

The result (108) interpolates between these cases, the interpolation parameter being t_1/t_0. Note that for the second, purely gaussian, regime to exist one needs $\log(t_1/t_0) \gg 1$. If not the PDF saturates at some non-universal value.

The study of the transverse wandering of the DP is a bit more delicate. Let us focus on the model $t_1 = +\infty$, i.e. $c(t) = 1/\sqrt{1 + \frac{t}{t_0}}$, studied in the main text. We study the case $t_0 \gg 1$. Going back to the height $H(y,\tau)$, i.e. the DP in quadratic well $-Ay^2/2$, let us separate the height from its mean profile, using the result from the STS symmetry (applying (65) to the problem for $H(y,\tau)$). One defines

$$H(y,\tau) = \tilde{H}(y,\tau) - \frac{y^2}{8t_0} \coth\left(\frac{\tau}{2t_0}\right) = \tilde{H}(y,\tau) - \frac{x^2}{8(t+t_0)} \left(1 + \frac{2t_0}{t}\right) \tag{111}$$

such that the one point PDF of $\tilde{H}(y,\tau)$ is independent of y. We recall that $\tau = t_0 \log(1 + \frac{t}{t_0})$ and that $y = x/\sqrt{1 + \frac{t}{t_0}}$. It implies for the original problem that

$$h(x,t) = H(y,\tau) - \frac{x^2}{8(t+t_0)} = \tilde{H}(y,\tau) - \frac{x^2}{4t} \tag{112}$$

The fact that the quadratic terms add up to $-x^2/(4t)$ is a consequence of the STS symmetry, i.e. since $a(t)=0$, the profile is the usual parabola independently of $c(t)$, see (62). We know little about the process $\tilde{H}(y,\tau)$, but we can heuristically assume that

$$\tilde{H}(y,\tau) \simeq \tau^{1/3}\mathcal{A}_2(\frac{y}{2\tau^{2/3}}) \quad , \quad 1 \ll \tau \ll t_0 \tag{113}$$

$$\simeq t_0^{1/3}\left(\frac{\tau}{t_0}\right)^{1/2}\omega + t_0^{1/3}\mathcal{A}\left(\frac{y}{2t_0^{2/3}}\right) \quad , \quad \tau \gg t_0 \tag{114}$$

where $\mathcal{A}(z)$ is a $O(1)$ unknown process, with a one-point PDF independent of z. The picture is that for $\tau \gg t_0$ the DP is bounded by the quadratic well and only the last independent segment of length $\propto t_0$ contributes to the spatial fluctuations. This is reasonable as it reproduces the two limits in (106), and for $\tau \gg t_0$, using the large τ limit of (111), it expresses the PDF $P_\xi^A(y,\tau)$ of the endpoint y for the DP in the quadratic well $-Ay^2/2$ in a given sample as

$$P_\xi^A(y,\tau) \propto e^{t_0^{1/3}\mathcal{A}\left(\frac{y}{2t_0^{2/3}}\right)-\frac{y^2}{8t_0}} = e^{t_0^{1/3}[\mathcal{A}(\tilde{y})-\frac{1}{2}\tilde{y}^2]} \quad , \quad \tilde{y} = y/(2t_0^{2/3}) \tag{115}$$

Since $t_0 \gg 1$, the mean endpoint PDF $P_A = \overline{P_\xi^A}$ identifies with the distribution of the arg-max of the term in the exponential, its variance being related to the prefactor y_2 in (106).

Going back to the original problem we see that:

- In the first regime $1 \ll t \ll t_0$, $\tau \simeq t$ and $c(t) \simeq 1$ and $h(x,t) \simeq t^{1/3}[\mathcal{A}_2(\hat{x}) - \hat{x}^2]$ with $\hat{x} = \frac{x}{2t^{2/3}}$ leading to the endpoint distribution of the standard DP problem.
- In the second regime $t \gg t_0 \gg 1$ we obtain

$$h(x,t) \simeq t_0^{1/3}(\frac{t}{t_0})^{1/2}\omega + t_0^{1/3}[\mathcal{A}(\tilde{x}) - \tilde{x}^2] \quad , \quad \tilde{x} = \frac{x}{2t_0^{1/6}t^{1/2}} \tag{116}$$

  and the endpoint PDF can be written as $P(x,t) \propto e^{t_0^{1/3}[\mathcal{A}(\tilde{x})-\tilde{x}^2]}$ leading for $t_0 \gg 1$ to a related, but slightly different, maximization problem from (115). This shows that the DP wandering length obeys $\delta x(t)^2 < c(t)^2\langle y^2\rangle$, but that both sides are of the same order, $\propto t_0^{4/3}t$, i.e. diffusive, as indicated in the main text.

II INHOMOGENEOUS DISCRETE MODEL

1) Preliminaries on the gamma distribution

Before analyzing the directed polymer model with inverse gamma weights discussed in the letter, we gather here some useful facts about the (inverse) gamma distribution. Let w be a random variable with inverse gamma distribution of parameter γ, i.e. the random variable with PDF

$$P(w) = \frac{1}{\Gamma(\gamma)}w^{-\gamma-1}e^{-1/w}. \tag{117}$$

The moments of w are given by $\mathbb{E}[w^n] = \Gamma(\gamma-n)/\Gamma(\gamma)$, hence the mean and variance of G are given by

$$\mathbb{E}[w] = \frac{1}{\gamma-1} = \frac{1}{\gamma} + \frac{1}{\gamma^2} + o(1/\gamma^2), \text{ for } \gamma > 1, \tag{118}$$

$$\mathrm{Var}[w] = \frac{1}{(\gamma-2)(\gamma-1)^2} = \frac{1}{\gamma^3} + o(1/\gamma^3), \text{ for } \gamma > 2, \tag{119}$$

where the approximations hold for large γ.

In the study of directed polymer models, one is often led to consider not only the distribution of Boltzmann weights but also on site energies. Let us define E such that $w = e^{E/\theta}$, so that here the PDF of the on site energy is thus $P(E) = \frac{1}{\theta\Gamma(\gamma)}e^{-\gamma E/\theta}e^{-e^{-E/\theta}}$. If we scale $\gamma = \theta\tilde{\gamma}$, in the zero temperature limit $\theta \to 0$, $P(E)$ converges to $P(E) = \tilde{\gamma}e^{-\tilde{\gamma}E}\mathbb{1}_{E>0}$, i.e. the PDF of an exponential random variable of parameter $\tilde{\gamma}$. We will adress below in which sense this limit corresponds to a zero temperature limit. For the moment, we simply remark that although γ (or θ)

can be physically interpreted as a temperature for γ close to 0, the relation between γ and the physical temperature is more complicated in general. In particular, in Section II 6) we will explain that γ should be interpreted as the square of the temperature T^2 as $\gamma \to \infty$.

We will also need the cumulants of on site-energies $E = \log w$ (where w is still an inverse gamma random variable of parameter γ, and we have set $\theta = 1$ for simplicity). A direct computation shows that

$$\mathbb{E}\left[e^{uE}\right] = \mathbb{E}\left[w^u\right] = \frac{\Gamma(\gamma - u)}{\Gamma(\gamma)}. \tag{120}$$

This implies that the cumulants of E (in the sequel of the paper, we will use indifferently the notations $\kappa_n(X)$ or $\langle X^n\rangle_c$ to denote the n-th cumulant of a random variable X) are given by

$$\kappa_n(E) = \partial_u^n \log \mathbb{E}\left[e^{uE}\right]\Big|_{u=0} = (-1)^n \psi^{(n-1)}(\gamma), \tag{121}$$

where ψ is the digamma function (127), and in particular, $\mathbb{E}[E] = -\psi(\gamma)$ and $\mathrm{Var}[E] = \psi'(\gamma)$. Thus, for large γ, we have the approximations

$$\mathbb{E}[E] \simeq -\log\gamma, \quad \mathrm{Var}[E] \simeq \frac{1}{\gamma}, \quad \kappa_3(E) = \frac{1}{\gamma^2}, \quad \kappa_4(E) \simeq \frac{2}{\gamma^3}, \tag{122}$$

while for $\gamma \to 0$, we have

$$\mathbb{E}[E] \simeq \frac{1}{\gamma}, \quad \mathrm{Var}[E] \simeq \frac{1}{\gamma^2}, \quad \kappa_3(E) = \frac{2}{\gamma^3}, \quad \kappa_4(E) \simeq \frac{6}{\gamma^4}. \tag{123}$$

2) Fredholm determinant formula

The three next sections are based on the following result from [23, Corollary 1.8]. Fix $n \geqslant m \geqslant 1$ and real parameters α_i, β_j such that $\alpha_i + \beta_j > 0$ for all i, j. For $u \in \mathbb{C}$ such that $\Re[u] > 0$,

$$\mathbb{E}[e^{-uZ(n,m)}] = \det(I + K)_{\mathbb{L}^2(\mathcal{C})}, \tag{124}$$

where $\mathcal{C}$ is a positively oriented closed contour enclosing the set of $-\beta_j$ and no other singularity, and

$$K(v, v') = \int_{\delta - \mathbf{i}\infty}^{\delta + \mathbf{i}\infty} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{\pi}{\sin(\pi(v - z))} \frac{1}{z - v'} \frac{H(z)}{H(v)} \tag{125}$$

where

$$H(z) = u^z \frac{\prod_{i=1}^n \Gamma(\alpha_i - z)}{\prod_{j=1}^m \Gamma(z + \beta_j)}. \tag{126}$$

and the integration contour $\delta + i\mathbb{R}$ is such that $\delta < \alpha_j$, the contour $\mathcal{C}$ lies to the left of $\delta + i\mathbb{R}$ and all the poles at $z = v+1, v+2, \dots$ lie to the right of $\delta + i\mathbb{R}$. In order to match the above result with [23, Corollary 1.8], we have simply set $\beta_j = -a_j$ ($\{a_j\}$ is a set of parameters used in [23]). Note that [23] assumes that $a_j \geqslant 0$ but this is unnecessary, the formula can be analytically continued to negative a_j as long as $\alpha_i - a_j = \alpha_i + \beta_j > 0$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

We will focus on the case $n = m$ (though the asymptotics when n/m is an arbitrary constant are very similar). Asymptotic analysis of (124) in the homogeneous case, i.e $a = 0$, have been performed in a number of works [23, 46–48], by Laplace's method. We adapt the same approach to the inhomogeneous case. It should be noted that the following asymptotic results do not constitute mathematical theorems. Mathematical proofs would require performing a more careful analysis of the function H along the tails of the contours, and proving a number of estimates to justify the convergence. Such justifications can be found for instance in [23, 46, 47] in a similar context. Making all these justifications in the present context (inhomogeneous weights) constitutes a mathematical challenge, but from the physical point of view it does not seem necessary, and thus we will proceed via saddle point analysis without attempting to prove rigorously a theorem.

