ASEP and the KPZ equation in a half-space

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Joint work with Alexei Borodin, Ivan Corwin and Michael Wheeler.

Consider the solution $Z(\tau,x)$ to the multiplicative SHE,

$$\partial_\tau Z = \frac{1}{2}\partial_{xx} Z + Z\,\dot{\mathscr{W}}, \quad \text{where } x \in \mathbb{R}, \tau > 0,$$

with delta initial condition $Z(0,\cdot) = \delta_0$, where $\dot{\mathscr{W}}$ is a Gaussian space time white noise. Then $H(\tau,x) = \log(Z(\tau,x))$ is a solution to the KPZ equation

$$\partial_\tau H = \frac{1}{2}\partial_{xx} H + \frac{1}{2}\left(\partial_x H\right)^2 + \dot{\mathscr{W}}.$$

One point distribution of the solution is characterized by the identity, for $u \in \mathbb{C}$ with $\mathfrak{Re}(u) > 0$,

$$\mathbb{E}\left[e^{\frac{-u}{4}Z(\tau,0)\,e^{\tau/24}}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{1+u\,e^{(\tau/2)^{1/3}\mathfrak{a}_i^{\mathrm{GUE}}}}\right],$$

where $\{\mathfrak{a}_i^{\mathrm{GUE}}\}_{i\geqslant 1}$ are the limiting eigenvalues of the GUE scaled at the edge [Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn 2011, see also Borodin-Gorin 2016].

Half-space analogue?

Consider now the solution $Z(\tau,x)$ to the multiplicative SHE in a half-space,

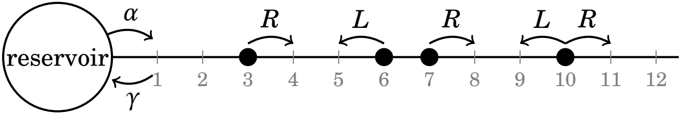
$$\partial_\tau Z = \frac{1}{2}\partial_{xx} Z + Z\,\dot{\mathscr{W}}, \quad \text{where } x \in \mathbb{R}_{\geqslant 0}, \tau > 0,$$

with delta initial condition $Z(0,\cdot) = \delta_0$ for some boundary condition at $x = 0$ (Neumann, Dirichlet, mixed...).
What is the law of the solution? Can one find a function f_u and a matrix ensemble G?E such that

$$\mathbb{E}\left[e^{-uZ(\tau,0)}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} f_u\left(\mathfrak{a}_i^{\mathrm{G?E}}\right)\right]?$$

In order to investigate this, one needs an exactly solvable regularization of the multiplicative SHE.

Let $R > L \geqslant 0$, and consider the asymmetric simple exclusion process (**ASEP**) on the positive integers with open boundary condition:



One can characterize the system by the function

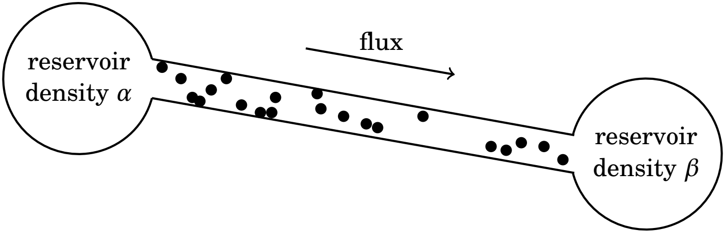
$$N_x(\tau) = \#\{\text{particles on the right of site } x \text{ at time } \tau\}.$$

In a certain weakly asymmetric scaling ($R - L \to 0$), [Corwin-Shen 2016] showed that $N_x(t)$ converges to the KPZ equation on the positive reals with Neumann boundary condition,

$$\begin{cases} \partial_\tau H = \frac{1}{2}\partial_{xx}H + \frac{1}{2}\left(\partial_x H\right)^2 + \dot{\mathscr{W}} \\ \partial_x H(\tau,x)\Big|_{x=0} = a \in \mathbb{R}. \end{cases}$$

It corresponds to mixed Robin boundary condition for Z,
$\partial_x Z(\tau,x)\Big|_{x=0} = a\, Z(\tau,0)$.

