KPZ scaling theory for integrable exclusion processes

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Introduction

What is KPZ?
Kardar, Parisi, Zhang, in 1986, study the random growth of rough interfaces. Propose a SPDE to describe the height \( h(t,x) \) of the interface

\[
\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \dot{\mathcal{W}},
\]

where \( \dot{\mathcal{W}} \) is a white noise. They made scaling predictions and claimed universality.

In this talk

- We focus on *exactly solvable* discrete random models.
- More precisely exclusion processes.
- We start from the most simple initial condition and study different dynamics.
Exclusion process

Description of the system

- Coordinates $X_n(t)$,
- Current (integrated)

$$N_x(t) = \#\{n \mid X_n(t) \geq x\},$$

- (Height function via Rost’s mapping,

$$h(x, t) = x + 2N_x(t).$$)
Limit theorems: Heuristics

Step initial data $x_n(0) = -n$:

Law of large numbers
One expects: for $n$ and $t$ going to infinity with $n/t = \kappa$,

$$\frac{X_n(t)}{t} \xrightarrow{a.s. \quad t \to \infty} \pi(\kappa).$$

Tracy-Widom Central limit theorem
For models in the KPZ universality class, one expects

$$\frac{X_n(t) - \pi(\kappa)t}{\sigma(\kappa) \cdot t^{1/3}} \xrightarrow{t \to \infty} \mathcal{L}_{TW},$$

where $\mathcal{L}_{TW}$ is the Tracy-Widom law from the fluctuations of the largest eigenvalue of Gaussian Unitary Ensemble.
**KPZ scaling theory : Heuristics**

KPZ scaling theory (Krug, Meakin, Halpin-Healy 1992) constitutes an educated guess to predict the value of the constants $\pi(\kappa)$ and $\sigma(\kappa)$ arising in the limit theorems.

**Assumptions**

- Dynamics are local and space homogeneous.
- Translation invariant stationary measures $\mu_\rho$ are labelled by the average density of particles $\rho = \lim_{a \to \infty} \frac{\# \text{ part. between } -a \text{ and } a}{2a+1}$.
- The function $j(\rho) := \mathbb{E}_{\mu_\rho} \left[ \frac{d}{dt} N_0(t) \right]$ is such that $j''(\rho) \neq 0$.

**Macroscopic density profile**

Let $\rho(x, \tau) = \lim_{t \to \infty} \mathbb{P}(\text{There is a particle at site } xt \text{ at time } t\tau)$ be the macroscopic density profile. It satisfies the conservation equation

$$\frac{\partial}{\partial t} \rho(x, t) + \frac{\partial}{\partial x} j(\rho(x, t)) = 0,$$

with $\rho(x, 0) = 1_{x<0}$ for step initial condition.
We choose \( n/t = \kappa(\rho) \) such that \( X_n(t) \) has a local environment given by \( \mu_\rho \). We expect \( \frac{X_n(t)}{t} \rightarrow \pi(\rho) \). If \( \bar{\rho}(x, t) \) solves the conservation PDE, then \( \bar{\rho}(\pi(\rho), 1) = \rho \).

\[
\pi(\rho) = \frac{\partial j(\rho)}{\partial \rho}.
\]

The function \( \kappa(\rho) \) can then be calculated by integrating the density, and one finds for step initial condition

\[
\kappa(\rho) = -\rho \frac{\partial j(\rho)}{\partial \rho} + j(\rho).
\]

**Magnitude of fluctuations**

Let \( \lambda = -j''(\rho) \) and \( A = \sum_{j \in \mathbb{Z}} \text{Cov}_{\mu_\rho}(\eta_0, \eta_j) \) where \( \eta_0, \eta_j \in \{0, 1\} \) are occupation variables at sites 0 and \( j \). Then

\[
\sigma(\rho) = \left( \frac{-\lambda A^2}{2\rho^3} \right)^{1/3}.
\]
Integrated covariance $A$

Consider $X_i, i \in \mathbb{Z}$ a stationary sequence of mean zero r.v. Under some assumptions, $S_n/\sqrt{n}$ converges to a Gaussian of variance $\sigma^2$ where

$$
\sigma^2 = \lim_{N \to \infty} \mathbb{E} \left[ \frac{S_N^2}{N} \right] = \lim_{N \to \infty} \mathbb{E} \left[ \frac{\left( \sum_{i=1}^{N} X_i \right) \left( \sum_{i=1}^{N} X_i \right)}{N} \right] = \mathbb{E} \left[ \sum_{i \in \mathbb{Z}} X_0 X_i \right] = \sum_{i \in \mathbb{Z}} \text{Cov}(X_0, X_i).
$$