3) Asymptotic analysis

Analysis of the amplitude σ_n. Let us fix $\theta > 0$ and set $\alpha_i = \beta_i = \theta i^a$. For that particular choice of parameters, we set $G(z) = \log H(z)$ in accordance with (21). Recall the definition of the digamma function

$$\psi(x) = \tfrac{d}{dx}\log(\Gamma(x)) \tag{127}$$

Since $G''(0) = G^{(4)}(0) = 0$, we have by Taylor expansion,

$$G(z) = z\log(u) + z\frac{f_n}{\theta} + z^3\frac{\sigma_n^3}{3\theta^3} + o(z^4), \tag{128}$$

where

$$f_n = -2\theta\sum_{i=1}^{n}\psi(\theta i^a), \quad \sigma_n^3 = \theta^3\sum_{i=1}^{n} -\psi''(\theta i^a). \tag{129}$$

We claim that the criterium to determine if we will observe Tracy-Widom fluctuations or a stabilization of the free energy (i.e. fluctuations of the free energy on the constant scale according to a non-universal distribution) is whether the quantity σ_n diverges to infinity or not. In principle, the true criterium should be that the term of order 3 in the Taylor expansion is dominant with respect to the remainder. This will happen if and only if σ_n diverges. Assume for the moment that θ is fixed. The digamma function satisfies the asymptotics $\psi(x) \simeq \frac{-1}{x^2}$ as x goes to $+\infty$. Thus, for $0 \leqslant a < 1/2$,

$$\sigma_n^3 \simeq \sum_{i=1}^{n}\frac{\theta}{i^{2a}} \simeq \frac{Tn^{1-2a}}{1-2a}, \quad n \to \infty. \tag{130}$$

For $a > 1/2$, σ_n converges to a constant, and for $a = 1/2$, $\sigma_n^3 \simeq \theta\log n$.

Asymptotic analysis when $a > 1/2$ Assume $a > 1/2$ and $\theta > 0$ is fixed. We will first show that the kernel (125) converges to some kernel $\tilde{K}$. Let $\tilde{u} = ue^{-2\sum_{i=1}^{n}\psi(\theta i^a)}$. Then,

$$G(z) = z\log(\tilde{u}) + \sum_{i=1}^{n}\log\Gamma(\theta i^a - z) - \log\Gamma(\theta i^a + z) + 2z\psi(\theta i^a). \tag{131}$$

Using the series expansions

$$\log\Gamma(z) = -\gamma z - \log z + \sum_{k=1}^{\infty}\frac{z}{k} - \log\left(1 + \frac{z}{k}\right) \tag{132}$$

and

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\frac{1}{k} - \frac{1}{z+k}, \tag{133}$$

we obtain that

$$\lim_{n\to\infty} G(z) - z\log(\tilde{u}) = \sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\log\left(1 + \frac{z}{\theta i^a + k}\right) - \log\left(1 - \frac{z}{\theta i^a + k}\right) - \frac{2z}{\theta i^a + k} =: G_\infty(z). \tag{134}$$

Note that this double sum is well-defined because

$$\sum_{i=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{(\theta i^a + k)^3} < \infty. \tag{135}$$

Using the exponential growth of the sine function towards $\mathbf{i}\infty$, and the boundedness of $\Re[G_\infty(z)]$ on the contour $\mathbf{i}\mathbb{R}$, we deduce by dominated convergence that $K(v, v')$ converges as n goes to infinity to $K_\infty(v, v')$ where

$$K_\infty(v, v') = \int_{-\mathbf{i}\infty}^{\mathbf{i}\infty}\frac{\mathrm{d}z}{2\mathbf{i}\pi}\frac{\pi}{\sin(\pi(v - z))}\frac{1}{z - v'}\tilde{u}^{z-v}e^{G_\infty(z) - G_\infty(v)}. \tag{136}$$

The kernel K (depending on n) is acting on $\mathbb{L}^2(\mathcal{C})$, where the contour $\mathcal{C}$ is a positively oriented contour containing all points $-\theta i^a$ for $1 \leqslant i \leqslant n$. We may deform $\mathcal{C}$ to some infinite contour $\mathcal{C}_\infty$, independent from n, which contains all points $-\theta i^a$ for $i \geqslant 1$ and we have $\det(I+K)_{\mathbb{L}^2(\mathcal{C})} = \det(I+K)_{\mathbb{L}^2(\mathcal{C}_\infty)}$. Applying dominated convergence to the Fredholm determinant expansion, we deduce the convergence of Fredholm determinants from the convergence of kernels, that is we arrive at

$$\det(I+K)_{\mathbb{L}^2(\mathcal{C})} \xrightarrow[n\to\infty]{} \det(I+K_\infty)_{\mathbb{L}^2(\mathcal{C}_\infty)}. \tag{137}$$

A rigorous mathematical justification would require some bounds on the kernel $K(v,v')$ valid along the contour $\mathcal{C}_\infty$ towards infinity, we assume without justification that such bounds hold. Finally, we conclude that the random variable $\tilde{\mathcal{Z}} = \mathcal{Z}(n,n)e^{2\sum_{i=1}^n \psi(\theta i^a)}$ weakly converges to some probability distribution characterized by its Laplace transform $\mathbb{E}[e^{-\tilde{u}\tilde{\mathcal{Z}}}] = \det(I+K_\infty)_{\mathbb{L}^2(\mathcal{C}_\infty)}$. Note that in order to check that the limit is indeed a probability distribution, i.e. no mass has been lost in the limit, it is enough to check that $\det(I+K_\infty)$ goes to 1 as u goes to 0, which is readily verified.

Asymptotic analysis when $0 \leqslant a \leqslant 1/2$ Assume now that $0 \leqslant a \leqslant 1/2$ and $\theta > 0$ is fixed. Let $u = e^{2\sum_{i=1}^n \psi(\theta i^a) - r\sigma_n/\theta}$. Then, since σ_n goes to infinity,

$$\mathbb{E}\left[e^{-u\mathcal{Z}}\right] \simeq \mathbb{P}\left(\frac{\log \mathcal{Z}(n,n) + 2\sum_{i=1}^n \psi(\theta i^a)}{\sigma_n/\theta} \leqslant r\right), \tag{138}$$

as n goes to infinity, provided the left-hand-side converges to some probability distribution function (see e.g. [49, Lemma 4.1.39]).

We may analyze the Fredholm determinant $\det(I+K)_{\mathbb{L}^2(\mathcal{C})}$ by Laplace's method. Let us define $\mathcal{C}_a^\varphi$ to be an infinite contour in the complex plane going straight from $\infty e^{-\mathbf{i}\varphi}$ to a and then to $\infty e^{\mathbf{i}\varphi}$. Using (128) and rescaling variables near 0 by a factor σ_n/θ, we see that $\det(I+K)_{\mathbb{L}^2(\mathcal{C})}$ converges to $\det(I+K^{\mathrm{GUE}})_{\mathbb{L}^2(\mathcal{C}_0^{2\pi/3})}$ where

$$K^{\mathrm{GUE}}(v,v') = \int_{\mathcal{C}_1^{\pi/3}} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{v-z} \frac{1}{z-v'} \exp\left(\frac{z^3}{3} - rz - \frac{v^3}{3} + rv\right). \tag{139}$$

Note that the contours may be deformed to vertical lines as in the letter (26). We recognize a well-known kernel such that $\det(I+K^{\mathrm{GUE}}) = F_{\mathrm{GUE}}(r)$, the CDF of the Tracy-Widom GUE distribution. Indeed, using $\frac{-1}{v-z} = \int_0^\infty \exp(\lambda(v-z))\mathrm{d}\lambda$ for $\Re[v-z]<0$, we may factorize the kernel K^{GUE} as $K^{\mathrm{GUE}} = -AB$ where

$$A(v,\lambda) = \exp\left(\frac{-v^3}{3} + (r+\lambda)v\right), \quad B(\lambda',v') = \int_{\mathcal{C}_1^{\pi/3}} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{z-v'} \exp\left(\frac{z^3}{3} - (r+\lambda')z\right). \tag{140}$$

Using the identity $\det(I-AB) = \det(I-BA)$, we find that $\det(I+K^{\mathrm{GUE}}) = \det(I-K_{\mathrm{Ai}})_{\mathbb{L}^2(r,\infty)}$, where K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(\lambda,\lambda') = \int_{\mathcal{C}_0^{2\pi/3}} \frac{\mathrm{d}v}{2\mathbf{i}\pi} \int_{\mathcal{C}_1^{\pi/3}} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{z-v} \exp\left(\frac{z^3}{3} - \lambda z - \frac{v^3}{3} + \lambda' v\right). \tag{141}$$

Thus,

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\log \mathcal{Z}(n,n) + 2\sum_{i=1}^n \psi(\theta i^a)}{\sigma_n/\theta} \leqslant r\right) = \det(I+K^{\mathrm{GUE}}) = F_{\mathrm{GUE}}(r). \tag{142}$$

Recall that $\sigma_n^3 \simeq \frac{\theta}{(1-2a)} n^{1-2a}$, hence defining (minus) the free energy as $\mathcal{F}_n = \theta \log(\mathcal{Z}(n,n))$, the free energy fluctuations at large n are

$$\delta\mathcal{F}_n \simeq \frac{\theta^{1/3}}{(1-2a)^{1/3}} n^{\frac{1-2a}{3}} \chi_2. \tag{143}$$

As we have already mentioned, we omit here the mathematical details to prove the convergence of the Fredholm determinant. Let us simply observe that it is reasonable to replace $G(z)$ by the first terms in its Taylor expansion: indeed $G^{(5)}(0) = \mathcal{O}(n^{1-4a} + cst)$ which is negligible compared to σ_n^3.

