A short proof of a symmetry identity for the $q$-Hahn distribution

Guillaume Barraquand

Abstract

We give a short and elementary proof of a symmetry identity for the $q$-moments of the $q$-Hahn distribution arising in the study of the $q$-Hahn Boson process and the $q$-Hahn TASEP. This identity discovered by Corwin in "The $q$-Hahn Boson Process and $q$-Hahn TASEP", Int. Math. Res. Not., 2014, was a key technical step to prove an intertwining relation between the Markov transition matrices of these two classes of discrete-time Markov chains. This was used in turn to derive exact formulas for a large class of observables of both these processes.

Keywords: Markov duality; $q$-Hahn process.

AMS MSC 2010: 60J10; 33D45.


Introduction

Zero-range and exclusion processes are generic stochastic models for transport phenomena on a lattice. Integrability of these models is an important question. In a short letter [5], Evans, Majumdar and Zia considered spatially homogeneous discrete time zero-range processes on periodic domains. They addressed and solved the question of characterizing the jump distributions for which invariant measures are product measures. Povolotsky [7] further examined the most general form of jump distributions allowing solvability by Bethe ansatz, and found a family depending on three real parameters $q$, $\mu$ and $\nu$, later called the $q$-Hahn distribution. In the same article [7], he also studied the corresponding $q$-Hahn Boson process and $q$-Hahn TASEP, and conjectured exact formulas for the models on the infinite lattice.

Using a Markov duality between the $q$-Hahn Boson process and the $q$-Hahn TASEP, Corwin [4] showed a variant of these formulas and provided a method to compute a large class of observables. This can be seen as a generalization of a similar work on $q$-TASEP and $q$-Boson process performed in [3, 2]. In his proof, the intertwining relation between the two Markov transition matrices essentially boils down to a symmetry identity verified by the $q$-moments of the $q$-Hahn distribution [4, Proposition 1.2]. The proof was adapted from [2, Lemma 3.7] which is the $\nu = 0$ case, and required the use of Heine’s summation formula for the basic hypergeometric series $\phi_1$. In the following, we give a new proof of this identity.

*Université Paris-Diderot, France. E-mail: barraquand@math.univ-paris-diderot.fr
A short proof of a symmetry identity for the \( q \)-Hahn distribution

A symmetry property for the \( q \)-moments of the \( q \)-Hahn distribution

First, we define the three parameter deformation of the Binomial distribution introduced in [7].

**Definition 0.1.** For \(|q| < 1, 0 \leq \nu \leq \mu < 1\) and integers \(0 \leq j \leq m\), define the function

\[
\varphi_{q,\mu,\nu}(j|m) = \mu^{j} \frac{(\nu/\mu; q)_j (\mu; q)_{m-j}}{(\nu; q)_m} \left[ \begin{array}{c} m \\ j \end{array} \right]_q,
\]

where

\[
\left[ \begin{array}{c} m \\ j \end{array} \right]_q = \frac{(q; q)_m}{(q; q)_j (q; q)_{m-j}}
\]

are \( q \)-Binomial coefficients with, as usual,

\[
(z; q)_n = \prod_{i=0}^{n-1} (1 - q^i z).
\]

It happens that for each \( m \in \mathbb{N} \cup \{\infty\}\), this defines a probability distribution on the set \( \{0, \ldots, m\} \). The weights \( \varphi_{q,\mu,\nu}(j|m) \) are very closely related to the weights associated with the \( q \)-Hahn orthogonal polynomials (see (7.2.22) in [6]), hence the use of the name \( q \)-Hahn.

**Lemma 0.2** (Lemma 1.1, [4]). For any \(|q| < 1\) and \(0 \leq \nu \leq \mu < 1\),

\[
\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1.
\]

**Proof.** As shown in [4], this equation is equivalent to a specialization of some known summation formula for basic hypergeometric series \( _2\phi_1 \) (Heine’s \( q \)-generalizations of Gauss’ summation formula).

We now state and prove the main identity.

**Proposition 0.3** (Proposition 1.2, [4]). Fix \(|q| < 1\) and \(0 \leq \nu \leq \mu < 1\). Let \( X \) (resp. \( Y \)) be a random variable following the \( q \)-Hahn distribution on \( \{0, \ldots, x\} \) (resp. \( \{0, \ldots, y\} \)). We have

\[
E[ q^{Y} ] = E[ q^{X} ].
\]

**Proof.** Let \( S_{x,y} := \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x)q^{jy} \). We have to show that \( S_{x,y} = S_{y,x} \) for all integers \( x, y \geq 0 \). Our proof is based on the fact that \( S_{x,y} \) satisfies a recurrence relation which is invariant when exchanging the roles of \( x \) and \( y \). First notice that by Lemma 0.2, \( S_{x,0} = 1 \) for all \( x \geq 0 \), and by definition \( S_{0,y} = 1 \) for all \( y \geq 0 \).