ASEP is also interesting for itself:



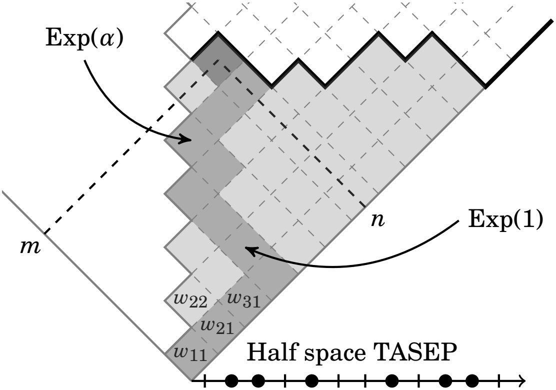
- **(KPZ) universality**. We expect that large scale statistics of the current of interacting particles travelling between reservoirs at different densities are universal under mild conditions. ASEP is a toy model to probe these statistics. Large time statistics of ASEP without reservoirs are well understood [Tracy-Widom 2008]. What is the influence of the boundary?
- The fluctuations of **TASEP** ($L = 0$) in a half-space (equivalently last-passage percolation in a half-quadrant) are known. Are those of ASEP similar?

Plan of the talk

1. The totally asymmetric case is equivalent to **Last Passage Percolation in a half-quadrant**, which is the simplest benchmark model for understanding KPZ growth or exclusion processes in a half space.
2. Results on **half-line ASEP**: Tracy-Widom GOE asymptotics of the current at the origin.
3. **KPZ equation** on $\mathbb{R}_{>0}$.
4. Ideas of the **proof** using 3 ingredients:
   - Stochastic six-vertex model in a half-quadrant.
   - Half-space Macdonald processes and Littlewood identities for Macdonald symmetric functions.
   - Pfaffian point processes techniques.

Last Passage Percolation in a half quadrant

Let w_{ij} a family of i.i.d. exponential random variables with rate 1 when $i > j$ and α when $i = j$.



Consider directed paths π from the box $(1,1)$ to (n,m) in the half quadrant. We define the last passage percolation time $H(n,m)$ by

$$H(n,m) = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}.$$

Passage-times on the boundary

Theorem (Baik-Rains 2001 / Baik-B.-Corwin-Suidan 2016)

- *When* $\alpha > 1/2$,

$$\frac{H(n,n)-4n}{2^{4/3}n^{1/3}} \Longrightarrow \mathcal{L}_{\mathrm{GSE}},$$

- *When* $\alpha = 1/2$,

$$\frac{H(n,n)-4n}{2^{4/3}n^{1/3}} \Longrightarrow \mathcal{L}_{\mathrm{GOE}},$$

- *When* $\alpha < 1/2$,

$$\frac{H(n,n)-cn}{c'n^{1/2}} \Longrightarrow \mathcal{N},$$

In particular, if $N_x(\tau)$ is the current in half-line TASEP (right jump rate 1, insertion of particles at rate $\alpha = 1/2$, no particle moving to the left), starting from the empty initial condition,

$$\frac{N_0(\tau)-\frac{\tau}{4}}{2^{-4/3}\tau^{1/3}} \xrightarrow[\tau\to\infty]{} -\mathcal{L}_{GOE}.$$

Understanding the phase transition

- The fact that $H(n,n) \sim 4n$ shows that the weights along the optimal path have size 2 in average. Thus, the disorder on the boundary becomes competitive when it has average at least 2, hence the transition at $\alpha = 1/2$.
- Algebraic considerations show that for any α, one can exchange the weights $w_{ii} \sim Exp(\alpha)$ on the boundary with the weights $w_{i1} \sim Exp(1)$ on the first row without changing the law of $H(n,n)$. This makes the previous argument rigorous.
- In the critical case, we expect that geodesics take $\mathcal{O}(n^{2/3})$ weights on the diagonal. Where?

