Fluctuations of the first particle in exclusion processes

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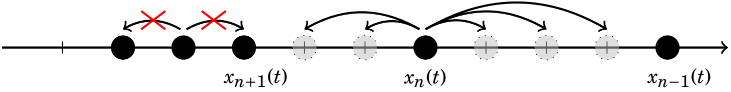
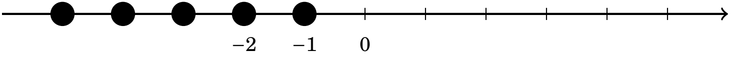
Motivations

We consider continuous-time **exclusion processes** on $\mathbb{Z}$,



starting from the step initial condition



Under mild hypotheses, we expect that for $\kappa \in (0, \kappa^*)$,

$$\frac{x_{\lfloor \kappa t \rfloor} - ct}{\sigma t^{1/3}} \Longrightarrow -\mathscr{L}_{\mathrm{GUE}},$$

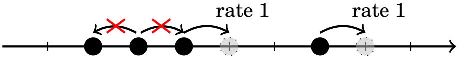
the Tracy-Widom **GUE** distribution.

Question

Is the behaviour of $x_1(t)$ universal as well?

Answer: NO

TASEP:

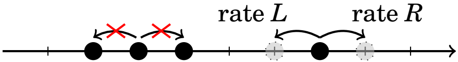


By the CLT, we have

$$\frac{x_1(t)-t}{\sqrt{t}} \Longrightarrow \mathcal{N}.$$

The same limit theorem holds for any totally asymmetric exclusion processes.

ASEP: Let $R > L > 0, R+L=1$ be asymmetry parameters



Theorem (Tracy-Widom 2009)

$$\frac{x_1(t)-(R-L)t}{\sigma\sqrt{t}} \Longrightarrow \mathscr{X},$$

where $\mathscr{X}$ *is not a Gaussian.* $\mathbb{P}(\mathscr{X} \leqslant x) = \det(I-K)_{\mathbb{L}^2(x,\infty)}$ *where*

$$K(x,y) = \frac{R}{\sqrt{2\pi}} e^{-(R^2+L^2)\frac{x^2+y^2}{4}+RLxy}.$$

MADM

The Multi-particle Asymmetric Diffusion Model (Sasamoto-Wadati 1998) is another exactly solvable partially asymmetric exclusion process.

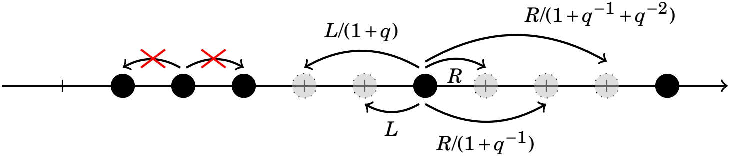
Fix a parameter $q \in (0,1)$, asymmetry parameters $R > L > 0$, $R + L = 1$.

The particle at $x_n(t)$ jumps to

- $x_n(t) + j$ at rate $\frac{R}{[j]_{q^{-1}}}$ for any $j \in \{1, \ldots, x_{n-1}(t) - x_n(t) - 1\}$,
- $x_n(t) - j$ at rate $\frac{L}{[j]_q}$ for any $j \in \{1, \ldots, x_n(t) - x_{n+1}(t) - 1\}$,

where the q- deformed integer $[j]_q$ is given by

$$[j]_q = 1 + q + \cdots + q^{j-1},$$
$$[j]_{q^{-1}} = 1 + q^{-1} + \cdots + q^{-j+1}.$$



Limit Theorem

Theorem (B.-Corwin 2014)

There exist constants c, σ, L^* *such that for* $0 < L < L^*$

$$\frac{x_1(t) - ct}{\sigma t^{1/3}} \Longrightarrow -\mathscr{L}_{\mathrm{GUE}}.$$

The result should hold with $L^* = 1/2$. The first particle behaves as in the bulk. Indeed, one can prove the one-point asymptotics predicted by KPZ universality,

Theorem (B.-Corwin 2014)