In the case $a = 1/2$, σ_n^3 still diverges to $+\infty$ but slowly, in the scale $\log(n)$. The limit theorem still holds.

4) Zero-temperature limit

In the log-gamma polymer, weights w are distributed as inverse gamma random variables. Recall that if we write Boltzmann weights as $w = e^{E/\theta}$, where w is an inverse gamma random variable of parameter γ, then scaling $\gamma = \theta\tilde{\gamma}$, the variable E converges as θ goes to zero to an exponential random variable of parameter $\tilde{\gamma}$. We now study this limit for the inhomogeneous model.

Fix some $n \geqslant 1$. Consider the log-gamma polymer model with weights with parameter $\gamma_{i,j} = \theta(a_i + b_j)$. When θ goes to zero, $\mathcal{F}_n = \theta \log(\mathcal{Z}(n,n))$ weakly converges to $L(n,n)$, the last passage time from $(1,1)$ to (n,n) in a model with energies distributed as exponential random variables. More precisely,

$$L(n,n) = \max_{\pi:(1,1)\to(n,n)} \sum_{(i,j)\in\pi} E_{i,j} \tag{144}$$

where $E_{i,j}$ are independent exponential random variables with parameter $a_i + b_j$. This is the model studied in [13], which considers in particular the case $a_i = b_j = i^a$.

Let us scale u in (125) as $u = e^{-s\theta^{-1}}$. Then $\mathbb{E}[e^{-uZ}]$ converges to $\mathbb{P}(L(n,m) \leqslant s)$. It can be shown that the Fredholm determinant (124) converges as well so that

$$\mathbb{P}(L(n,m) \leqslant s) = \det(I + K^{\mathrm{LPP}})_{\mathbb{L}^2(\mathcal{C})}, \tag{145}$$

where

$$K^{\mathrm{LPP}}(v,v') = \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{v-z} \frac{1}{z-v'} e^{-s(z-v)} \prod_{i=1}^{n} \frac{i^a - v}{i^a - z} \prod_{j=1}^{m} \frac{j^a + z}{j^a + v}. \tag{146}$$

The contour $\mathcal{C}$ encloses all poles at $-j^a$ for all $j \geq 1$ and δ is chosen so that $\delta + \mathbf{i}\mathbb{R}$ passes to the right of the contour $\mathcal{C}$. Again, let us write $\frac{1}{v-z} = -\int_0^{+\infty} e^{x(v-z)}dx$, so that we can factorize the kernel as $K^{\mathrm{LPP}} = -AB$ with

$$A(v,x) = e^{xv+sv} \frac{\prod_{i=1}^{n} i^a - v}{\prod_{j=1}^{m} j^a + v}, \quad B(x',v') = \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{z-v'} e^{-sz-x'z} \frac{\prod_{j=1}^{m} j^a + z}{\prod_{i=1}^{n} i^a - z}. \tag{147}$$

Using the identity $\det(I - AB) = \det(I - BA)$ for Hilbert-Schmidt kernels, we may write $\mathbb{P}(L(n,m) \leqslant s) = \det(I - \tilde{K}^{\mathrm{LPP}})_{\mathbb{L}^2(s,\infty)}$ where

$$\tilde{K}^{\mathrm{LPP}}(x,y) = \int_{\mathcal{C}} \frac{\mathrm{d}v}{2\mathbf{i}\pi} \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{z-v} e^{-xz+yv} \prod_{i=1}^{n} \frac{i^a - v}{i^a - z} \prod_{j=1}^{m} \frac{j^a + z}{j^a + v}. \tag{148}$$

Note that $\tilde{K}^{\mathrm{LPP}}$ is the same kernel as in [13, Eq. 1.11]. We may deform the contours in (148) so that the contour for the variable z becomes $\mathcal{C}_{1/2}^{\pi/4}$ and the contour for the variable v becomes $\mathcal{C}_{-1/2}^{3\pi/4}$. The angles chosen do not matter much as long as the real part is increasing (resp. decreasing) along the tails of the z contour (resp. v contour). It was shown in [13] that the large time asymptotics of $L(n,n)$ depend on the value of a. Let $c_n = 2\sum_{i=i}^{n} i^{-a}$. For $a \in (1/3, 1)$, fluctuations of $L(n,n)$ are of order 1, and their distribution is characterized by the kernel

$$\lim_{n\to\infty} \tilde{K}^{\mathrm{LPP}}(x + c_n, y + c_n) = \int_{\mathcal{C}_{-1/2}^{3\pi/4}} \frac{\mathrm{d}v}{2\mathbf{i}\pi} \int_{\mathcal{C}_{1/2}^{\pi/4}} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{e^{-xv+yz}}{v-z} \frac{e^{F_{\theta\to 0}(v)}}{e^{F_{\theta\to 0}(z)}}, \tag{149}$$

where

$$F_{\theta\to 0}(z) = \sum_{k=1}^{\infty} \log\left(1 + \frac{z}{k^a}\right) - \log\left(1 - \frac{z}{k^a}\right) - \frac{2z}{k^a}. \tag{150}$$

We recover the cases a) and b) of [13, Theorem 1.1]. Note that in that work it is proved that the spatial behavior is different according to $1/3 < a < 1/2$ (non-trivial extended kernel) and $1/2 < a < 1$ (trivial extended kernel).

For $a \in [0, 1/3]$, the kernel $\tilde{K}^{\mathrm{LPP}}$ converges to the Airy kernel is the sense that

$$\lim_{n\to\infty} d_n \tilde{K}^{\mathrm{LPP}}(c_n + d_n x, c_n + d_n y) = \int_{\mathcal{C}_{-1}^{2\pi/3}} \frac{\mathrm{d}v}{2\mathbf{i}\pi} \int_{\mathcal{C}_{1}^{\pi/3}} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{z-v} \exp\left(\frac{z^3}{3} - xz - \frac{v^3}{3} + yv\right), \tag{151}$$

where $d_n = (2\log n)^{1/3}$ when $a = 1/3$ and $d_n = \left(\frac{2n^{1-3a}}{1-3a}\right)^{1/3}$ otherwise. This means that for $a \in [0, 1/3]$, $L(n,n) \simeq d_n \chi_2$ where χ_2 follows the Tracy-Widom GUE distribution.

5) Low temperature crossover

In this Section, we study the case where the parameter θ goes to zero simultaneously as n goes to infinity. Let us scale u as $u = e^{-s/\theta}$. Then $\mathbb{E}[e^{-uZ}]$ can be approximated by $\mathbb{P}(\theta \log(Z(n,n)) \leqslant s)$ with an error of order $\mathcal{O}(e^{-\theta^{-1}})$. It is convenient to rescale variables in the kernel (20) so that $\mathbb{E}[e^{-uZ}] = \det(I+K)$, where

$$K(v,v') = \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{\theta\pi}{\sin(\theta\pi(v-z))} \frac{1}{z-v'} e^{F_n(z)-F_n(v)} \tag{152}$$

with

$$F_n(z) = -zs + \sum_{i=1}^{n} \log\Gamma(\theta i^a - \theta z) - \log\Gamma(\theta i^a + \theta z). \tag{153}$$

We have already seen that by Taylor approximation,

$$F_n(z) = -zs - zf_n - \frac{z^3}{3}\sigma_n^3 + \mathcal{O}(\theta^5 z^5). \tag{154}$$

Let $a \in (1/3, 2/3)$. We know from the previous results that if θ goes to zero sufficiently fast, we should expect the free energy to behave as in the zero-temperature model, that is, converge to a non-universal distribution. If, however, the temperature goes to 0 slowly enough, we expect the free energy to behave still as if θ was fixed and thus have fluctuations following the Tracy-Widom distribution. We will see that the threshold arises for θ of order $\mathcal{O}(n^{-1+2a})$.

Let us scale θ as $\theta = An^{-c}$. We need to determine for which range of c the quantity σ_n converges or diverges. For $c \geqslant a$, it is not difficult to show that σ_n converges to a constant, using the asymptotics of the digamma function ($\psi''(x) \simeq -2/x^3$ as x goes to zero). Consider now $c < a$. We decompose the series as a sum $\sigma_n^3 = S_1 + S_2$ as

$$\sigma_n^3 = \underbrace{\sum_{i=1}^{n^{c/a}} -\theta^3 \psi''\left(A\left(\frac{i}{n^{c/a}}\right)^a\right)}_{(S_1)} + \underbrace{\sum_{i=n^{c/a}+1}^{n} -\theta^3 \psi''(Ai^a n^{-c})}_{(S_2)}. \tag{155}$$

Again, using approximation of the digamma function near 0 and Taylor-Maclaurin formula, one readily obtains that the first sum S_1 converges to a constant. Since $\psi''(x) \simeq \frac{-1}{x^2}$ as x goes to infinity, the sum S_2 is divergent only if

$$\sum_{i=n^{c/a}+1}^{n} \frac{\theta^3}{(Ai^a n^{-c})^2} = \sum_{i=n^{c/a}+1}^{n} \frac{An^{-c}}{i^{2a}} \simeq \frac{An^{-c}}{1-2a} n^{1-2a} \tag{156}$$

is divergent (recall that $a < 1/2$). Thus, the sum S_2, and consequently σ_n as well, is divergent only if $c < 1-2a$, in which case

$$\sigma_n^3 \simeq \frac{A}{1-2a} n^{1-2a-c}. \tag{157}$$

Otherwise, when $c \geqslant 1-2a$, σ_n converges to a constant. We may now adapt the asymptotic analysis performed above in the finite temperature case. For $a \in (1/3, 1/2)$ and $\theta = An^{-c}$, we have the following.