Passage times away from the boundary

Theorem (Sasamoto-Imamura 2005/Baik-B.-Corwin-Suidan 2016)

For $\kappa \in (0,1)$ *and* $\alpha > \sqrt{\kappa}/(1+\sqrt{\kappa})$*,*

$$\frac{H(n,\kappa n) - (1+\sqrt{\kappa})^2 n}{\sigma n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GUE}}.$$

- We recover the exact same fluctuations as for LPP in a full quadrant. The boundary has no influence as long as the boundary weights are not too large.
- BBP transition arises at $\alpha = \sqrt{\kappa}/(1+\sqrt{\kappa})$.

Crossover

Consider two parameters $\varpi \in \mathbb{R}, \eta > 0$.

Theorem (Baik-B.-Corwin-Suidan 2016)

When the boundary parameter scales as

$$\alpha = \frac{1}{2} + 2^{-4/3} \varpi n^{-1/3},$$

and one consider passage times at distance $\eta\, n^{2/3}$ from the boundary,

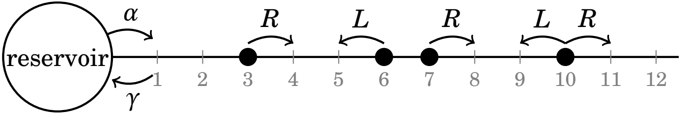
$$H_n(\eta, \varpi) := \frac{H\left(n + 2^{2/3} \eta n^{2/3}, n - 2^{2/3} \eta n^{2/3}\right) - 4n + n^{1/3} 2^{4/3} \eta^2}{2^{4/3} n^{1/3}},$$

The (multipoint) limiting distribution of $H_n(\eta, \varpi)$ is a two-parametric distribution that interpolates between GUE, GOE and GSE Tracy-Widom distributions, characterized by a correlation kernel $K^{crossover}_{\omega,\eta}$, it is not TW_β.

Random matrix interpretations

- When $\omega \to +\infty$, $K^{crossover}_{+\infty,\eta}$ becomes the correlation kernel of a point configuration corresponding to the limiting eigenvalues of a Hermitian complex matrix X_η for $\eta \in (0, +\infty)$ with density proportional to
$$\exp\left(\frac{-\mathrm{Tr}\left((X_\eta - e^{-\eta}X_0)^2\right)}{1-e^{-2\eta}}\right),$$
where X_0 is a GSE matrix [Forrester-Nagao-Honner 1999, Sasamoto-Imamura 2004].
- When $\omega = 0$, $K^{crossover}_{0,\eta}$ has an analogous interpretation with X_0 being a GOE matrix.
- The largest eigenvalue of a rank 1 perturbation of the GSE has Tracy-Widom GOE fluctuations in the critical scaling [Wang], so that one expects that in general, $K^{crossover}_{\omega,\eta}$ corresponds to the eigenvalues of
$$GSE + DBM(\eta) + \text{rank 1 perturbation}(\omega).$$

Back to the asymmetric case

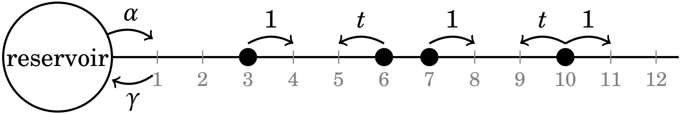


Notations

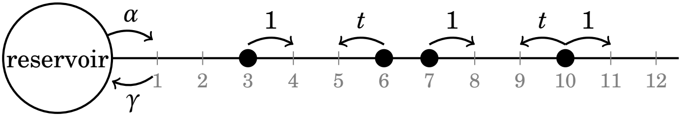
- Without loss of generality, one can assume $R = 1$.
- We denote the parameter L by $t \in [0,1)$.
- Denote time by τ.
- Recall

$$N_x(\tau) = \#\{\text{particles on the right of } x \text{ at time } \tau\},$$

at large times τ.