Product form invariant measures
If $\mu_\alpha(\text{gap} = k) \propto \alpha^k/(g(1) \ldots g(k))$ for some positive increasing function $g$, then

$$
A = -\alpha \rho \frac{d\rho}{d\alpha}
$$

where $\rho(\alpha)$ is the density of particles under law $\mu_\alpha$. 
Example: TASEP

Description of the dynamics

Properties
One finds that the invariant measures are such that each site is occupied independently with probability $\rho$.
This yields $j(\rho) = \rho(1 - \rho)$, $\pi(\rho) = 1 - 2\rho$ and $\kappa(\rho) = \rho^2$, so that $\pi = 1 - 2\sqrt{\kappa}$. One finds $\sigma(\rho) = \left(\frac{(1-\rho)^2}{\rho}\right)^{1/3}$.

Theorem (Johansson 2000)
For $n/t = \kappa \in (0, 1)$,

$$\frac{X_n(t) - (1 - 2\sqrt{\kappa})t}{\sigma(\rho)t^{1/3}} \xrightarrow{t \to \infty} \mathcal{L}_{TW}.$$
A brief introduction to $q$-analogues I

Newton binomial formula:

$$(X + Y)^n = \sum_{k=0}^{n} \binom{n}{k} X^k Y^{n-k}.$$  

If $YX = qXY$, one can a priori write

$$(X + Y)^n = \sum_{k=0}^{n} C^k_n(q) X^k Y^{n-k}.$$  

Definitions

- $q$-deformed integer $[n]_q := 1 + q + \cdots + q^{n-1}$.
- $q$-deformed factorial $n!_q := [n]_q[n - 1]_q \cdots [1]_q$.
- $q$-Pochhammer symbol: $(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$.

Then the $q$-binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{n!_q}{k!_q(n - k)!_q} = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} = C^k_n(q).$$
A brief introduction to $q$-analogues II

Fix $0 < q < 1$ for the rest of the talk.

**Definition**

The $q$-exponential is defined by

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{k_q!}$$

Then we have the identity

$$e_q(x) = \sum_{k=0}^{\infty} \frac{(x(1-q))^k}{(q; q)_k} = \frac{1}{(x(1-q); q)_\infty}.$$ 

The $q$-Laplace transform of a random variable $X$ is

$$\mathbb{E}\left[ \frac{1}{(\zeta(1-q)X; q)_\infty} \right]$$
Definition of the $q$-TASEP

Introduced by Borodin and Corwin in the context of Macdonald processes (2011). Set $q \in (0, 1)$.

Stationary measures
Translation invariant stationary measures are such that gaps are distributed according to $q$-geometric random variables:

$$
\mathbb{P}(X_n - X_{n+1} - 1 = k) = \frac{\alpha^k}{(q; q)_k}(\alpha; q)_{\infty},
$$

for $\alpha \in (0, 1)$. 

Main result

- For the system at equilibrium given by the stationary measure 
  \( \mu_\alpha(k) = \frac{\alpha^k}{(q;q)_k} (\alpha; q)_\infty \), the average density is given by

  \[
  \rho_\alpha = \frac{1}{1 + \mathbb{E}[\text{gap}]} = \frac{1}{1 + \sum_{k=0}^{\infty} \frac{\alpha q^k}{1-\alpha q^k}}.
  \]

- The speed of a particle is \( \mathbb{E}_{\mu_\alpha} [1 - q^{\text{gap}}] = \alpha \).

- This implies that \( j(\rho_\alpha) = \alpha \rho_\alpha \).

- This yields formulas for \( \kappa(\rho_\alpha), \pi(\rho_\alpha) \) and \( \sigma(\rho_\alpha) \) given by KPZ scaling theory. (involves \( q \)-deformed special functions)

**Theorem (Ferrari-Vető, 2013 / B. 2014)**

*For \( \alpha \in (0, 1) \), \( n/t = \kappa(\alpha) \) ranges in \((0, 1)\) and

\[
\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha) \cdot t^{1/3}} \xrightarrow{(d)} \mathcal{L}_{TW}.
\]
Exclusion process vs Zero Range

- Coupling $x_k - x_{k+1} - 1 \sim y_k$
- Exclusion processes $\leftrightarrow$ Zero range processes
- here, $q$-totally asymmetric zero range process, also called $q$-Boson model.