- If $c < 1-2a$, then

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{\mathcal{F}_n - f_n}{\sigma_n} \leqslant s\right) = F_{\mathrm{GUE}}(s). \tag{158}$$

  Hence the free energy $\mathcal{F}_n$ fluctuates as

$$\delta\mathcal{F}_n \simeq \frac{A^{1/3}}{(1-2a)^{1/3}} n^{\frac{1-2a-c}{3}} \chi_2. \tag{159}$$

- If $c > 1-2a$, then

$$\lim_{n\to\infty} \mathbb{P}\left(\mathcal{F}_n - f_n \leqslant s\right) = \det(I + K^{\theta\to 0}), \tag{160}$$

where

$$K^{\theta\to 0}(v,v') = \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{v-z}\frac{1}{z-v'} e^{-s(z-v)} e^{F_{\theta\to 0}(z)-F_{\theta\to 0}(v)} \tag{161}$$

with

$$F_{\theta\to 0}(z) = \lim_{n\to\infty} \sum_{i=1}^{n} \log\Gamma(\theta i^a - \theta z) - \log\Gamma(\theta i^a + \theta z) + 2z\theta\psi(\theta i^a), \tag{162}$$

$$= \sum_{i=1}^{\infty} \log\left(1+\frac{z}{i^a}\right) - \log\left(1-\frac{z}{i^a}\right) - \frac{2z}{i^a}. \tag{163}$$

Indeed, using the series representations (132) and (133),

$$\sum_{i=1}^{n} \log\Gamma(\theta i^a - \theta z) - \log\Gamma(\theta i^a + \theta z) + 2z\theta\psi(\theta i^a) = \sum_{i=1}^{n}\sum_{k=0}^{\infty} \log\left(1+\frac{\theta z}{\theta i^a + k}\right) + \log\left(1-\frac{\theta z}{\theta i^a+k}\right) - \frac{2\theta z}{\theta i^a + k}, \tag{164}$$

and only the terms corresponding to $k=0$ remain in the limit. We recover exactly (149) (150) which shows that the free energy fluctuations have the same distribution as in the zero temperature case.

- If $c = 1-2a$, we set $\theta = An^{-1+2a}$. Again we have that

$$\lim_{n\to\infty} \mathbb{P}\left(\mathcal{F}_n - f_n \leqslant s\right) = \det(I + K^{\mathrm{cross}})_{\mathbb{L}^2(\mathcal{C}_{-1/2}^{3\pi/4})}, \tag{165}$$

where

$$K^{\mathrm{cross}}(v,v') = \int_{\delta-\mathbf{i}\infty}^{\delta+\mathbf{i}\infty} \frac{\mathrm{d}z}{2\mathbf{i}\pi} \frac{1}{v-z}\frac{1}{z-v'} e^{-s(z-v)} e^{F_{\mathrm{cross}}(z)-F_{\mathrm{cross}}(v)} \tag{166}$$

with

$$F_{\mathrm{cross}}(z) = \lim_{n\to\infty} \sum_{i=1}^{n} \log\Gamma(\theta i^a - \theta z) - \log\Gamma(\theta i^a + \theta z) + 2z\theta\psi(\theta i^a). \tag{167}$$

The function F_{cross} interpolates between the zero temperature case (163) and a cubic behaviour as in the Airy kernel. It depends on A as

$$F_{\mathrm{cross}}(z) = \frac{Az^3}{3(1-2a)} + F_{\theta\to 0}(z). \tag{168}$$

Indeed,

$$\sum_{i=1}^{n} \log\Gamma(\theta i^a - \theta z) - \log\Gamma(\theta i^a + \theta z) + 2z\theta\psi(\theta i^a) =$$

$$\sum_{i=1}^{n}\sum_{k=0}^{\infty} \log\left(1+\frac{An^{-1+2a}z}{An^{-1+2a}i^a + k}\right) + \log\left(1-\frac{An^{-1+2a}z}{An^{-1+2a}i^a+k}\right) - \frac{2An^{-1+2a}z}{An^{-1+2a}i^a + k}. \tag{169}$$

In order to determine the limit, consider separately the case $k=0$, for which the main contribution is given by terms corresponding to small i, and the terms corresponding to $k \geqslant 0$ for which the main contribution comes from large i. The term $k=0$ simplifies and yields $F_{\theta\to 0}(w)$. In the terms corresponding to $k\geqslant 1$, we may use the Taylor expansion of the logarithm and series expansion of the polygamma function so that

$$(169) = -2\sum_{i=1}^{n}\sum_{j=1}^{\infty} \frac{z^{2j+1}(An^{-1+2a})^{2j+1}}{(2j+1)!} \psi^{(2j)}(An^{-1+2a}i^a + 1), \tag{170}$$

where we used the identity $\sum_{k=1}^{+\infty} \frac{1}{(k+x)^{2j+1}} = -\frac{1}{(2j)!}\psi^{(2j)}(1+x)$. Using the large x asymptotics $\psi^{(2j)}(x) \simeq -(2j-1)!/x^{2j}$, it is equivalent to

$$\sum_{j=1}^{\infty}\sum_{i=1}^{n} An^{-1+2a}\frac{z^{2j+1}}{j(2j+1)}\frac{1}{i^{2aj}}. \tag{171}$$

We see that only the term corresponding to $j=1$ will contribute to the limit and this yields

$$\sum_{i=1}^{n} An^{-1+2a}\frac{z^{2j+1}}{3}\frac{1}{i^{2a}} \xrightarrow[n\to\infty]{} \frac{Az^3}{3(1-2a)}. \tag{172}$$

Remark. As in Section II 2), the Fredholm determinant $\det(I+K^{\mathrm{cross}})$ can be written, using $\det(I+AB) = \det(I+BA)$ as a Fredholm determinant $\det(I+\tilde{K}^{\mathrm{cross}})$ where the kernel $\tilde{K}^{\mathrm{cross}}$ acts on $\mathbb{L}^2(\mathbb{R})$. The kernel $\tilde{K}^{\mathrm{cross}}$ is a limit of the Schur process correlation kernel, which usually occurs in zero temperature models. More specifically, $\tilde{K}^{\mathrm{cross}}$ corresponds to a limit of the so called Airy kernel with two sets of parameters introduced in [51, Remark 2].

6) Discrete model with arbitrary weight distribution and KPZ scaling theory

For a generic interface model in the KPZ univerality class, the interface height $h(x,t)$, starting from an initial condition in the droplet class, is expected under mild hypotheses to obey a limit theorem of the form [52]

$$h(vt,t) \simeq t\phi(v) + \left(\tfrac{1}{2}\lambda A^2 t\right)^{1/3}\chi_2, \tag{173}$$

for large times, where the function $\phi(v)$ and the coefficients λ, A are model-dependent and we provide their definition below. A necessary condition for this limit to hold is that the limit profile ϕ is curved at v. In certain cases, these coefficients can be computed explicitly. We refer the reader to [52] and [53] for details about KPZ scaling theory. The aim of this section is to explain how the KPZ scaling theory needs to be modified in the time dependent inhomogeneous case. We start by recalling KPZ scaling theory for directed polymers in the homogeneous case.

Homogeneous case. Directed polymer models fit in the KPZ scaling theory framework. Consider the partition function of a polymer model $\mathcal{Z}(n,m)$ as defined in the main text of the letter (18). As in the text of the letter, it will be more convenient to work with space-time coordinates $\tau = n+m$ and $\varkappa = n-m$ and we define $Z_d(\varkappa,\tau) = \mathcal{Z}(n,m)$.

In this context, the analogue of the interface height is the free energy, so that in this section,

$$h(\varkappa,\tau) = \log Z_d(\varkappa,\tau). \tag{174}$$

We define the slope field associated to h as $u(\varkappa,\tau) = \frac{1}{2}\left(h(\varkappa+1,\tau) - h(\varkappa-1,\tau)\right)$. Let us assume that translation invariant and stationary distributions of the slope field are known and parametrized by the density $\rho = \mathbb{E}[u(0,\tau)]$, and let us denote the corresponding measure by μ_ρ. For the log-gamma polymer with parameter γ, these stationary measures [21] are parametrized by a real number $\vartheta \in (0,\gamma)$, such that under μ_ρ, the slope field $u(\varkappa)$ is i.i.d. as $\varkappa$ varies, and distributed as

$$\frac{1}{2}\left(\log G(\gamma-\vartheta) - \log G(\vartheta)\right), \tag{175}$$

where $G(\gamma-\vartheta)$ and $G(\vartheta)$ are independent gamma distributed random variables with parameters respectively $\gamma-\vartheta$ and ϑ. Hence the density ρ is related to the parameter ϑ via

$$\rho = \frac{1}{2}\left(\psi(\gamma-\vartheta) - \psi(\vartheta)\right). \tag{176}$$

More precisely, the stationary measures introduced in [21] are such that increments $h(\tau,\varkappa) - h(\tau-1,\varkappa+1)$ are distributed as $-\log G(\gamma-\vartheta)$, and increments $h(\tau,\varkappa) - h(\tau,\varkappa-1)$ are distributed as $-\log G(\vartheta)$. For a fixed time τ, all these increments are independent as $\varkappa$ varies.

One also defines the instantaneous current $j(\rho)$, which equals the increment of $h(\varkappa,\tau)$ per unit of time under the stationary slope field μ_ρ. For the log-gamma polymer with parameter γ, we have

$$j(\rho(\vartheta)) = \frac{-1}{2}\mathbb{E}[\log\tilde{G}(\gamma-\vartheta) + \log\tilde{G}(\vartheta)] = \frac{-1}{2}\left(\psi(\gamma-\vartheta) + \psi(\vartheta)\right). \tag{177}$$

(Note that increments $-\log \tilde{G}(\gamma-\vartheta), -\log \tilde{G}(\vartheta)$ along the time direction are not independent, but this does not matter for the computations.)