Previous results



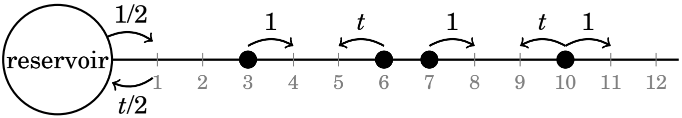
- [Liggett 1975] classified the stationary measures when

$$\alpha + \frac{\gamma}{t} = 1.$$

  Then α is the average density enforced at the boundary. There is a **phase transition** at $\alpha = 1/2$ between product-form Bernoulli measure and spatially correlated stationary measures (which can be expressed using Matrix Product Ansatz [Derrida-Evans-Hakim-Pasquier 1993]).
- [Tracy-Widom 2013] used coordinate **Bethe ansatz** to find formulas for the transition probabilities, but these do not seem amenable for asymptotic analysis.
- We analyze half-line ASEP through a half space version of the **stochastic six-vertex model**, that will be defined later. (analogously as in the full-space [Borodin-Corwin-Gorin 2014, Aggarwal-Borodin 2016, Aggarwal 2016, Borodin-Olshanski 2016])

We assume

1. Ligget's condition.
2. The boundary enforces a density of particles $\alpha = 1/2$ at the origin.



Theorem (B.-Borodin-Corwin-Wheeler 2017)

For any $t \in [0,1)$, starting from the empty initial condition,

$$\frac{N_0\left(\frac{T}{1-t}\right) - \frac{T}{4}}{2^{-4/3}T^{1/3}} \xrightarrow[T\to\infty]{} -\mathcal{L}_{GOE}.$$

Recall $N_0(\tau)$ is the number of particles in the system at time τ.

- Based on the prediction that ASEP fluctuations are the same as TASEP modulo a rescaling by the asymmetry, one expects diffusive scaling in the low density phase $\alpha < 1/2$ and GSE fluctuations in the high density phase $\alpha > 1/2$.

KPZ equation in a half-space

Consider

$$(SHE)\quad \begin{cases} \partial_\tau Z = \frac{1}{2}\partial_{xx} Z + Z\dot{W} \\ \partial_x Z(x,\tau)\Big|_{x=0} = a\ Z(\tau,0) \end{cases}$$

on $\mathbb{R}_+$ with delta initial data at the origin, in the mild sense [Corwin-Shen 2016]:

$$Z(x,\tau) = p^a_\tau(x,0) + \int_0^\tau \int_0^\infty p^a_{\tau-s}(x,y) Z(y,s)\,\mathrm{d}W_s(\mathrm{d}Y)$$

where the last integral is the Itô integral with respect to Wiener process W, and p^a is the heat kernel satisfying the Robin boundary condition

$$\partial_x p^a_\tau(x,y)\Big|_{x=0} = a\ \ p^a_\tau(0,y) \qquad (\forall \tau > 0, y > 0)\,.$$

One can show that a.s. $Z(x,\tau) > 0$ and we define the solution of the KPZ equation

$$(KPZ)\quad \begin{cases} \partial_\tau h = \frac{1}{2}\partial_{xx} h + \left(\partial_x h\right)^2 + \dot{W} \\ \partial_x h(x,\tau)\Big|_{x=0} = a. \end{cases}$$

in the Cole-Hopf sense, i.e. as $h = \log(Z)$. (see [Gerencsér-Hairer 2017] about the meaning of the boundary condition)

Weakly asymmetric scaling of ASEP

Theorem (B.-Borodin-Corwin-Wheeler 2017)

Under the scalings

$$t = e^{-\varepsilon}, \qquad \tau \approx 8\varepsilon^{-4}\hat{\tau},$$

the random variable

$$\mathcal{Z}_\varepsilon(\tilde{\tau}) = \frac{4\exp\left[-\varepsilon N(\tau) - 2\varepsilon^{-2}\hat{\tau})\right]}{1-t^2}$$

weakly converges as $\varepsilon \to 0$ to a positive random variable $\mathcal{Z}(\tilde{\tau})$. For any $z > 0$,

$$\mathbb{E}\left[\exp\left(\frac{-z}{4}\mathcal{Z}(\tau)\right)\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty}\sqrt{\frac{1}{1+z\exp\left((\tau/2)^{1/3}\mathfrak{a}_i^{\mathrm{GOE}}\right)}}\right],$$

where $\{\mathfrak{a}_i^{\mathrm{GOE}}\}_{i=1}^{\infty}$ forms the GOE point process (i.e. the sequence of rescaled eigenvalues of a large Gaussian real symmetric matrix).

