ASEP on a half-space with an open boundary

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Joint work with Alexei Borodin, Ivan Corwin and Michael Wheeler.
Many two-dimensional statistical mechanics systems can be characterized by the space time evolution of a height function \( h \).

**Universality**

Models in 1 + 1 dimensions having a local radial growth mechanism driven by uncorrelated noise form the **KPZ universality class**, and share the same large time/space scale behavior (under mild hypotheses) characterized by 1/3 scaling exponents and Tracy-Widom type statistics.

**Kardar-Parisi-Zhang SPDE (1986)**

\[
\partial_t h = \partial_{xx} h + (\partial_x h)^2 + \mathcal{W}
\]
The nature of large time/scale fluctuations of $h(x, \tau)$ does not depend on the specifics of the growth mechanism, but does depend on:

- Initial conditions,
- The spatial domain, it can be $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $[0,1]$, $\mathbb{Z}_{>0}$, $\mathbb{R}_{>0}$, ...).

Three main cases are the full line, the circle, bounded spaces.

In this talk, we focus on half-spaces $\mathbb{Z}_{>0}$ or $\mathbb{R}_{>0}$ and explain how the fluctuations depend on the boundary condition and the distance to the boundary.

**Stochastic integrable systems**

So far, very few systems are amenable for detailed analysis. These systems are exactly solvable, using techniques from representation theory (symmetric functions) and integrable systems (Yang-Baxter equation).

We will focus on last passage percolation and ASEP.
Last passage percolation

Let $w_{ij}$ a family of iid exponential random variables with rate 1.

Last passage percolation is described by last passage times

$$H(n,m) := \max_{\pi:(1,1) \to (n,m)} \sum_{(i,j) \in \pi} w_{ij}.$$ 

**Theorem (Johannson 2001)**

*For the model in a quadrant,*

$$H(n,n) - 4n \xrightarrow{2^{4/3}n^{1/3}} n \to \infty \mathcal{L}_{\text{GUE}}.$$
Let $w_{ij}$ a family of iid exponential random variables with rate 1.

TASEP is described by the integrated current

$$N_x(\tau) := \# \{ \text{particles on the right of } x \text{ at time } \tau \}.$$ 

**Theorem (Johannsson 2001)**

Starting from step initial condition,

$$\frac{N_0(\tau) - \frac{\tau}{4}}{2^{-4/3} \tau^{1/3}} \xrightarrow[\tau \to \infty]{} -\mathcal{L}_{\text{GUE}}.$$
Let $t \in (0, 1)$ be an asymmetry parameter. For the talk, time will be denoted $\tau$.

**Theorem (Tracy-Widom 2008)**

*Starting from step initial condition,*

$$
\frac{N_0 \left( \frac{\tau}{1-t} \right) - \frac{\tau}{4}}{2^{-4/3} \tau^{1/3}} \xrightarrow{\tau \to \infty} -\mathcal{L}_{GUE}.
$$

Tracy and Widom’s proof relies on Bethe ansatz and a tricky analysis of Fredholm determinants. Recently, ASEP has been approached via a more general stochastic integrable system: the **stochastic six-vertex model** (Borodin-Corwin-Gorin 2014, Borodin-Olshanski 2016, Aggarwal 2016).
KPZ growth in a half space

KPZ equation on the positive reals (Corwin-Shen 2016)

Scaling limit of ASEP suggest that a natural boundary condition is of Neumann type:

\[
\begin{align*}
\partial_t h &= \partial_{xx} h + (\partial_x h)^2 + \dot{\mathcal{W}} \\
\partial_x h(x, \tau) \bigg|_{x=0} &= \alpha.
\end{align*}
\]
Let $w_{ij}$ a family of i.i.d. exponential random variables with rate 1 when $i > j$ and $\alpha$ when $i = j$.

Consider directed paths $\pi$ from the box $(1, 1)$ to $(n, m)$ in the half quadrant. We define the last passage percolation time $H(n,m)$ by

$$H(n,m) = \max_{\pi} \sum_{(i,j) \in \pi} w_{ij}.$$
What is known for LPP in a half-quadrant?