There exist constants $c(\kappa), \sigma(\kappa), L^*, \kappa^*$ *such that for* $0 \leqslant L < L^*$ *and* $\kappa \in (0, \kappa^*)$,

$$\frac{x_{\lfloor \kappa t \rfloor}(t) - c(\kappa)t}{\sigma(\kappa) t^{1/3}} \Longrightarrow -\mathscr{L}_{\mathrm{GUE}}.$$

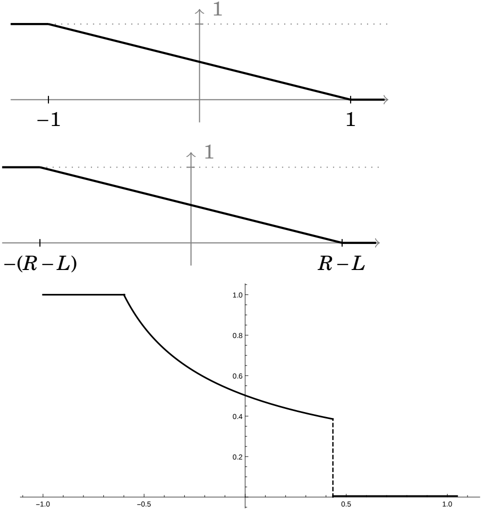
Why so different than ASEP?

Let $\rho(x) :=$ density of particles around xt at time t as t goes to infinity.

TASEP

ASEP

MADM



Universality?

Question

For exclusion processes such that the density around the first particle is positive, are the $t^{1/3}$ scaling and GUE distribution universal?

In order to test the universality, one needs at least one other such process.

Question

When is the density of particles positive around the first particle?

The density profile has a jump discontinuity when the drift (average speed of a tagged particle) is not decreasing as a function of the local density.

Hydrodynamic limit

- Assume that there exists a family of translation invariant stationary measures indexed by the average density of particles ϱ.
- Define the flux $j(\varrho)$ as the expected (for that measure) number of particles crossing a given bound per unit of time, counted algebraically.
- Assume that the following limit exists

$$\rho(x,t) := \lim_{\tau\to\infty} \mathbb{P}\big(\text{There is a particle at site } x\tau \text{ at time } t\tau\big).$$

It satisfies the conservation equation

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} j(\rho(x,t)) = 0.$$

heuristic result: Let ϱ_0 be the density of particles around the first particle. The density profile is discontinuous at the first particle (i.e. $\varrho_0 > 0$) when the function $j(\varrho)/\varrho$ is not decreasing. Actually ϱ_0 locally maximizes the drift, $j(\varrho)/\varrho$.

Heuristic proof

Assume $\varrho_0 > 0$.

(1) On the one hand, the macroscopic position of the first particle is its drift $j(\varrho_0)/\varrho_0$.

(2) On the other hand the characteristics method (applied to the conservation PDE) yields a function $\pi(\varrho)$ s.t.

$$\rho\big(\pi(\varrho)t, t\big) = \varrho. \tag{1}$$

i.e. $\pi(\varrho)$ is the macroscopic position where particles have a local density ϱ. Differentiating (1) yields

$$\pi(\varrho) = \frac{\partial j(\varrho)}{\partial \varrho} = j'(\varrho).$$

Combining (1) and (2), we have that

$$j'(\varrho_0) = \frac{j(\varrho_0)}{\varrho_0} \implies \frac{\mathrm{d}}{\mathrm{d}\varrho} \frac{j(\varrho)}{\varrho}\bigg|_{\varrho=\varrho_0} = 0,$$

which suggests that ϱ_0 is a maximizer of $j(\varrho)/\varrho$.

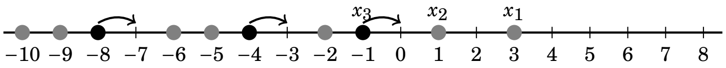
Facilitated TASEP

Question

For exclusion processes such that the density around the first particle is positive, are the $t^{1/3}$ scaling and GUE distribution universal?

We consider the Facilitated TASEP (FTASEP): the particle at $x_n(t)$ moves by +1 at rate 1 provided that

- the site $x_n(t)+1$ is empty (exclusion),
- the site $x_n(t)-1$ is occupied (facilitation).