The Pascal identity for \( q \)-Binomial coefficients, (see 10.0.3 in [1]),

\[
\left[ \begin{array}{c} x+1 \\ j \end{array} \right]_q = \left[ \begin{array}{c} x \\ j \end{array} \right]_q q^j + \left[ \begin{array}{c} x \\ j-1 \end{array} \right]_q,
\]

yields

\[
S_{x+1,y} = \sum_{j=0}^{x+1} \mu^{j} \frac{(\nu/\mu; q)_j (\mu; q)_{x+1-j}}{(\nu; q)_{x+1}} \left[ \begin{array}{c} x \\ j \end{array} \right]_q q^j q^j + \sum_{j=0}^{x+1} \mu^{j} \frac{(\nu/\mu; q)_j (\mu; q)_{x+1-j}}{(\nu; q)_{x+1}} \left[ \begin{array}{c} x \\ j-1 \end{array} \right]_q q^j q^j,
\]

\[
= \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) \frac{1 - \mu q^{x-j}}{1 - \nu q^j} q^j q^j + \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) \mu \frac{1 - \nu q^j}{1 - \nu q^j} q^j q^j.
\]
A short proof of a symmetry identity for the \(q\)-Hahn distribution

The last equation can be rewritten

\[
(1 - \nu q^x) S_{x+1,y} = (S_{x,y+1} - \mu q^y S_{x,y}) + (\mu q^y (S_{x,y} - \nu/\mu S_{x,y+1}))
\]
\[
= (S_{x,y+1} + \mu (q^y - q^x) S_{x,y}).
\]

Thus, the sequence \((S_{x,y})_{(x,y)\in\mathbb{N}^2}\) is completely determined by

\[
\begin{cases}
(1 - \nu q^x) S_{x+1,y} = (1 - \nu q^y) S_{x,y+1} + \mu (q^y - q^x) S_{x,y}, \\
S_{x,0} = S_{0,y} = 1.
\end{cases}
\tag{0.1}
\]

Setting \(T_{x,y} = S_{y,x}\), one notices that the sequence \((T_{x,y})_{(x,y)\in\mathbb{N}^2}\) enjoys the same recurrence, which concludes the proof. \(\square\)

**Remark 0.4.** To completely avoid the use of basic hypergeometric series, one would also need a similar proof of the Lemma above. One can prove the result by recurrence on \(m\) (as in the proof of [2, Lemma 1.3]), but the calculations are less elegant when \(\nu \neq 0\).

More precisely, fix some \(m\) and suppose that for any \(0 \leq \nu \leq \mu < 1\), \(S_{m,0}(q,\mu,\nu) := \sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1\). Pascal’s identity yields

\[
S_{m+1,0}(q,\mu,\nu) = \frac{1 - \mu}{1 - \nu} S_{m,0}(q,\mu,\nu) + \sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) \mu \frac{1 - \nu/\mu q^j}{1 - \nu q^m},
\]
\[
= \frac{1 - \mu}{1 - \nu} S_{m,0}(q,\mu,\nu) + \frac{\mu}{1 - \nu q^m} (S_{m,0}(q,\mu,\nu) - \nu/\mu S_{m,1}(q,\mu,\nu)).
\]

Then, using the recurrence formula (0.1) for \(S_{m,1}(q,\mu,\nu)\), and applying the recurrence hypothesis, one obtains \(S_{m+1,0}(q,\mu,\nu) = 1\).

**References**


**Acknowledgments.** The author would like to thank his advisor Sandrine Péché for her support.
Advantages of publishing in EJP-ECP

• Very high standards
• Free for authors, free for readers
• Quick publication (no backlog)

Economical model of EJP-ECP

• Low cost, based on free software (OJS\textsuperscript{1})
• Non profit, sponsored by IMS\textsuperscript{2}, BS\textsuperscript{3}, PKP\textsuperscript{4}
• Purely electronic and secure (LOCKSS\textsuperscript{5})

Help keep the journal free and vigorous

• Donate to the IMS open access fund\textsuperscript{6} (click here to donate!)
• Submit your best articles to EJP-ECP
• Choose EJP-ECP over for-profit journals

\textsuperscript{1}OJS: Open Journal Systems http://pkp.sfu.ca/ojs/
\textsuperscript{2}IMS: Institute of Mathematical Statistics http://www.imstat.org/
\textsuperscript{3}BS: Bernoulli Society http://www.bernoulli-society.org/
\textsuperscript{4}PK: Public Knowledge Project http://pkp.sfu.ca/
\textsuperscript{5}LOCKSS: Lots of Copies Keep Stuff Safe http://www.lockss.org/
\textsuperscript{6}IMS Open Access Fund: http://www.imstat.org/publications/open.htm