Definition
Two Markov processes $\vec{X}(t) \in \mathcal{X}$ and $\vec{Y}(t) \in \mathcal{Y}$ are said dual w.r.t $H : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ if for any initial data,

$$
\mathbb{E}[H(\vec{X}(t), \vec{Y}(0))] = \mathbb{E}[H(\vec{X}(0), \vec{Y}(t))] \iff L^X H(\vec{x}, \vec{y}) = L^Y H(\vec{x}, \vec{y})
$$

Proposition (Borodin-Corwin-Sasamoto, 2012)
A direct calculation shows that for $H(\vec{x}, \vec{y}) = \prod_{i=0}^N q^{(x_i+i)y_i}$,

$$
L^{q-\text{TASEP}} H = L^{q-\text{Boson}} H.
$$
Remark
The duality is useful if $H$ characterizes enough the law of the process. Here $\mathbb{E}[H(\vec{X}(t), \vec{y})]$ are mixed moments of the variables $q^{X_i(t)}$.

What one can do with duality?
We compute the probability distribution function of $X_n(t)$ (cf Borodin-Corwin-Sasamoto 2012).

1. Find a closed system of ODEs for $\mathbb{E}[\prod_i q^{y_i X_i(t)}]$. Using the duality, one writes Kolmogorov equations for a $q$-Boson with $k$ particles.
2. Solve the system of equations using Bethe ansatz.
3. It yields formulas for $\mathbb{E}[q^{kX_n(t)}]$ for $k \in \mathbb{N}$ which characterize the law of $X_n(t)$.
4. Take generating function to express the $q$-Laplace transform $\mathbb{E}\left[\frac{1}{(\zeta q^{X_n(t)})_q}\right]$.
5. Can be inverted to find the probability distribution function.
Fredholm determinant representation

Theorem (Borodin-Corwin, 2011)

Fix $0 < q < 1$. For all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, if $X_n(t)$ are coordinates of particles of the $q$-TASEP with step initial data,

$$
\mathbb{E} \left[ \frac{1}{(\zeta q X_n(t); q)_{\infty}} \right] = \det(I + K_{\zeta})_{L^2(C)},
$$

where $\det(I + K_{\zeta})_{L^2(C)}$ is the Fredholm determinant of $K_{\zeta}$ defined by its integral kernel

$$
K_{\zeta}(w, w') = \frac{1}{2i\pi} \int_{1/2 + i\mathbb{R}} \frac{\pi}{\sin(\pi s)} (-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{ds}{q^s w - w'}
$$

with

$$
g(w) = \left( \frac{w}{(w; q)_{\infty}} \right)^n e^{-tw},
$$

and the integration contour $C$ is a small circle around 1.
Asymptotic analysis I

- One expects $X_n(t) \sim \pi(\alpha)t + t^{1/3}\sigma(\alpha)\chi_{TW}$ where $\chi_{TW}$ is a Tracy-Widom distributed random variable.

- The function $x \mapsto 1/(−q^x; q)_{\infty}$ have limits 0 in $−\infty$ and 1 in $+\infty$. If one sets $\zeta = −q^{−\pi(\alpha)t−t^{1/3}\sigma(\alpha)x}$ for $x \in \mathbb{R}$,

$$\lim_{t \to \infty} \mathbb{E} \left[ \frac{1}{(\zeta q^{X_n(t)}; q)_{\infty}} \right] = \lim_{t \to \infty} \mathbb{P} \left( \frac{X_n(t) − \pi(\alpha)t}{\sigma(\alpha)t^{1/3}} \leq x \right).$$

- One needs to prove that

$$\lim_{t \to \infty} \det(I + K_\zeta) = F_{TW}(x),$$

where $F_{TW}$ is the distribution function of a Tracy-Widom r.v.
Asymptotic analysis II

Fredholm Determinant

\[ \det(I+K)_{L^2(C)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int C \ldots \int C \det(K(w_i, w_j))_{1 \leq i, j \leq n} \, dw_1 \ldots dw_j. \]

Fredholm determinant representation of \( F_{TW}(x) \)

\[ F_{TW}(x) = \det(I + K_{Ai})_{L^2(\Gamma)} \] where

\[ K_{Ai}(w, w') = \frac{1}{2i\pi} \int_{\Xi} dz \frac{e^{z^3/3-zx}}{e^{w^3/3-wx}} \frac{1}{z-w} \frac{1}{z-w'}, \]

where \( \Gamma \) and \( \Xi \) are some flexible contours.

Idea of the proof

One applies Laplace’s method (saddle point analysis) on each \( n \)-fold integral in the Fredholm determinant series expansion.
Question
Can we prove a Tracy-Widom central limit theorem for the most general exclusion process?