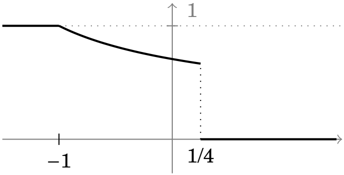
Introduced in physics literature, Basu-Mohanty 2009, and studied further by Gabel-Krapivsky-Redner 2010. The flux

$$j(\rho) = \frac{(1-\rho)(2\rho-1)}{\rho} \mathbb{1}_{\rho>1/2}$$

is such that $j(\rho)/\rho$ has a maximum for $\rho = 2/3$.

The density profile is given by

$$\rho(x) = \frac{1}{\sqrt{2+x}} \text{ for } x \in (-1, 1/4).$$



Theorem (Baik-B.-Corwin-Suidan)

$$\frac{x_1(t) - t/4}{2^{-4/3} t^{1/3}} \Longrightarrow -\mathscr{L}_{\mathrm{GSE}},$$

where $\mathscr{L}_{\mathrm{GSE}}$ is the Tracy-Widom ***GSE*** *distribution.*

The FTASEP is in the KPZ universality class in the sense that

Theorem (Baik-B.-Corwin-Suidan)

For all $r \in (0,1)$, there exist (explicit) constants $\pi(r), \sigma(r)$ such that

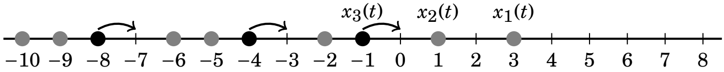
$$\frac{x_{\lfloor rt \rfloor}(t) - t\pi(r)}{\sigma(r) t^{1/3}} \Longrightarrow -\mathscr{L}_{\mathrm{GUE}},$$

as the KPZ scaling theory predicts.

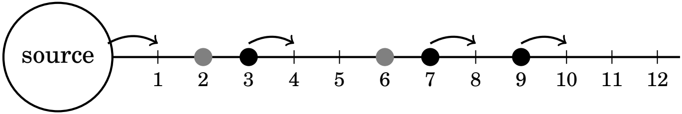
Proofs

- **MADM:** it can be studied via a method initially designed by Borodin-Corwin-Sasamoto 2012 for the q-TASEP and ASEP, using Markov duality and Bethe ansatz.
- **FTASEP:** the solvability comes from a coupling with last passage percolation on a half-quadrant.

FTASEP and OpenTASEP



We use first a coupling between the FTASEP and a TASEP on a semi-infinite lattice with a source at the origin (we call it the OpenTASEP)



Define the current at site x by

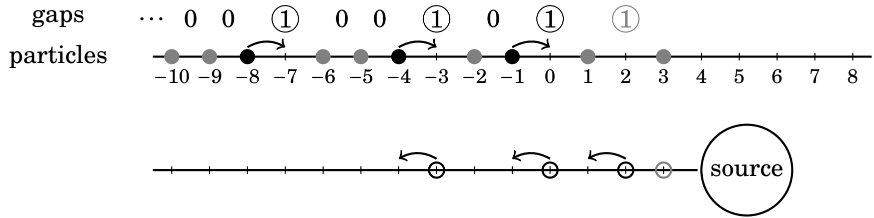
$$N_x(t) = \#\{i \geqslant x \,|\, \text{site } i \text{ is occupied}\}.$$

The coupling

Consider the gaps between consecutive particles in the FTASEP

$$g_i(t) := x_i(t) - x_{i+1}(t) - 1.$$

For all $i \geqslant 1$, the rules of the dynamics implies that $g_i \in \{0, 1\}$.



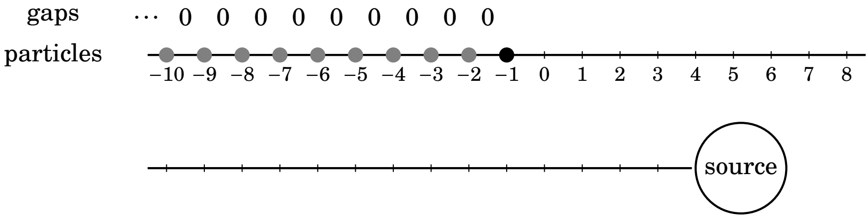
The current at site n in the OpenTASEP corresponds to the number of jumps done by the nth particle in FTASEP, i.e. $x_n(t) + n$.