In general, the function ϕ appearing in (173) is the Legendre transform of the function $j(\rho)$, that is [52, Eq. (3.13)]

$$\phi(v)=\inf_{\rho\in\mathbb{R}}\{v\rho-j(\rho)\}. \tag{178}$$

For the log-gamma polymer model, it can be computed explicitly, and we find that $\phi(0)=-\psi(\gamma/2)$. At velocity $v=0$, the density ρ that optimizes the variational problem above is $\rho=0$ and the corresponding value of ϑ is $\vartheta=\gamma/2$. More generally, the relation between v and ρ is determined by $v=\partial j/\partial\rho$.

Now we may explain the coefficients λ and A which appear in the magnitude of fluctuations in (173). We define the curvature of the limit shape λ by $\lambda\equiv\lambda(\rho)=j''(\rho)$. Implicitly, λ depends on the velocity v through the local density ρ around the location $\varkappa=\tau v$. The coefficient A is the integrated covariance of the slope field,

$$A\equiv A(\rho)=2\left(\sum_{j\in\mathbb{Z}}\mathbb{E}_{\mu_\rho}\left[u(0)u(j)\right]-\rho^2\right). \tag{179}$$

For the log-gamma polymer model, the stationary slope field is i.i.d. in space so that

$$A(\rho)=2\mathrm{Var}\left[\frac{1}{2}\left(\log G(\gamma-\vartheta)-\log G(\vartheta)\right)\right]=\frac{1}{2}\left(\psi'(\gamma-\vartheta)+\psi'(\vartheta)\right). \tag{180}$$

Note that our definition of A in (179) differs from the definition given in Eq. (2.8) of [52] by a factor 2. This is due to the fact that we work on the square lattice and strictly speaking, our height field $h(\varkappa,\tau)$ is defined only when $\varkappa$ and τ have the same parity. In any case, the coefficient A should measure the size of lateral increments under the stationary measure. More precisely, if the density field is distributed under μ_ρ, then the variance of the height difference $h(\varkappa)-h(\varkappa+d)$ between two points at distance d should scale as $A(\rho)d$ as d grows. It is then easy to check that (179) is the correct definition for discrete directed polymers.

Using (180) and (177), one obtains that for the log-gamma polymer model,

$$\frac{1}{2}\lambda A^2=\frac{A^2}{2}\frac{1}{\partial_\vartheta\rho}\frac{\partial}{\partial\vartheta}\left(\frac{1}{\partial_\vartheta\rho}\frac{\partial j}{\partial\vartheta}\right)=\frac{-1}{2}\frac{\psi'(\gamma-\vartheta)\psi''(\vartheta)+\psi'(\vartheta)\psi''(\gamma-\vartheta)}{\psi'(\gamma-\vartheta)+\psi'(\vartheta)}. \tag{181}$$

At $v=0$, the corresponding ϑ is $\vartheta=\gamma/2$ and one finds $\frac{1}{2}\lambda A^2=\frac{-1}{2}\psi''(\gamma/2)$, so that

$$h(0,t)\simeq -t\psi(\gamma/2)+\left(\frac{-1}{2}\psi''(\gamma/2)t\right)^{1/3}\chi_2, \tag{182}$$

as t goes to infinity, where we recall that h was defined in (174). The asymptotics (182) was first proved in [23] for $v=0$ and obtained for arbitrary v in [48, 65].

Let us consider a general polymer model with weights $w=e^{\mathcal{E}/T}$, where we assume that the distribution of energies $\mathcal{E}$ has variance 1. We will keep henceforth the notation $\mathcal{E}$ for on-site energies with are assumed to be normalized to have variance 1, while we use the letter E to denote on-site energies which may depend on some parameter γ, the location, etc. To put the log-gamma polymer in this framework, one has to assume that $P(\mathcal{E})$ depends on T (indeed, $\mathcal{E}$ is distributed as T times the log of an inverse gamma random variable with parameter γ). In any case, we define the temperature T as

$$T=\frac{1}{\sqrt{\mathrm{Var}\log w}}. \tag{183}$$

Since the variance of the log of an inverse gamma variable of parameter γ is given by $\psi'(\gamma)$ (see (121)), the parameter γ is related to the temperature T via $\psi'(\gamma)=1/T^2$, so that $\gamma\propto T$ for $\gamma\to 0$ and $\gamma\propto T^2$ for $\gamma\to\infty$.

For an arbitrary distribution on weights w with 5 finite moments, one expects the limit theorem (173) to still hold. Stationary distributions μ_ρ should exist under mild assumptions but they are in general not known explicitly so that we cannot compute exactly the coefficients λ and ρ.

However, in the large temperature regime, one expects, that as T goes to infinity

$$\frac{1}{2}\lambda A^2 \simeq \frac{2}{T^4}, \tag{184}$$

provided the distribution of weights w has a sufficient number of finite moments [64]. This estimate is based on the universality of convergence of directed polymer free energy at high temperature to the KPZ equation [3, 24] (see also [47]). In the case of the log-gamma polymer, one can check that indeed as γ goes to infinity,

$$\frac{1}{2}\lambda A^2 = \frac{-1}{2}\psi''(\gamma/2) \simeq \frac{2}{\gamma^2} \simeq \frac{2}{T^4}. \tag{185}$$

Inhomogeneous case. Let us consider now a polymer model with weights $w = e^{\mathcal{E}/T}$, where we assume that $T = T(\tau)$ may depend onthe location of the site through the time τ. Again, for the log-gamma polymer to fit in this framework, one has to assume that the distribution of E also depends on τ. The parameter γ of the log-gamma polymer now depends on τ, and it is related to the temperature via

$$\psi'(\gamma(\tau)) = \frac{1}{T(\tau)^2}. \tag{186}$$

In any case, $1/T(\tau) = \sqrt{\operatorname{Var}\log w}$ where the distribution of w is now τ dependent.

Let us focus on the fluctuations of $h(\varkappa = 0, \tau)$. In the time dependent case, we expect that the scalings in (173) will be modified, and the Tracy-Widom GUE limit distribution will occur only when the size of fluctuations of $h(0,\tau)$ grows to infinity as τ goes to infinity. Otherwise, the fluctuations of h would be determined by a finite number of weights and we expect a non-universal distribution.

The simplest functional of the fluctuations that is linear in time is the third cumulant, denoted $\langle h^3\rangle_c$. In the homogeneous case, $\langle h^3\rangle_c \propto \lambda A^2\tau$. We expect that in the time dependent case,

$$\langle h^3\rangle_c \propto \sum_{t=1}^{\tau} \frac{1}{2}\lambda(t)A(t)^2, \tag{187}$$

at least when the functions $\lambda(t)$ and $A(t)$ vary slowly enough, and the divergence of this quantity as τ goes to infinity is a necessary criteria for $h(0,\tau)$ to have Tracy-Widom distributed fluctuations. In particular, for the log-gamma polymer model,

$$\sum_{t=1}^{\tau} \frac{1}{2}\lambda(t)A(t)^2 = \frac{1}{2}\sum_{t=1}^{\tau} -\psi''(\gamma(t)/2). \tag{188}$$

Modulo some constant (due to the fact that we use the parameter τ instead of n), this sum is asymptotically equivalent to the quantity σ_n defined in (23), and we have seen that for the solvable inhomogeneous log-gamma polymer model studied in the previous sections, Tracy-Widom fluctuations occur if and only if σ_n diverges.

However, as we have already mentioned, explicit expressions of the quantities $\lambda(t)$ and $A(t)$ are, in general, not available. If inhomogeneities are chosen so that $T(\tau)$ goes to infinity as τ goes to infinity, we may use the estimate (184), and we find that

$$\sum_{t=1}^{\tau} \frac{1}{2}\lambda(t)A(t)^2 \simeq \sum_{t=1}^{\tau} \frac{2}{T(t)^4}, \tag{189}$$

whenever the series are divergent (when the series are convergent the series may converge to different values). Recall that $1/T^4 = (\operatorname{Var}\log w)^2$. Hence, a general criteria to predict the occurrence of Tracy-Widom fluctuations is whether the sum of the squares of logarithms of weights converges or diverges along the polymer path.

Example 1: For a model with weights $w = e^{\mathcal{E}/T}$ where $\mathcal{E}$ has a fixed distribution with variance 1 and $T(t) = t^{a'}$, the series $\sum \frac{1}{T(t)^4}$ is divergent for $a' \leqslant 1/4$ and convergent for $a' > 1/4$. Hence we expect that the free energy has Tracy-Widom fluctuations when $a' \leqslant 1/4$ and non-universal fluctuations determined by weights close to the origin when $a' > 1/4$.

Example 2: For the log-gamma polymer model with $\gamma(t) = t^a$. We have that $(\mathrm{Var}\log w)^2 = \psi'(\gamma)^2$. Since $\psi'(\gamma) \sim 1/\gamma$ as $\gamma \to \infty$, the series $\sum \psi'(\gamma)^2$ is divergent for $a \leqslant 1/2$ and convergent for $a > 1$. Hence we expect that the free energy has Tracy-Widom fluctuations when $a \leqslant 1/2$ and non-universal fluctuations determined by weights close to the origin when $a > 1/2$. This is exactly what we have proved for the model with inhomogeneities given as $\gamma_{i,j} = i^a + j^a$.

Example 3: For the (homogeneous) KPZ equation itself (i.e. (38) with all coefficients time independent) the KPZ scaling holds with $A = D/(2\nu)$ and here $D = 2c$. In the units used here (i.e. in (1) of the Letter) we thus have $\nu = 1$, $\lambda = 2$ and $A = c$, hence $\frac{1}{2}A^2\lambda = c^2$. The analogue of (187) for the time inhomogeneous KPZ equation $c \to c(t)$ thus yields $\langle h^3 \rangle_c \propto \int_0^t c(u)^2 du$. We have seen in some cases that the divergence of this quantity is the exact criterion for TW type fluctuations at large times (and believed to hold more generally).