Mostly everything.

- Baik-Rains (2001) studied the model with geometric weights and proved limit theorems for $H(n,n)$. There is a phase transition between Tracy-Widom GSE, GOE and Gaussian fluctuations as the size of weights on the boundary varies. The proof relies on RSK algorithm, orthogonal polynomials and asymptotic analysis of Riemann-Hilbert problems.

- Imamura-Sasamoto (2004) use Pfaffian point processes to study the spatial correlations of the height function off diagonal, and obtain (via formal asymptotic analysis) correlation kernels related to models of random matrices interpolating between the classical Gaussian ensembles.

- Baik-B-Corwin-Suidan (2016) extend results to the exponential case and a larger set of parameters using Pfaffian Schur processes. Initial motivation was related to the non-universal behavior of the leading particle in an exclusion processes.
What is known for ASEP in a half-space?

Much less...

- Liggett 1975 classified the stationary measures when \( \alpha = \lambda \) and \( \gamma = t(1 - \lambda) \). There is a phase transition at \( \lambda = 1/2 \) between product-form Bernoulli measure and spatially correlated stationary measures.

- Tracy-Widom 2013 used Bethe ansatz to find formulas for the transition probabilities, not amenable to asymptotic analysis though.

- We will attack the problem through a half space version of the stochastic six-vertex model (that will be defined later).
Main result

Recall $N_x(\tau)$ is the number of particles on the right of site $x$ at time $\tau$.

Theorem (B.-Borodin-Corwin-Wheeler 2017+)

For $\lambda = 1/2$ and any $t \in [0, 1)$, starting from the empty initial condition,

$$
\frac{N_0 \left( \frac{T}{1-t} \right) - \frac{T}{4}}{2^{-4/3} T^{1/3}} \xrightarrow{T \to \infty} -\mathcal{L}_{\text{GOE}}.
$$

- Our method also allows to access the distribution of the KPZ equation on $\mathbb{R}_{>0}$ with Neumann boundary condition $h'(0) = 0$. (in preparation)

- One expects diffusive fluctuations in the low density phase $\lambda < 1/2$ and GSE fluctuations in the high density phase $\lambda > 1/2$. 
Hierarchy of integrable structures at play

Pfaffian Macdonald processes
(Borodin-B-Corwin-Wheeler)

Refined Littlewood identity

Pfaffian Schur process
(Borodin-Rains 2005)

Pfaffian Hall-Littlewood process

Stochastic six-vertex model in a half space

Pfaffian Higher spin system (in progress)

Last passage percolation, TASEP, ...

Half space ASEP

KPZ on $\mathbb{R}_{>0}$

More general structures

More specific models

$q = t$

$q = 0$

Is related to

Limits to
Plan for the rest of the talk

- **LPP in a half-quadrant** is the simplest benchmark model for understanding KPZ growth in a half space. We will state limit theorems and discuss the **Baik-Rains phase transition**.
- Introduce **Pfaffian Schur and Macdonald processes**.
- Then we will **reduce the study of the current in ASEP to the asymptotic analysis of Schur process correlation functions**.
Let $w_{ij}$ a family of i.i.d. exponential random variables with rate 1 when $i > j$ and $\alpha$ when $i = j$.

\[ H(n, m) = \max_\pi \sum_{(i,j) \in \pi} w_{ij}. \]
Passage-times on the diagonal

The fluctuations of the last passage time on the boundary depend on the size of boundary weights.

**Theorem (Baik-B.-Corwin-Suidan 2016)**

- **When \( \alpha > 1/2 \),**
  \[
  \frac{H(n,n) - 4n}{2^{4/3} n^{1/3}} \Rightarrow \mathcal{L}_{\text{GSE}},
  \]

- **When \( \alpha = 1/2 \),**
  \[
  \frac{H(n,n) - 4n}{2^{4/3} n^{1/3}} \Rightarrow \mathcal{L}_{\text{GOE}},
  \]

- **When \( \alpha < 1/2 \),**
  \[
  \frac{H(n,n) - cn}{c' n^{1/2}} \Rightarrow \mathcal{N},
  \]

The results are easily adapted to TASEP current fluctuations.