Proposition

We have the equality in law of the processes

$$\{x_n(t) + n\}_{n \geqslant 1, t \geqslant 0} = \{N_n(t)\}_{n \geqslant 1, t \geqslant 0}.$$

Let us see how this works dynamically



- gray: particle that cannot move,
- black: particle that can move.

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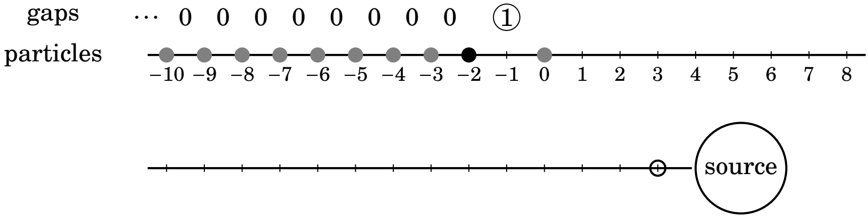

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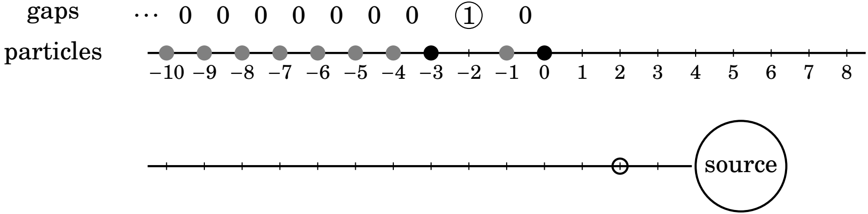

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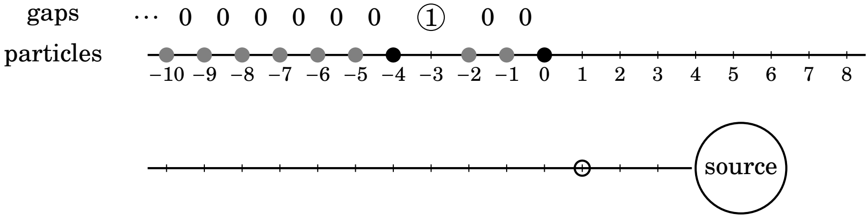

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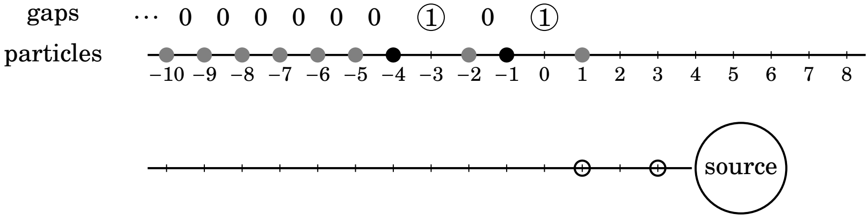

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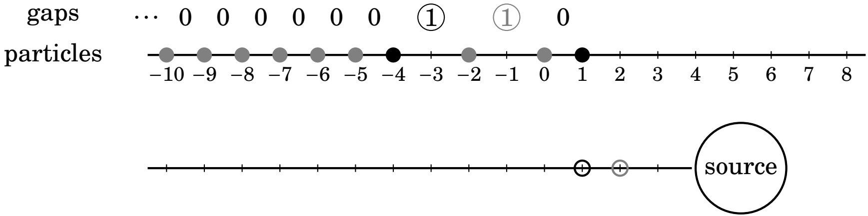

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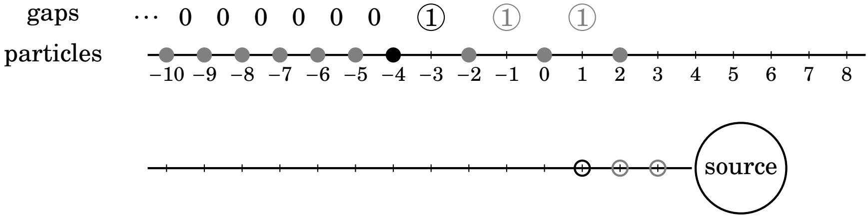