Partial answers

• CLT for ASEP (Asymmetric simple exclusion process) (Tracy-Widom 2008).
• Discrete time version of $(q)$-TASEP. (Borodin-Corwin 2013).
• Many other partial answers in the literature, namely proving fluctuation exponents under hypotheses.
• Exactly solvable long-range exclusion process: The $q$-Hahn TASEP (Povolotsky 2013 / Corwin 2014).
The \( q \)-Hahn process

**\( q \)-Hahn Boson process**

Discrete-time Markov chain. Particles live on \( N \) sites. From a site occupied by \( y \) particles, \( j \leq y \) particles move to the left with probability \( \varphi(j|y) \).

Introduced by Povolotsky 2013

**\( q \)-Hahn distribution**

For \( 0 < q < 1 \) and \( 0 \leq \nu \leq \mu \leq 1 \),

\[
\varphi_{q,\mu,\nu}(j|y) := \mu^j \frac{(\nu/\mu; q)_j (\mu; q)_{y-j}}{(-\nu; q)_y} \left[ y \right]_q^j,
\]

defines a probability distribution on \( \{0, 1, \ldots, y\} \). (This is also the weight function for the \( q \)-Hahn orthogonal polynomials)
Duality with $q$-Hahn TASEP

The $q$-Hahn process can be described by an exclusion process:

$\text{Prob. } \varphi(2|3)$

The $q$-Hahn TASEP and the $q$-Hahn Boson are dual w.r.t. $H(\vec{x}, \vec{y}) = \prod_{i=1}^{N} q^{y_i(x_i+i)}$.

$$E[H(\vec{X}(t), \vec{Y}(0))] = E[H(\vec{X}(0), \vec{Y}(t))] .$$

It relies on a symmetry of the $q$-Hahn distribution:

$$\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m)q^{jy} = \sum_{j=0}^{y} \varphi_{q,\mu,\nu}(j|y)q^{jm}.$$
Almost the same Fredholm determinant

Theorem (Corwin 2014)
Fix $0 < q < 1$ and $0 \leq \nu \leq \mu \leq 1$. For all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$,

$$
E \left[ \frac{1}{(\zeta q^{X_n(t)}; q)_\infty} \right] = \det(I + K_\zeta)_{L^2(C)},
$$

where $\det(I + K_\zeta)_{L^2(C)}$ is the Fredholm determinant of $K_\zeta$ defined by its integral kernel

$$
K_\zeta(w, w') = \frac{1}{2i\pi} \int_{1/2 + i\mathbb{R}} \frac{\pi}{\sin(\pi s)} (-q^{-n}\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} \, ds
$$

with

$$
g(w) = \left( \frac{(\nu w; q)_\infty}{(w; q)_\infty} \right)^n \left( \frac{(\mu w; q)_\infty}{(\nu w; q)_\infty} \right)^t \frac{1}{(\nu w; q)_\infty},
$$

and the integration contour $C$ is a small circle around 1.
Degenerations

- $\nu = 0$: Corresponds to a discrete-time $q$-TASEP: Geometric $q$-TASEP.
- If $\nu = 0$ and scaling $\mu = (1 - q)\epsilon$ and rescaling time by $\tau = \epsilon^{-1}t$, one recovers the $q$-TASEP.
- Many other degenerations

Translation invariant stationary measures

$$
\mu_\alpha(\text{gap} = k) = \alpha^k \frac{(\nu; q)_k}{(q; q)_k} \frac{(\alpha; q)_\infty}{(\alpha \nu; q)_\infty}.
$$

Theorem (Vető (2014))

Under some restrictions on the range of parameters $q, \mu$ and $\nu$, and for $\alpha > 2q/(1 + q)$,

$$
\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha) \cdot t^{1/3}} \overset{(d)}{\longrightarrow} \mathcal{L}_{TW}.
$$
Two sided $q$-Hahn process

Question
Is it possible to generalize the processes allowing jumps in both directions, preserving duality and Bethe ansatz solvability?

Continuous time process:

Rates
Let $R, L \in \mathbb{R}_+$ be asymmetry parameters, with $R + L = 1$. We define

$$\phi^R_{q,\nu}(j|m) := R \frac{\nu^{j-1}}{[j]_q} \frac{(\nu;q)_{m-j}}{(\nu;q)_m} \frac{(q;q)_m}{(q;q)_{m-j}} \sim R \lim_{\mu \to \nu} \varphi_{q,\mu,\nu}(j|m)$$

$$\phi^L_{q,\nu}(j|m) := L \frac{1}{[j]_q} \frac{(\nu;q)_{m-j}}{(\nu;q)_m} \frac{(q;q)_m}{(q;q)_{m-j}} \sim L \lim_{\mu \to \nu} \varphi_{q^{-1},\mu^{-1},\nu^{-1}}(j|m).$$
Duality

Two-sided $q$-Hahn Boson

- Sites indexed by $\mathbb{N}$.
- For each $j, j' \leq y_i$, $j$ particles move to site $i - 1$ with rate $\phi_{q,\nu}^R(j | y_i)$ and $j'$ particles move to site $i + 1$ with rate $\phi_{q,\nu}^L(j' | y_i)$.