Interpretation

- Using results from [Corwin-Shen 2016], $\log \mathcal{Z}(\tau) - \tau/24$ is expected to have the law of the solution to KPZ equation $h(0,\tau)$ with boundary parameter $a = -1/2$ (though [Corwin-Shen] work with $a \geqslant 0$).
- The result should be compared with the analogous full-space result ([Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn 2011, Borodin-Gorin 2016]) where

$$(full-space) \qquad \mathbb{E}\left[\exp\left(\frac{-z}{4}\mathcal{Z}(\tau)\right)\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \frac{1}{1+z\exp\left((\tau/2)^{1/3}\mathfrak{a}_i^{\mathrm{GUE}}\right)}\right],$$

$$(half-space) \qquad \mathbb{E}\left[\exp\left(\frac{-z}{4}\mathcal{Z}(\tau)\right)\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} \sqrt{\frac{1}{1+z\exp\left((\tau/2)^{1/3}\mathfrak{a}_i^{\mathrm{GOE}}\right)}}\right],$$

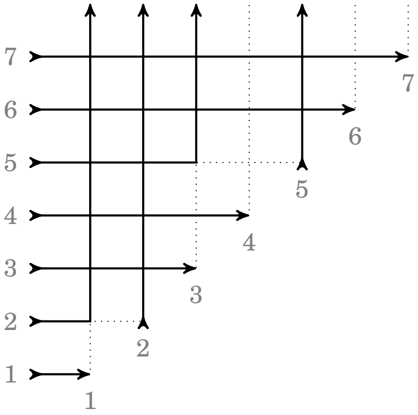
- In the cases $a = +\infty$ [Le Doussal-Gueudre 2012] and $a = 0$ [Borodin-Bufetov-Corwin 2015] there exist non rigorous results about the law of $\log(\mathcal{Z}(\tau))$, though only when $\tau \to \infty$.

Proof sketch

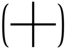
1. We access ASEP through the stochastic six-vertex model in a half-quadrant.
2. The latter is a marginal of Half-space Macdonald processes. A variant of Borodin-Corwin's Macdonald processes.
3. Exploit properties of Macdonald symmetric functions to compute observables.
4. Asymptotic analysis of Fredholm Pfaffians in the two asymptotic regimes (ASEP height function at large times, weakly asymmetric regime)

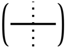
Stochastic six vertex model in a half space

Let $a_1, a_2, \dots \in (0,1)$.

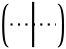


6 vertex configurations:

$\mathbb{P}\Big($  $\Big) = 1,$

$\mathbb{P}\Big($  $\Big) = \dfrac{1-a_x a_y}{1-t a_x a_y},$

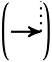
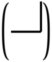
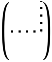
$\mathbb{P}\Big($  $\Big) = \dfrac{(1-t)a_x a_y}{1-t a_x a_y},$

$\mathbb{P}\Big($  $\Big) = \dfrac{t(1-a_x a_y)}{1-t a_x a_y},$

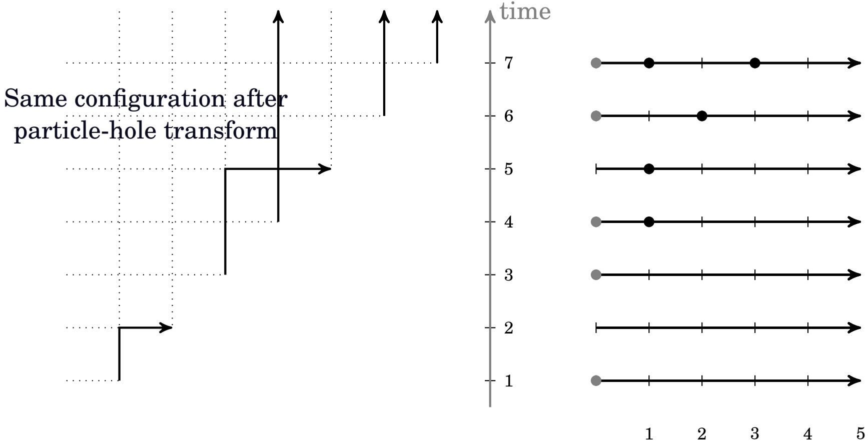
$\mathbb{P}\Big($  $\Big) = \dfrac{1-t}{1-t a_x a_y},$