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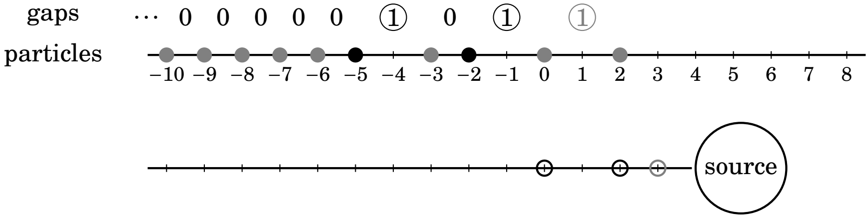

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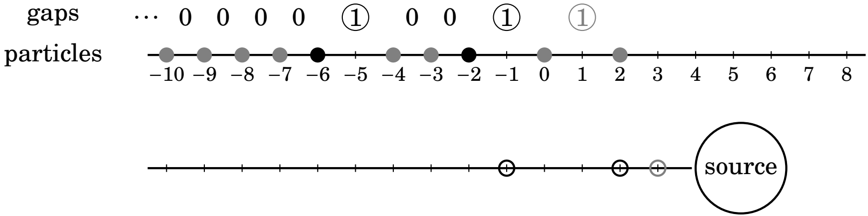

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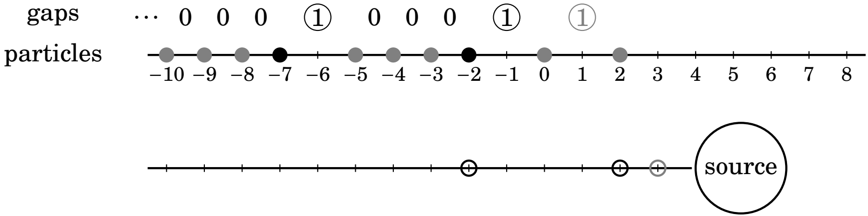

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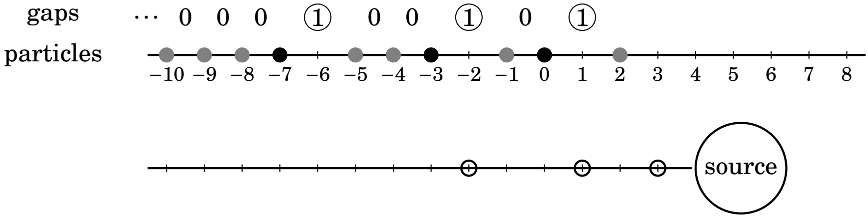

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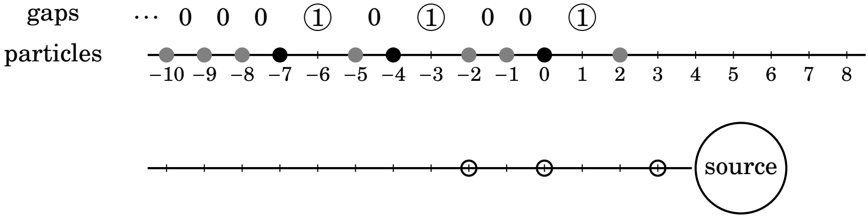

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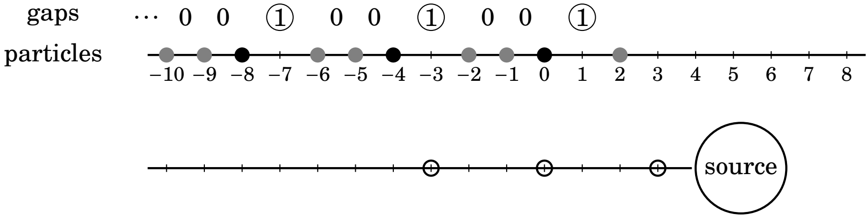

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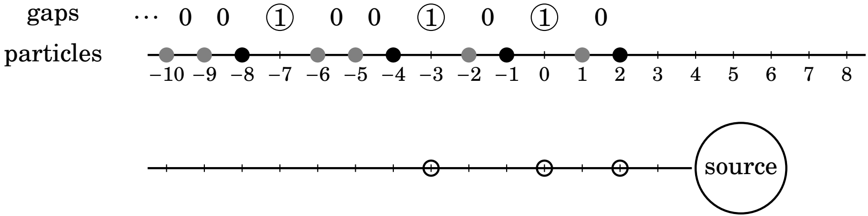