One expects to see the same phase transition in ASEP (consistent with the change of stationary measures already observed by Liggett).
The fact that $H(n,n) \sim 4n$ shows that the weights along the optimal path have size 2 in average. Thus, the disorder on the boundary becomes competitive when it has average at least 2, hence the transition at $\alpha = 1/2$.

Algebraic considerations show that for any $\alpha$, the law of $H(n,n)$ in the model with weight $\text{Exp}(\alpha)$ on the diagonal is the same as the law of $H(n,n)$ in a modified model where the weights on the boundary are $\text{Exp}(1)$ and the weights on the first row are $\text{Exp}(\alpha)$.

Open question: Is there a probabilistic proof?

Open question: In the critical case, geodesics take $\mathcal{O}(n^{1/3})$ weights on the diagonal. Where?
Passage times away from the boundary

**Theorem (Baik-B.-Corwin-Suidan 2016)**

For \( \kappa \in (0, 1) \) and \( \alpha > \sqrt{\kappa}/(1 + \sqrt{\kappa}) \),

\[
\frac{H(n, \kappa n) - (1 + \sqrt{\kappa})^2 n}{\sigma n^{1/3}} \Rightarrow \mathcal{L}_{\text{GUE}}.
\]

- One recovers the exact same result as for LPP in a full quadrant. The boundary has no influence as long as the boundary weights are not too big.
- If \( \alpha \) decreases (i.e. boundary weights increase) the fluctuations should transition between GUE Tracy-Widom and Gaussian, with \( F_{\text{GOE}}^2 \) fluctuations when \( \alpha = \sqrt{\kappa}/(1 + \sqrt{\kappa}) \). This is the Baik-Ben Arous-Péché (2005) phase transition already observed in the full space case.
Crossovers

Consider two parameters $\omega \in \mathbb{R}, \eta > 0$.

**Theorem (BBCS 2016)**

*When the boundary parameter scales as*

$$\alpha = \frac{1}{2} + 2^{-4/3} \omega n^{-1/3},$$

*and one consider passage times at distance $\eta n^{2/3}$ from the boundary,*

$$H_n(\eta, \omega) := \frac{H(n + 2^{2/3} \eta n^{2/3}, n - 2^{2/3} \eta n^{2/3}) - 4n + n^{1/3} 2^{4/3} \eta^2}{2^{4/3} n^{1/3}},$$

*The (multipoint) limiting distribution of $H_n(\eta, \omega)$ is a new two-parametric distribution that interpolates between GUE, GOE and GSE Tracy-Widom distribution.*

- It is related to RMT models interpolating between Unitary, Orthogonal and Symplectic Gaussian ensembles.
Symmetric functions

For integer partitions $\lambda_1 \geq \lambda_2 \geq \ldots$, and $\mu_1 \geq \mu_2 \geq \ldots$, we will consider skew-Schur functions

$$s_{\lambda/\mu} = \det [h_{\lambda_i - \mu_j + j - i}]_{i,j},$$

where $h_k$ are complete homogeneous symmetric functions

$$h_k(x) = \sum_{i_1 \leq \ldots \leq i_k} x_{i_1} \ldots x_{i_k}.$$

We also define

$$\tau_\lambda = \sum_{\kappa' \text{ even}} s_{\lambda/\kappa}$$

where $\kappa'$ even means that $\kappa_1 = \kappa_2 \geq \kappa_3 = \kappa_4 \geq \ldots$

$$\tau_\lambda = \text{Pf} \left[ \sum_{a \in \mathbb{Z}} h_{\lambda_i - i - a - 1} h_{\lambda_j - j - a} - h_{\lambda_j - j - a - 1} h_{\lambda_i - i - a} \right]_{i,j}$$
The **Schur measure** (Okounkov 2001) is a probability measure on partitions $\lambda := \lambda_1 \geq \lambda_2 \geq \ldots \geq 0$ such that

$$
\mathbb{P}(\lambda) = \frac{1}{\Pi(a;b)} s_\lambda(a_1, \ldots, a_n) s_\lambda(b_1, \ldots, b_n).
$$

where

$$
\Pi(a, b) = \prod_{i,j} \frac{1}{1 - a_i \lambda_j}.
$$

The **Pfaffian Schur measure** (Borodin-Rains 2005) is a probability measure on partitions $\lambda$ such that

$$
\mathbb{P}(\lambda) = \frac{1}{\Pi(a;b)\Phi(a)} s_\lambda(a_1, \ldots, a_n) \tau_\lambda(b_1, \ldots, b_n),
$$

where

$$
\Phi(a) = \prod_{i<j} \frac{1}{1 - a_i a_j}.
$$
Geometric last passage percolation