III NUMERICAL RESULTS

We consider in this section two models:

- A zero temperature model, with on site energies distributed as exponential variables of parameter $\gamma_{i,j} = (i+j)^a$. This corresponds to the zero temperature limit of the log-gamma polymer model discussed in the letter. We denote by $L(n,n)$ (see (144)) the optimal energy (last passage time).
- A positive temperature model, with Boltmann weights $w = e^{E/T}$ with $T = 1$ where on site energies E are distributed as exponential variables of parameter $\gamma_{i,j} = (i+j)^{a'}$. We denote by $\mathcal{Z}(n,m)$ or $Z_d(\varkappa, \tau)$ its partition function, as in the letter.

The positive temperature model converges (in the limit $T \to 0$) to the zero temperature model, i.e. $a = a'$.

1) Zero-temperature model

In Fig. 1 in the letter, we have shown the difference between the empirical CDF of the optimal energy $L(n,n)$ (for the zero temperature model with $\gamma_{i,j} = (i+j)^a$) and the CDF of the Tracy-Widom GUE distribution, for various polymer lengths n, and $a = 0.3, 0.4$. The empirical CDF have been centered and scaled to compare to the Tracy-Widom CDF. We have reproduced the results along with the additional case $a = 0.2$ in Fig. 5. One can clearly see

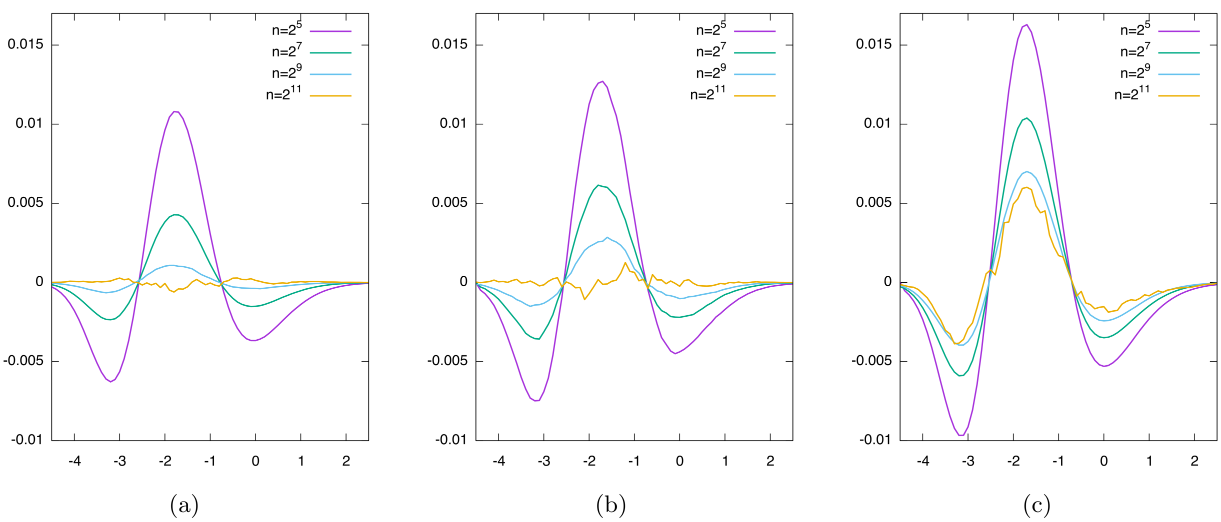


FIG. 5. Difference between the empirical CDF of the optimal energy in the zero temperature model (with exponential energies with parameter $\gamma_{i,j} = (i+j)^a$) and the CDF of the GUE TW distribution. The optimal energy is centered and scaled to have the same mean and variance as the GUE TW distribution. (a): case $a = 0.2$, for various polymer lengths n. (b): case $a = 0.3$ and the same polymer lengths. (c): case $a = 0.4$.

that regarding the bulk of the distribution, the convergence to the Tracy-Widom distribution seems to hold in the cases $a = 0.2$ and $a = 0.3 < 1/3$ but not in the case $a = 0.4 > 1/3$ (where the difference of CDF seems to converge to a non-zero limit).

We have also investigated the tail of the distributions in Fig. 6. We find that for $a = 0.2$ the tails of the empirical PDF of $L(n, n)$ (centered and scaled) seem to match the tails of the Tracy-Widom GUE distribution, while they are slightly different in the case $a = 0.4$.

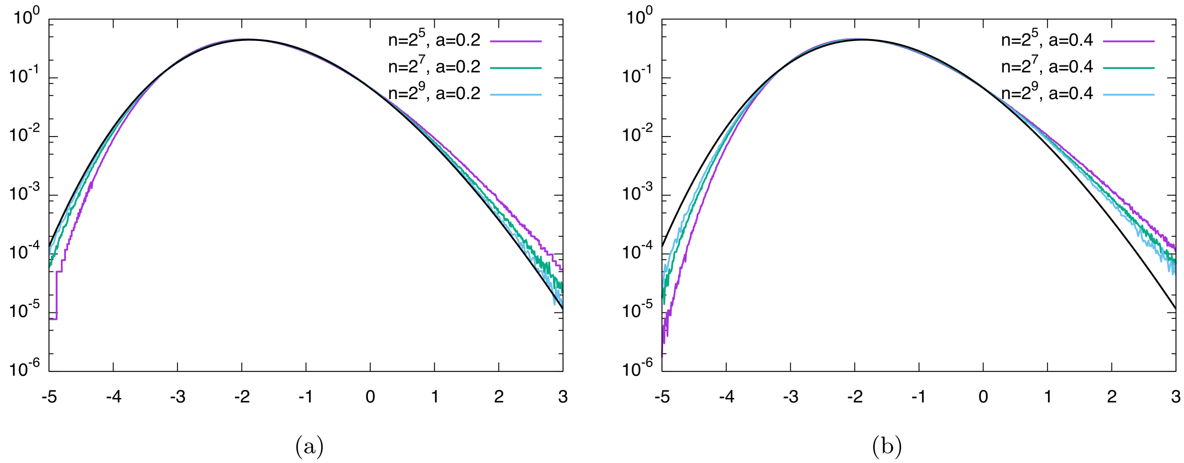


FIG. 6. Tails of the empirical PDF of the optimal energy in the zero temperature model (same as in Fig. 5), centered and scaled to the same mean and variance as the Tracy-Widom distribution (whose PDF is drawn in black for comparison). (a): case $a = 0.2$, for various polymer lengths n. (b): case $a = 0.4$.

2) Positive temperature model

Now we consider the positive temperature model. Let us emphasize that instead of simulating the log-gamma polymer model (which demands a lot of computational resources in order to reach large polymer sizes), we have performed simulations of the polymer model with Boltzmann weights $w = e^E$ where on-site energies E are distributed as exponential random variables with parameter $\gamma_{i,j} = (i + j)^{a'}$. According to the discussion made in Section II 6), we may view this parameter $\gamma_{i,j}$ as a (local) temperature, so that the model corresponds to example 1 of section II 6).

The results are shown in in Fig. 7. There is a strong evidence that the fluctuations are TW distributed for $a' = 0.2 < 1/4$ and converge to another limit when $a' = 0.3 > 1/4$ and $a' = 0.4$.

3) Profile of the third-cumulant as the temperature varies

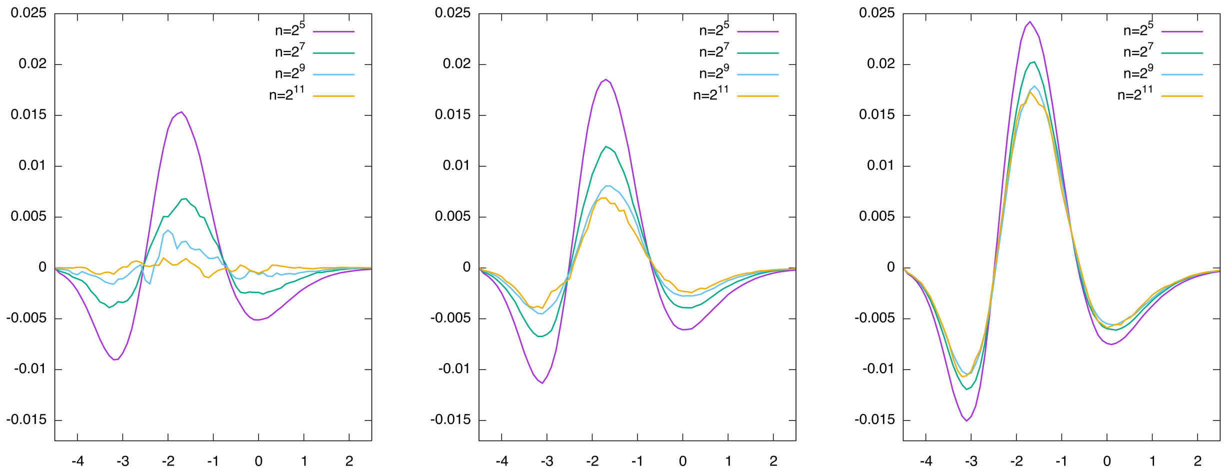
Previous sections indicate that the transition between Tracy-Widom fluctuations or non-universal ones occurs between $a = 0.3$ and $a = 0.4$ (we expect $a_c = 1/3$) for the zero temperature model considered in Section III 1), and between $a' = 0.2$ and $a' = 0.3$ (we expect $a'_c = 1/4$) for the positive temperature model considered in Section III 2).

Let us consider first the positive temperature model (with exponential on-site energies of parameter $\gamma_{i,j} = (i+j)^{a'}$). In order to confirm the critical value $a'_c = 1/4$, we use the criterium that Tracy-Widom fluctuations should occur if and only if $\langle \log Z_d(\varkappa = 0, \tau)^3 \rangle_c$ diverges as τ goes to infinity. Recall that for the homogeneous model with $\gamma_{i,j} \equiv \gamma$, we know that

$$\left\langle \log Z_d(\varkappa = 0, \tau)^3 \right\rangle_c \simeq \tau\, B(\gamma)\, \langle \chi_2^3 \rangle_c. \tag{190}$$

For the positive temperature model with inhomogeneous parameters $\gamma_{i,j} = (i + j)^{a'}$, we expect (as in Section II 6)) that

$$\left\langle \log Z_d(\varkappa = 0, \tau)^3 \right\rangle_c \propto \langle \chi_2^3 \rangle_c \sum_{t=1}^{\tau} B(t^{a'}). \tag{191}$$



(a) (b) (c)

FIG. 7. Difference between the empirical CDF of the positive temperature free energy and the CDF of the GUE TW distribution (centered and scaled to the same mean and variance). (a): case $a' = 0.2$, for various polymer lengths n. (b): case $a' = 0.3$ and the same polymer lengths. (c): case $a' = 0.4$.