Duality

For any initial conditions $\vec{X}(0)$ being a finite perturbation of the step, and $\vec{Y}(0)$ with a finite number of particles,

$$
\mathbb{E} \left[ \prod_{i=1}^\infty q^{Y_i(0)(X_i(t)+i)} \right] = \mathbb{E} \left[ \prod_{i=1}^\infty q^{Y_i(t)(X_i(0)+i)} \right].
$$
Fredholm determinant

Theorem (B.-Corwin (in prep.))
Fix $0 < q < 1$ and $0 \leq \nu < 1$. For all $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$,

$$
\mathbb{E}\left[\frac{1}{(\zeta q^X_n(t); q)_\infty}\right] = \det(I + K_\zeta)_{L^2(C)},
$$

where $\det(I + K_\zeta)_{L^2(C)}$ is the Fredholm determinant of $K_\zeta$ defined by its integral kernel

$$
K_\zeta(w, w') = \frac{1}{2i\pi} \int_{1/2+i\mathbb{R}} \frac{\pi}{\sin(\pi s)} (-q^{-n} \zeta)^s \frac{g(w)}{g(q^s w)} \frac{ds}{q^s w - w'}
$$

with

$$
g(w) = \left(\frac{(\nu w; q)_\infty}{(w; q)_\infty}\right)^n \exp\left((q - 1)t \sum_{k=0}^{\infty} R \frac{wq^k}{1 - \nu wq^k} - L \frac{wq^k}{1 - wq^k}\right) \frac{1}{(\nu w; q)_\infty},
$$

and the integration contour $C$ is a small circle around 1.
Scaling theory

Translation invariant stationary measures

$$\mu_\alpha(\text{gap} = k) = \alpha^k \frac{(\nu; q)_k}{(q; q)_k} \frac{\alpha q}{(\alpha q; q)_\infty},$$

same as for $q$-Hahn TASEP.

Model dependent constants

One can still find expressions for $\rho$ as a function of $\alpha$, and then $\kappa(\alpha)$, $\pi(\alpha)$ and $\sigma(\alpha)$. (involves $q$-deformed special functions).

Tracy-Widom Central limit theorem

Fix $0 < q, \nu < 1$ and $R > L$. For all meaningful $\alpha$, keeping $n/t = \kappa(\alpha)$ we expect

$$\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha) \cdot t^{1/3}} \xrightarrow{(d)} t \to \infty \mathcal{L}_{TW}.$$
Multi-particle Asymmetric Diffusion Model

When $\nu = q$ the rates no longer depend on the gap and become $R/[j]_{q-1}$ and $L/[j]_q$.

- Translation invariant stationary measures are products of i.i.d. Bernoulli.
Simulations

One can check the predictions of KPZ scaling theory (here only the LLN) with simulations:

Figure: $N_{xt}(t)/t$ in function of $x$ for $t = 1500$. Left: $R = 0.8$, Right: $R = 1$. $L = 1 - R$
Result

Theorem (B.-Corwin, in preparation)

Fix $0 < q < 1$ and $R > L$. For $\alpha \geq 2q/(1 + q)$, keeping $n/t = \kappa(\alpha)$ we have

$$\frac{X_n(t) - \pi(\alpha)t}{\sigma(\alpha) \cdot t^{1/3}} \xrightarrow{t \to \infty} \mathcal{L}_{TW}.$$ 

The saddle point analysis is computationally difficult for $\alpha < 2q/(1 + q)$.

Surprising phenomena

- The density profile has a discontinuity at the first particles.
- Because of long range jumps on the left, the position of the first particle does not satisfy a classical CLT but a Tracy-Widom CLT. (It is not the case for ASEP)
Conclusion

We have seen

- General expression of model-dependent constant for the renormalization theory of models in the KPZ universality class, in the context of exclusion processes.

- Exactly solvable examples: TASEP and the $q$ deformed exclusion processes: $q$-TASEP and $q$-Hahn TASEP.

- Exact solvability of the $q$-Hahn process extends to two-sided jumps. Some degenerations were already known to be integrable.
Thank you for your attention