For $n \geq m$, Consider the Pfaffian Scur measure with weight

$$\mathbb{P}(\lambda) \propto \tau_{\lambda}(c, a_{m+1}, \ldots, a_n) s_{\lambda}(a_1, \ldots, a_m),$$

then

$$\lambda_1 \overset{(d)}{=} G(n, m)$$

for a family of random variables $G(n, m)$ that satisfies the recursion

$$\begin{cases} G(n, m) = \max \{G(n-1, m), G(n, m-1)\} + \text{Geom}(a_n a_m) & \text{for } n > m \\ G(n, n) = G(n, n-1) + \text{Geom}(ca_n). \end{cases}$$

- To prove this, one defines more general Pfaffian Schur processes, that couple together several partitions marginally distributed as a Pfaffian Schur measure.
- One studies dynamics on them (following Borodin-Ferrari 2004) and the first coordinate marginal corresponds to last-passage percolation.
- This more general setup allows to prove multipoint limit Theorems.
As the geometric distribution converges to the exponential,

**Proposition**

If we set $c = \sqrt{q}(1 + (\alpha - 1)(q - 1))$ and $a_i = \sqrt{q}$, then as $q \to 1$,

$$\{(1 - q)G(n,m) \Rightarrow H(n,m)\}$$

where $H(n,m)$ are the passage times in LPP with exponential weights on a half quadrant (with parameter $\alpha$ on the diagonal).

Of course this works as well for the joint distribution of passage times at different points (along a space-like path).
Pfaffian Point process

A random configuration $X \subset \mathbb{X}$ (state space) is a **Pfaffian point process** if one can write the correlation function as

$$\rho(Y) = \mathbb{P}(Y \subset X) = \text{Pf}[K(x,y)]_{x,y \in Y},$$

where

$$K(x,y) = \begin{pmatrix} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{pmatrix}$$

is a skew-symmetric matrix indexed by elements in $\mathbb{X}$; called the correlation kernel.

Multiplicative functionals are given by Fredholm Pfaffians

$$\mathbb{E}\left( \prod_{x \in X} 1 + f(x) \right) = \text{Pf}(J + f \cdot K)_{L^2(\mathbb{X})}$$

$$\text{Pf}(J + f \cdot K)_{L^2(\mathbb{X})} := 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int_Y \cdots \int_Y dx_1 \cdots dx_k f(x_1) \cdots f(x_k) \text{Pf}[K(x_i, x_j)]_{i,j=1}^k.$$ 

In particular for a subset $Y \subset \mathbb{X}$,

$$\mathbb{P}(\text{no point in } Y) = \text{Pf}(J - K)_{L^2(Y)}$$
The Pfaffian Schur process is Pfaffian

**Theorem (Borodin-Rains 2005)**

*The Pfaffian Schur measure is Pfaffian in the sense that

\[ \lambda_1 - 1 > \lambda_2 - 2 > \lambda_3 - 3 > \ldots \]

is a Pfaffian point process on \( \mathbb{Z} \) with an explicit correlation kernel \( K^{Schur} \). There is a more general multidimensional result for the Pfaffian Schur process.*

The variable \( G(n,m)^{(d)} = \lambda_1 \) is the extremal point (+1) in the Pfaffian point process, so that

\[ \mathbb{P}(G(n,m) \leq x) = \text{Pf}(J - K^{Schur})_{\ell^2[x,\infty)}. \]

- Sending \( q \to 1 \) yields the (multipoint) probability distribution of passage times in exponential LPP on the half-quadrant.
- Since the GSE/GOE/GUE distribution functions can be written as Fredholm Pfaffians, one concludes by asymptotic analysis of the above formula.
What is this point process?