$\mathbb{P}\Big($  $\Big) = 1.$

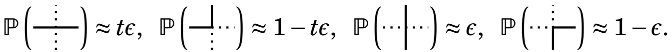
We use the boundary weights:

$\mathbb{P}\Big($  $\Big) = \mathbb{P}\Big($  $\Big) = 1, \qquad \mathbb{P}\Big($  $\Big) = \mathbb{P}\Big($  $\Big) = 0.$

Limit to ASEP on a half quadrant



If the parameters are scaled such that $a_x \equiv 1 - \frac{(1-t)\epsilon}{2}$,



and paths will almost always zig-zag and do something else at rates 1 and t.

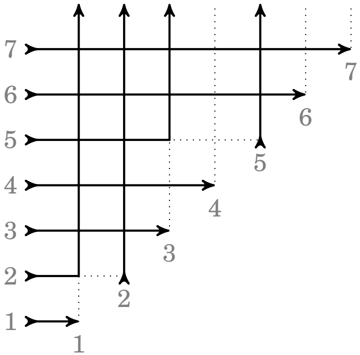
Half-space Macdonald measures

- An integer partition λ is a sequences of integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$. Symmetric Macdonald polynomials P_λ, Q_λ are symmetric polynomials in many variables whose coefficients are rational functions in two parameters $q, t \in (0,1)$. They degenerate to Schur functions s_λ when $q = t$.
- For a set of variables $a_1, \dots, a_n$, define the **Half-space Macdonald measure** as

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Phi(a_1, \dots, a_n)} P_\lambda(a_1, \dots, a_n)\, b_\lambda^{\mathrm{el}} \mathbb{1}_{\lambda' \mathrm{even}},$$

  where λ' even means $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$, and $\Phi(a)$ is an explicit normalization constant.
- It is a variant of the Macdonald measure [Borodin-Corwin 2011] which is a (q,t)-generalization of the Schur measure [Okounkov 2001]. As in the full-space case, one can define more general **half-space Macdonald processes**. Half-space Macdonald processes degenerate when $q = t$ to Pfaffian Schur processes.

Stochastic six vertex model in a half space and Hall-Littlewood functions



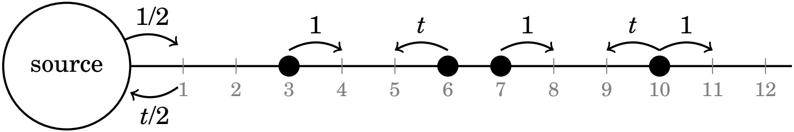
- Let $\mathfrak{h}(x,y)$ be the number of outgoing vertical arrows from the vertices on the left of (x,y).
- Let $\ell(\lambda)$ be the number of nonzero components in a partition λ following the Half-space Hall-Littlewood measure (i.e. Macdonald measure when $q=0$).

Theorem (B.-Borodin-Corwin-Wheeler 2017)

$$\mathfrak{h}(n,n) \overset{(d)}{=} \ell(\lambda).$$

Similar results exist for full-space models [Borodin 2016, Borodin-Bufetov-Wheeler 2016, Bufetov-Matveev 2017].

Laplace transform of ASEP current



Recall that $N_0(\tau)$ denotes the total number of particles in the system at time τ.

Theorem (B.-Borodin-Corwin-Wheeler 2017)

For any time $\tau > 0$ and $x \in \mathbb{R}$,

$$\mathbb{E}\left[\frac{1}{(-t^{x+N_0(\tau)}, t^2)_\infty}\right] = \mathrm{Pf}\left[\mathsf{J} + \mathsf{f}_x \cdot \mathsf{K}^{\mathrm{ASEP}}\right]_{\ell^2(\mathbb{Z}_{\geqslant 0})}$$

where $\mathsf{K}^{\mathrm{ASEP}}$ is a certain kernel expressible exactly as contour integrals.