To determine when the above series diverges, we need to estimate $B(\gamma)$. Note that from Section II 6), $B(\gamma) = \frac{1}{2}\lambda A^2 \left\langle \chi_2^3 \right\rangle_c$ but we cannot compute explicitly the coefficients λ, A for the positive temperature model with exponential on-site energies. However, we expect that

$$B(\gamma) \simeq \frac{2}{\gamma^4} \left\langle \chi_2^3 \right\rangle_c, \text{ as } \gamma \to \infty, \quad B(\gamma) \simeq \frac{8}{\gamma^3} \left\langle \chi_2^3 \right\rangle_c, \text{ as } \gamma \to 0. \tag{192}$$

Indeed, for the positive temperature model with homogeneous weights of parameter γ, we have that at large temperature $\frac{1}{2}\lambda A^2 \simeq \frac{2}{T^4}$ (see (184)). By our definition of the temperature (183), for this model the temperature is related to γ via $T = \gamma$, and we recover that $B(\gamma) \simeq \frac{2}{\gamma^4} \left\langle \chi_2^3 \right\rangle_c$ as in (192). Thus, we deduce that for the inhomogeneous positive temperature model with parameter $\gamma_{i,j} = (i+j)^{a'}$, the series (191) diverges if and only if $a' \leqslant 1/4$, hence the transition occurs at $a'_c = 1/4$.

Now we turn to the zero temperature model. We denote by $L(\varkappa, \tau)$ the optimal energy. Recall that for the homogeneous polymer model with $\gamma_{i,j} \equiv \gamma$, we have $\gamma \log Z_d(\varkappa, \tau) \simeq \mathcal{L}(\varkappa, \tau)$ for small γ, where $\mathcal{L}$ is the optimal energy in the homogeneous zero temperature model with exponential energies of parameter 1. For the latter model, it is known [12] that $\mathcal{L}(\varkappa = 0, \tau) \simeq 2\tau + 2\tau^{1/3}\chi_2$ at large times τ, hence for small γ, we have

$$B(\gamma) \simeq \frac{8}{\gamma^3} \left\langle \chi_2^3 \right\rangle_c, \tag{193}$$

as in (192). Going back to the inhomogeneous model with $\gamma_{i,j} = (i+j)^a$, we expect that

$$\left\langle L(\varkappa = 0, \tau)^3 \right\rangle_c \propto \langle \chi_2^3 \rangle_c \sum_{t=1}^{\tau} \frac{8}{t^{3a}}, \tag{194}$$

which diverges if and only if $a \leqslant 1/3$. Thus, for the zero temperature model, the transition occurs at $a_c = 1/3$.

We have checked that the estimates (192) are consistent with third cumulants obtained via numerical simulations in Fig. 8. Our theoretical predictions seem correct, although values of $B(\gamma)$ obtained by numerical simulation are slightly above the predicted ones – we believe that this is due to the small size of polymers that we have used (512) in order to produce Fig. 8.

IV CONVERGENCE TO THE KPZ EQUATION

In this section, we provide some details regarding the convergence of the discrete recurrence (32) to the SHE.

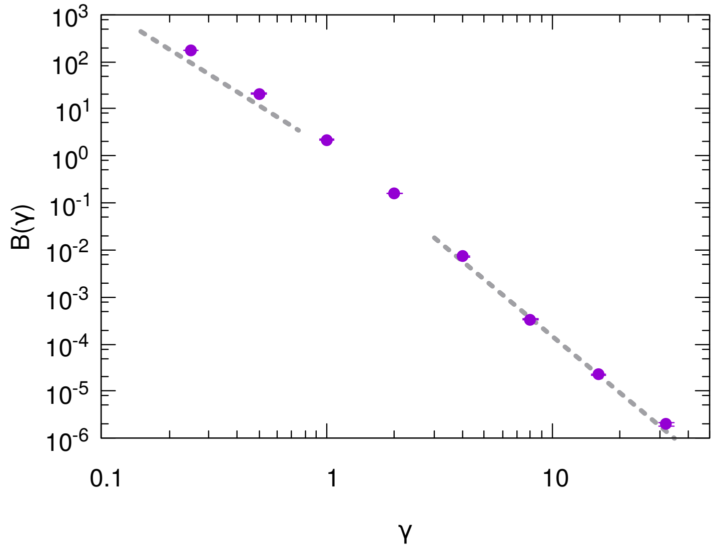


FIG. 8. Log-log plot of $B(\gamma)$ for various values of γ and directed polymers of length 512. The dotted lines correspond to the curves $\frac{2}{\gamma^4}\left\langle\chi_2^3\right\rangle_c$ for large γ and $\frac{8}{\gamma^3}\left\langle\chi_2^3\right\rangle_c$ for small γ.

1) Case $\gamma = \sqrt{n}/c\left(\frac{i+j}{2n}\right)$

Let us consider the scalings from (34) and assume that $\gamma_{\varkappa,\tau} = \sqrt{n}/c(t)$. Recall that we have set $\eta_{\varkappa,\tau} = \frac{2w_{\varkappa,\tau}}{C_\tau} - 1$ where $C_\tau = 2\mathbb{E}[w_{\varkappa,\tau}] = \frac{2}{\sqrt{n}/c(\tau/2n)-1}$. Using (119), we obtain that

$$\mathbb{E}[\eta_{\varkappa,\tau}] = 0, \quad \mathrm{Var}[n\eta_{\varkappa,\tau}] \simeq n^{3/2}c(t). \tag{195}$$

The family of independent variables $n\eta_{\varkappa,\tau}$ converges (in the sense of distributions) to a white noise $\xi(x,t)$ of variance $c(t)/2$. Recalling the rescaled partition function Z_r defined in the main text as $Z_r(\varkappa,\tau) = Z_d(\varkappa,\tau)\left(\prod_{s=1}^{\tau} C_s\right)^{-1}$, and multiplying (33) by n, we obtain

$$n\nabla_\tau Z_r(\varkappa,\tau) = \frac{1+\eta_{\varkappa,\tau}}{2}\, n\Delta_\varkappa Z_r(\varkappa,\tau-1) + n\eta_{\varkappa,\tau}\, Z_r(\varkappa,\tau-1). \tag{196}$$

Let us define $Z(x,t) = \lim_{n\to\infty} Z_r(2nt, x\sqrt{n})$. Assuming the limit exists and converges in a suitably strong sense, the continuum limit of (196) yields

$$\partial_t Z(x,t) = \partial_x^2 Z(x,t) + \sqrt{2c(t)}\xi(x,t)Z(x,t), \tag{197}$$

so that $h(x,t) = \log Z(x,t)$ solves

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \sqrt{2c(t)}\;\xi(x,t). \tag{198}$$

Our derivation of the continuum stochastic PDE (197) is somewhat formal. In order to make it more rigorous from the mathematical point of view, one should adapt the arguments of [24] dealing with the homogeneous case: write a a Feynman Kac-type representation for $Z_r(\tau,\varkappa)$, expand it as a chaos series, and justify that this series converges to the chaos series solution associated to (197).

2) Case $\gamma = \frac{\sqrt{n}}{2c\left(\frac{i}{n}\right)} + \frac{\sqrt{n}}{2c\left(\frac{j}{n}\right)}$

Previously, we had set up the renormalization factor C_τ so that the coefficient in front of the Laplacian – that is $\frac{1+\eta_{\varkappa,\tau}}{2}$ – has exactly mean 1/2. However now, the expectation of $w_{\varkappa,\tau}$ depends on both $\varkappa$ and τ, so that we cannot set $C_\tau = 2\mathbb{E}[w_{\varkappa,\tau}]$. We will choose C_τ as $C_\tau = \frac{2}{\sqrt{n}/c\left(\frac{i+j}{2n}\right)-1}$ so that it matches with $2\mathbb{E}[w_{\varkappa,\tau}]$ at the first order in n.

Recall that $i = (\tau + \varkappa)/2 = (2nt + \sqrt{n}x)/2$ and $j = (\tau - \varkappa)/2 = (2nt - \sqrt{n}x)/2$. Using (119) and Taylor expansion of the function $c(t)$, we find

$$\mathbb{E}[\eta_{\varkappa,\tau}] = x^2 \frac{c(t)c''(t) - 2(c'(t))^2}{8c(t)^2 n} + o(1/n), \quad \mathrm{Var}[n\eta_{\varkappa,\tau}] \simeq n^{3/2} c(t). \tag{199}$$

In this case, the family or random variables $n\eta_{\varkappa,\tau}$ converges to

$$\sqrt{c(t)/2}\xi(x,t) + \frac{c(t)c''(t) - 2(c'(t))^2}{8c(t)^2}. \tag{200}$$

so that in the continuum limit, (33) becomes

$$\partial_t Z(x,t) = \partial_x^2 Z(x,t) + \left(\sqrt{2c(t)}\xi(x,t) + x^2 \frac{c(t)c''(t) - 2(c'(t))^2}{4c(t)^2}\right) Z(x,t). \tag{201}$$

so that $h(x,t) = \log Z(x,t)$ solves

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + a_c(t)\frac{x^2}{2} + \sqrt{2c(t)}\,\xi(x,t), \tag{202}$$

where $a_c(t)$ was defined in (8).