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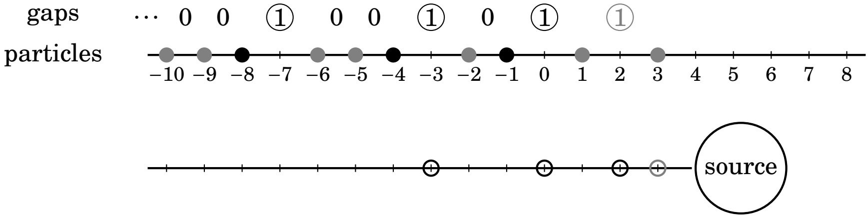

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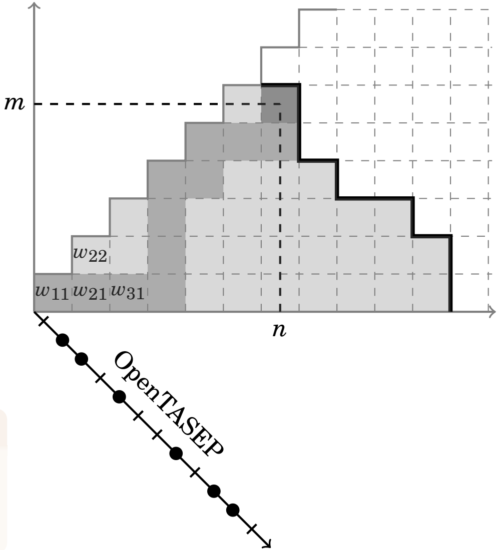
Last passage percolation

- Let w_{ij} a family of i.i.d. exponential random variables.
- Consider up-right paths π from the box $(1,1)$ to (n,m) in the half quadrant. We define the last passage percolation time $H(n,m)$ by

$$H(n,m) = \max_{\pi} \sum_{(i,j)\in\pi} w_{ij}.$$



Lemma

If $w_{ij} \sim \mathrm{Exp}(1)$,

$$\mathbb{P}\big(N_n(t) \leqslant x\big) = \mathbb{P}\big(H(n+x-1,x) \geqslant t\big)$$

$x_1(t)$ in FTASEP corresponds to $H(n,n)$.

Passage-times on the diagonal

LPP in a half-quadrant has first been studied by Baik and Rains (2001) with Geometric weights. In the model with exponential weights, we find similar limit theorems.

Theorem (Baik-B.-Corwin-Suidan)

Assume that $w_{ij} \sim \mathrm{Exp}(1)$ *for* $i > j$ *and* $w_{ii} \sim \mathrm{Exp}(\alpha)$ *for some parameter* $\alpha > 0$.

- *When* $\alpha > 1/2$,
$$\frac{H(n,n) - 4n}{2^{4/3} n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GSE}},$$
(implies the GSE limit theorem for $x_1(t)$ *in FTASEP, corresponding to* $\alpha = 1$.*)*
- *When* $\alpha = 1/2$,
$$\frac{H(n,n) - 4n}{2^{4/3} n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GOE}},$$
- *When* $\alpha < 1/2$,
$$\frac{H(n,n) - cn}{c' n^{1/2}} \Longrightarrow \mathscr{N},$$

The parameter α corresponds to the rate of the first particle in the FTASEP.

Away from the diagonal: KPZ typical behaviour

The fluctuations away from the diagonal have first been studied by Sasamoto-Imamura 2004 – for the discrete PNG model. In the model with exponential weights, we have

Theorem (Baik-B.-Corwin-Suidan)

For $\kappa \in (0,1)$ and $\alpha > \sqrt{\kappa}/(1+\sqrt{\kappa})$,

$$\frac{H(n,\kappa n) - (1+\sqrt{\kappa})^2 n}{\sigma n^{1/3}} \Longrightarrow \mathscr{L}_{\mathrm{GUE}}.$$

(implies the GUE limit theorem for $x_{(1-\kappa)t}$ in FTASEP)

Proofs?

(I) LPP in a half-quadrant is a marginal of a **Pfaffian Schur process**.