The whole point process

\[ \Lambda := \{\lambda_i - i\}_{i \geq 1} \]

has an interpretation when \( n = m \) (that is for the partition corresponding to passage times on the diagonal):

Consider last passage percolation in the full quadrant with weights symmetric with respect to the diagonal. Then

\[
\lambda_1 = \max_\pi \left\{ \sum_{(i,j) \in \pi} w_{ij} \right\},
\]

\[
\lambda_1 + \lambda_2 = \max_{\pi_1, \pi_2} \left\{ \sum_{(i,j) \in \pi_1 \cup \pi_2} w_{ij} \right\},
\]

where the maximum is taking over two non-intersecting directed paths \( \pi_1 \) and \( \pi_2 \).

\[
\lambda_1 + \lambda_2 + \lambda_3 = \max_{\pi_1, \pi_2, \pi_3} \left\{ \sum_{(i,j) \in \pi_1 \cup \pi_2 \cup \pi_3} w_{ij} \right\},
\]

where the maximum is taking over three non-intersecting directed paths \( \pi_1, \pi_2, \pi_3 \), etc. This is related to the application of RSK algorithm to a symmetric random matrix.
Pfaffian Macdonald measures

Skew Macdonald polynomials $P_{\lambda/\mu}, Q_{\lambda/\mu}$ are symmetric polynomials in many variables whose coefficients are rational functions in two parameters $q,t \in (0,1)$. They degenerate to skew Schur functions $s_{\lambda/\mu}$ when $q = t$.

For two sets of variables $a_1,\ldots,a_n$ and $b_1,\ldots,b_m$ in $(0,1)$, we consider the Pfaffian Macdonald measure

$$\mathbb{P}(\lambda) \propto P_{\lambda}(a) \mathcal{E}_\lambda(b),$$

where

$$\mathcal{E}_\lambda = \sum_{\mu' \text{ even}} b_\lambda^{\mu'} Q_{\lambda/\mu}.$$

In the following, we set $b_i \equiv 0$, so that the measure depends only on parameters $a_1,\ldots,a_n$. As in the Schur case, one can define more general Pfaffian Macdonald processes.

- It’s a variant of the Macdonald measure (Borodin-Corwin 2011) which is a $(q,t)$-generalization of the Schur measure.
When $q = 0$, Macdonald polynomials degenerate to Hall-Littlewood polynomials

$$P_\lambda(x_1, \ldots, x_n; t) = c(\lambda) \sum_{\sigma \in S_n} \sigma(x_1^{\lambda_1} \ldots, x_n^{\lambda_n} \prod_{i < j} \frac{x_i - tx_j}{x_i - x_j}).$$

- Hall Littlewood polynomials have been recently connected to the six-vertex model and spin systems (Korff 2011, Borodin 2014, Wheeler-Zinn-Justin 2014 & 2015).
- For the stochastic six vertex-model in a rectangular domain, the connection is very precise (Borodin 2016, Borodin-Bufetov-Wheeler 2016) One can use a spin model representation of Hall-Littlewood functions to relate Hall-Littlewood processes to a stochastic six-vertex model in a quadrant.
- We adapt this to the half space case using Pfaffian Hall-Littlewood processes.
Proposition (B.-Borodin-Corwin-Wheeler 2017+)

\[
P(\mathcal{h}(n,n) = k) = \mathbb{P}(\ell(\lambda) = k),
\]
where \(\mathcal{h}(n,n)\) is the number of outgoing vertical arrows from the vertices on the left of \((n,n)\), and \(\ell(\lambda)\) is the number of nonzero components in a partition \(\lambda\) following the Pfaffian Hall-Littlewood measure.
Relation Hall-Littlewood and Schur

A refined Littlewood identity for Macdonald functions (Rains 2015) shows that certain observables of Pfaffian Macdonald measures do not depend on $q$.