3) Comparison of the two approaches: change of variable and discretization

In this section, we explain how the results obtained from asymptotic analysis of exact formulas for the log-gamma polymer in Section II are consistent with the results obtained from the change of variables on the KPZ equation. Recall that we defined in the letter two inhomogeneous DP models with log-gamma weights whose free energy converges to the inhomogeneous KPZ equation, Model I and Model II defined in (36).

In the case $a = 1/2$, the inhomogeneity parameters of Model II become

$$\gamma_{i,j} = \frac{1}{2}\left((i + t_0 n)^a + (j + t_0 n)^a\right) \tag{203}$$

On one hand, as t_0 goes to zero, we recover the model studied above, i.e. $\gamma = i^a + j^a$ (the factor $1/2$ is inconsequential) and we have seen that the free-energy fluctuations are Tracy-Widom distributed for large n as long as $a \leqslant 1/2$ (see (27)).

One the other hand, we have also seen in the previous section that the free energy of Model II converges to the solution of the inhomogeneous KPZ equation (2) with $c(t) = \frac{1}{(t+t_0)^a}$ and $V(x,t) = a_c(t)\frac{x^2}{2}$. By the change of variables (4), (5), the solution $h(x,t) = \log Z(x,t)$ where Z solves (3) with $V(x,t) = a_c(t)\frac{x^2}{2}$, can be mapped to the droplet solution of the homogeneous KPZ equation (1) at time $\tau(t) = \log(1 + t/t_0)$. As $t_0 \to 0$, $\tau(t)$ goes to $+\infty$ and we recover Tracy-Widom GUE fluctuations as well.

V BETHE ANSATZ

Most of the results of this section, which we give here for completeness, were obtained by T. Thiery.

1) Time inhomogeneous evolution

Consider $Z(x,t)$ satisfying the SHE equation (3) in the text with $V(x,t) = 0$ and a white noise variance $c(t)$. The moments

$$\mathcal{Z}(x_1, \dots, x_n; t) = \overline{Z(x_1,t)\dots Z(x_n,t)} \tag{204}$$

satisfy the imaginary time quantum mechanical evolution equation $\partial_t \mathcal{Z} = -H_n(t)\mathcal{Z}$ where $H_n(t)$ is a time dependent version of the attractive Lieb-Liniger Hamiltonian

$$H_n(t) = -\sum_{i=1}^{n} \partial_{x_i}^2 - 2c(t)\sum_{i<j} \delta(x_i - x_j) \tag{205}$$

We consider the infinite line. Let us denote $\Psi_\mu(c) \equiv \Psi_\mu(\vec{x}, c)$, with $\vec{x} = \{x_1, \dots, x_n\}$, the eigenstates for a fixed value of $c(t) = c$. They are known from the Bethe ansatz to be the string states, parameterized by (i) the (integer) number of strings $1 \le n_s \le n$ (ii) the (integer) sizes of each string $m_j \ge 1$, $j = 1, \dots, n_s$ with $\sum_{j=1}^{n_s} m_j = n$, (iii) the real momenta of each strings k_j. Their corresponding eigenenergy is $E_\mu = \sum_{j=1}^{n_s}(m_j k_j^2 - \frac{c^2}{12} m_j(m_j^2 - 1))$. We denote these eigenstate labels collectively as μ. We denote $\hat{\Psi}_\mu(c) = \Psi_\mu(c)/||\Psi_\mu(c)||$ the normalized eigenstates .

For the time dependent problem we are interested in the solution $\Psi(t) \equiv \Psi(\vec{x}, t)$ of

$$\partial_t \Psi(\vec{x}, t) = -H_n(t)\Psi(\vec{x}, t) \tag{206}$$

Since the $\Psi_\mu(c)$ form a basis for any c we can always decompose, for each t

$$\Psi(t) = \sum_\mu a_\mu(t)\hat{\Psi}_\mu(c(t)) \quad , \quad a_\mu(t) = \langle \hat{\Psi}_\mu(c(t))|\Psi(t)\rangle \tag{207}$$

Inserting into (206) one finds

$$\sum_\mu (\partial_t a_\mu(t)\hat{\Psi}_\mu(c(t)) + a_\mu(t)\dot{c}(t)\partial_c\hat{\Psi}_\mu(c(t))) = -\sum_\mu a_\mu(t)E_\mu(c(t))\hat{\Psi}_\mu(c(t)) \tag{208}$$

We can decompose

$$\partial_c\hat{\Psi}_\mu(c) = \sum_{\mu'} A_{\mu',\mu}(c)\hat{\Psi}_{\mu'}(c) \quad , \quad A_{\mu',\mu}(c) = \langle \hat{\Psi}_{\mu'}(c)|\partial_c\hat{\Psi}_\mu(c)\rangle \tag{209}$$

Note that since the $|\hat{\Psi}_\mu\rangle$ form an orthonormal basis, one has $\partial_c\langle\hat{\Psi}_{\mu'}|\hat{\Psi}_\mu\rangle = 0$ and the matrix $A_{\mu',\mu}(c)$ is anti-hermitian, i.e. $A_{\mu',\mu}(c) = -A^*_{\mu,\mu'}(c)$.

Projected on the basis we obtain the evolution equation for the $a_\mu(t)$ as

$$\partial_t a_\mu(t) = -E_\mu(c(t))a_\mu(t) - \dot{c}(t)\sum_{\mu'} A_{\mu,\mu'}(c(t))a_{\mu'}(t) \tag{210}$$

which is an exact equation.

In time-dependent problems the adiabatic limit is often discussed. In that limit $\dot{c}(t) \to 0$ and one may want to approximate $a_\mu(t) \simeq e^{-\int_0^t ds E_\mu(c(s))}a_\mu(0)$, leading to the adiabatic approximation

$$\Psi(t) \simeq \sum_\mu e^{-\int_0^t ds E_\mu(c(s))}\hat{\Psi}_\mu(c(t))\langle\hat{\Psi}_\mu(c(0))|\Psi(0)\rangle \tag{211}$$

Here we note that the factor

$$e^{-\int_0^t ds E_\mu(c(s))} = e^{-\tau(t)\sum_{j=1}^{n_s}(m_j^3 - m_j) - t\sum_{j=1}^{n_s} m_j k_j^2} \quad , \quad \tau(t) = \int_0^t ds\, c(s)^2 \tag{212}$$

depends both on the original time, and the "new" time $\tau(t)$ defined in the main text. Usually the adiabatic limit can be controlled when there is a gap in the spectrum [54]. Here however because of the momenta k_j the spectrum has a continuous part, so the general validity of the approximation is unclear. We point out further references [10, 55] on related questions.

2) One-string states

The calculation of $A_{\mu,\mu'}(c)$ is usually quite non-trivial. Consider now the *1 string states*, i.e. $n_s = 1$, where all n bosons are in a single bound state of momentum k. They are parameterized by a single momentum $\Psi_\mu \equiv \psi_k$. We now show that for these states

$$A_{k,k'}(c) = 0 \tag{213}$$

Indeed, the un-normalized 1string states are given by

$$\psi_k(\vec{x}) = n! e^{-\frac{c}{2}\sum_{i<j}|x_i-x_j|} e^{ik\sum_i x_i} \tag{214}$$

with energy $E = nk^2 - \frac{c^2}{12}n(n^2-1)$. They are orthogonal for $k \neq k'$, and their inverse square norm is $||\psi_k||^{-2} = \frac{c^{n-1}}{n!n^2}$. It is easy to see that one has the explicit form

$$\partial_c \psi_k(\vec{x}) = -\frac{1}{2}\sum_{i<j}|x_i - x_j|\, \psi_k(\vec{x}) \tag{215}$$

Hence one has

$$\partial_c \hat{\psi}_k(\vec{x}) = f(\vec{x})\hat{\psi}_k(\vec{x}) \tag{216}$$

where the function $f(\vec{x}) = -\frac{1}{2}\sum_{i<j}|x_i - x_j| + ||\psi_k(c)||\partial_c||\psi_k(c)||^{-1} = -\frac{1}{2}\sum_{i<j}|x_i - x_j| + \frac{1}{2}(n-1)c$, is real and independent of k. From this it follows that the matrix $A_{k',k}(c) = \langle \hat{\psi}_{k'}(c)|\partial_c \hat{\psi}_k(c)\rangle = \int d\vec{x} \hat{\psi}^*_{k'}(\vec{x},c) f(\vec{x}) \hat{\psi}_k(\vec{x},c) = A^*_{k,k'}(c)$. Since it must also be anti-hermitian, it follows that $A_{k',k}(c) = 0$.

Tail of the one-point PDF. It has been found, in a number of situations in the time-independent case (see discussion in [56–58]) that if one restricts the sum over the eigenstates μ to the subspace of the single string states $n_s = 1$, one obtains, in the large time limit, the exact right tail of the PDF of $h(0,t)$ for large positive values. It is thus tempting to surmise, heuristically, that a similar feature holds in the time-dependent case. Since $A_{k,k'}(c) = 0$, if one projects the evolution onto this subspace, the evolution is then identical to the adiabatic one (211)-(212), where the sum over states $\sum_\mu$ becomes an integral $\int \frac{dk}{2\pi}$ since the states are labeled by k. It would then lead to the conjecture that the leading (stretched exponential) behavior of the tail is the same as the one for the time independent case, up to the replacement of $c^2 t$ by $\int_0^t du c(u)^2$, that is replacement of t by $\tau(t)$. Indeed the integration over the momentum k in (212) only leads to simple prefactors depending on the initial condition [58]. Note that, since there is a gap between the (low lying) 1-string states and the other eigenstates (with two or more strings), some further control on the full problem may even be possible in the adiabatic limit. This is the case notably for the flat initial condition, when there is a clean gap between the ground state (a single string with $m_1 = n$ bosons and zero momentum $k = 0$), and the first excited states (a two string state) [35].

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[64] It is conjectured that the minimum number of finite moments needed is 6 [24, 59].
[65] Note that in [48] the discrete space variable $\varkappa$ is defined as $(m-n)/2$ hence φ there is equivalent to $v/2$ there, and ϑ here identifies with $\gamma/2+k_\varphi$ there.