The L.H.S of the equation should be thought of as a deformed Laplace transform. Half-line ASEP and KPZ equation limit theorems result from an asymptotic analysis of the above identity.

How to extract information from Macdonald measures?

- Usual full space Macdonald measures are such that

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Pi(a,b)} P_\lambda(a_1,\dots,a_n) Q_\lambda(b_1,\dots,b_n).$$

- In general, one may act with difference operators diagonalized by Macdonald symmetric functions in order to compute observables [Borodin-Corwin 2011, Borodin-Corwin-Gorin-Shakirov 2012].
- In the $q=t$ case, the process is determinantal (Schur process).
- In the Hall-Littlewood ($q=0$) case, one may use that certain observables do not depend on q and exploit the determinantal structure of the $q=t$ case.

Refined Cauchy identity

Proposition ([Warnaar 2008])

For $u \in \mathbb{C}$,

$$\frac{1}{\Pi(a,b)} \sum_{\lambda} \prod_{i} \left(1 - uq^{\lambda_i} t^{n-i}\right) P_{\lambda}(a)\, Q_{\lambda}(b) = \frac{\det\left[\frac{1}{1-a_i b_j} - u \frac{1}{1-t a_i b_j}\right]}{\det\left[\frac{1}{1-a_i b_j}\right]}.$$

It implies that

$$\mathbb{E}^{q,t}\left[\prod_{i=1}^{n} \left(1 - uq^{\lambda_i} t^{n-i}\right)\right]$$

does not depend on q! Comparing the $q=0$ and $q=t$ cases yields identities relating functionals of Schur ($q=t$) and Hall-Littlewood ($q=0$) random partitions.

Refined Littlewood identity

For half-space Macdonald processes, recall

$$\mathbb{P}^{q,t}(\lambda) = \frac{1}{\Phi(a_1,\dots,a_n)} P_\lambda(a_1,\dots,a_n)\, b_\lambda^{\mathrm{el}} \mathbb{1}_{\lambda' \mathrm{even}},$$

Proposition ([Rains 2015], [Betea-Wheeler-Zinn-Justin 2015])

For $u \in \mathbb{C}$*,*

$$\frac{1}{\Phi(a)} \sum_{\lambda' \text{ even}} \prod_{i \text{ even}} \left(1 - u q^{\lambda_i} t^{n-i}\right) b_\lambda^{\mathrm{el}} P_\lambda(a_1,\dots,a_n) = \frac{\mathrm{Pf}\left[\frac{a_i - a_j}{1 - a_i a_j} - u \frac{a_i - a_j}{1 - t a_i a_j}\right]}{\mathrm{Pf}\left[\frac{a_i - a_j}{1 - a_i a_j}\right]}.$$

It implies that

$$\mathbb{E}^{q,t}\left[\prod_{i \text{ even}} \left(1 - u q^{\lambda_i} t^{n-i}\right)\right]$$

does not depend on q! Comparing the $q = 0$ and $q = t$ cases yields identities relating functionals of (half-space) Schur and Hall-Littlewood random partitions.

Conclusion

We have shown GOE asymptotics for ASEP and KPZ with a specific Neumann boundary condition at zero.

Further directions in progress

- **More general boundary conditions**. This requires going higher in the hierarchy of integrable structures.
- Other interesting models are coming from Half-space Macdonald processes: **Log gamma directed polymer in a half space**.
- **General approach** to extract the distribution of half-space Macdonald processes.

Ultimately, the Laplace transform of **KPZ equation in a half space at any space point and for general boundary condition** should be a multiplicative functional of a certain point process corresponding to the two-dimensional crossover kernel obtained in LPP,

$$\mathbb{E}\left[e^{-u\mathcal{Z}(\tau,x)}\right] = \mathbb{E}\left[\prod_{i=1}^{+\infty} f_u^x\left(\mathfrak{a}_i^{\text{crossover}}\right)\right]?$$

Thank you