Comparing the $q = 0$ and $q = t$ cases yields identities relating functionals of Schur and Hall-Littlewood random partitions:

**Proposition (B.-Borodin-Corwin-Wheeler 2017+)**

For any $x \in \mathbb{R}$, $n \in 2\mathbb{Z}_{>0}$, and $(a_1, \ldots, a_n) \in (0, 1)$ and $b \equiv 0$,

\[
E^{HL} \left[ \frac{1}{(-tx+n-\ell(\lambda), t^2)_{\infty}} \right] = E^{Schur} \left[ \prod_{p \in \mathbb{Z} \setminus \Lambda} (1 + f_x(p)) \right] = Pf \left[ J + f_x \cdot K^C \right]_{\ell^2(\mathbb{Z}_{\geq 0})},
\]

where $K^C = J - K^{Schur}$ is the correlation kernel of the complement of the Pfaffian Schur point process $\Lambda := \{\lambda_i - i\}_i$,

\[
f_x(j) = \frac{(-tx+j+1; t^2)_{\infty}}{(-tx+j; t^2)_{\infty}} - 1.
\]

where

\[
(a; t^2)_{\infty} = (1 - a)(1 - at^2)(1 - at^4)\ldots
\]
If the parameters are scaled such that $a_x \equiv 1 - \frac{(1-t)\epsilon}{2}$,

$$P\left(\right) \approx t\epsilon, \quad P\left(\right) \approx 1 - t\epsilon, \quad P\left(\right) \approx \epsilon, \quad P\left(\right) \approx 1 - \epsilon.$$ 

and paths will almost always zig-zag and do something else at rates 1 and $t$. 


Recall that $N_0(\tau)$ denotes the total number of particles in the system at time $\tau$.

**Theorem (B.-Borodin-Corwin-Wheeler 2017+)**

For any time $\tau > 0$ and $x \in \mathbb{R}$,

$$
\mathbb{E} \left[ \frac{1}{(-t^x+N_0(\tau),t^2)_{\infty}} \right] = \text{Pf} \left[ J + f_x \cdot K_{\text{ASEP}}^{\text{C}} \right]_{\ell^2(\mathbb{Z}_{\geq 0})}
$$

where $K_{\text{ASEP}}^{\text{C}}$ is a limit of $K^{\text{C}}$ from the previous slides, which can be expressed exactly as contour integrals.

The L.H.S of the equation should be thought of as a deformed Laplace transform.
Theorem (B.-Borodin-Corwin-Wheeler 2017+)

For any $t \in [0, 1)$, starting from the empty initial condition,

$$\lim_{T \to \infty} \mathbb{P}\left( \frac{N_0 \left( \frac{T}{1-t} \right) - T/4}{2^{-4/3} T^{1/3}} > -x \right) = \text{Pf}\left[ J - K^{\text{GOE}} \right]_{L^2(x, \infty)} = F^{\text{GOE}}(x).$$

- In the scaling limit above, $K^{\text{ASEP}}$ goes to the correlation kernel of the GOE, and the function $f$ goes to $-\mathbb{1}_{>x}$.

- In the scaling limit leading to the solution of the KPZ equation, $K^{\text{ASEP}}$ still go to the correlation kernel of the GOE, and $f$ converges to some non-trivial function. Hence, the Laplace transform of the solution to the KPZ equation equals a multiplicative functional of the GOE.
Summary

- Fluctuations of the height function in KPZ growth in a half space are predicted by the results for Last-Passage percolation in a half-quadrant.
- Last Passage percolation can be studied in much details using Pfaffian Schur processes.
- It turns out that the point process which is defined by the Pfaffian Schur measure is also related to the stochastic six-vertex model in a half quadrant.
- This leads to limit theorems for the current in half-line ASEP and half-space KPZ equation.
Further directions

- **More general boundary conditions.** This will probably require going higher in the hierarchy of integrable structures.

- Other interesting models are coming from Pfaffian Macdonald processes: **Log gamma directed polymer in a half space.**

- Ultimately one hopes to prove that the Laplace transform of **KPZ equation in a half space at any space point and for general boundary condition** is a multiplicative functional of a certain point process corresponding to the two-dimensional crossover kernel obtained in LPP.
Thank you