(II) By a theorem of Borodin-Rains 2005, it is hence a Pfaffian point process, with explicit correlation kernel.

(III) Saddle-point analysis of the correlation kernel yields the various limit theorems (in progress).

Symmetric functions

For integer partitions $\lambda_1 \geqslant \lambda_2 \geqslant \dots$, and $\mu_1 \geqslant \mu_2 \geqslant \dots$, we will consider skew-Schur functions

$$s_{\lambda/\mu} = \det\left[h_{\lambda_i - \mu_j + j - i}\right]_{i,j},$$

where h_k are complete homogeneous symmetric functions

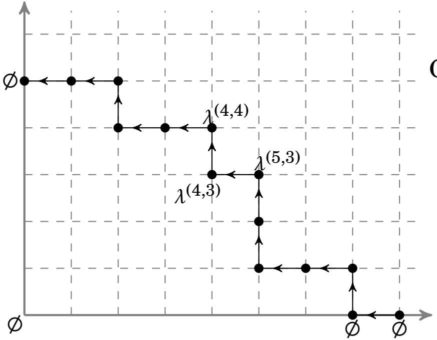
$$h_k(x) = \sum_{i_1 \leqslant \dots \leqslant i_k} x_{i_1} \dots x_{i_k}.$$

We also define

$$\tau_\lambda = \sum_{\kappa' \text{even}} s_{\lambda/\kappa} = \mathrm{Pf}[\dots]$$

where κ' even means that $\kappa_1 = \kappa_2 \geqslant \kappa_3 = \kappa_4 \geqslant \dots$.

Schur process



Consider a path γ as on the left

- vertex $v \mapsto \lambda^v$ a random partition,
- edge $e \mapsto \rho_e$ a set of variables. (More generally a specialization of the symmetric functions).

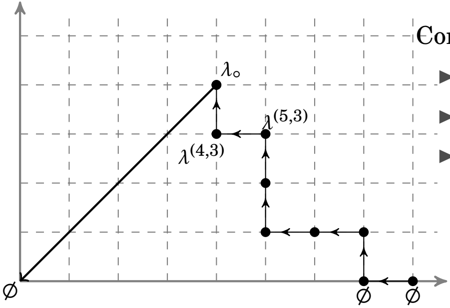
The **Schur process** (Okounkov-Reshetikhin 2003) is a probability measure on the sequence of partitions $\lambda := \left(\lambda^v\right)_{v \in \gamma}$ such that

$$\mathbb{P}(\lambda) = \frac{1}{Z} \prod_{e \in \gamma} \text{weight}(e) = \frac{1}{Z} \det[\ldots],$$

where

$$\text{weight}\left(e = {}_{v'}\!\leftarrow_{v}\right) = s_{\lambda^v/\lambda^{v'}}(\rho_e) \text{ and } \text{weight}\left(e = \uparrow^{v'}_{v}\right) = s_{\lambda^{v'}/\lambda^{v}}(\rho_e).$$

Pfaffian Schur process



Consider a path γ as on the left

- vertex $v \mapsto \lambda^v$ a random partition,
- edge $e \mapsto \rho_e$ a set of variables.
- Denote $\rho_\circ$ and $\lambda_\circ$ the specialization and the partition on the diagonal.

The **Pfaffian Schur process** is a probability measure on the sequence of partitions $\lambda := \left(\lambda^v\right)_{v\in\gamma}$ such that

$$\mathbb{P}(\lambda) = \frac{1}{Z}\tau_{\lambda_\circ}(\rho_\circ)\prod_{e\in\gamma}\mathrm{weight}(e) \;\; = \frac{1}{Z}\mathrm{Pf}[\ldots],$$

where the weight of off-diagonal edges are chosen as in the Schur process.

Geometric last passage percolation

Assume that all $\rho_e = \{\sqrt{q}\}$, and $\rho_\circ = \{c\}$. Then for $0 < n_1 \leqslant \dots \leqslant n_k$, $m_1 \geqslant \dots \geqslant m_k$, with $n_i \geqslant m_i$,

$$\left(\lambda_1^{(n_1,m_1)}, \dots, \lambda_1^{(n_k,m_k)}\right) \overset{(d)}{=} \left(G(n_1,m_1), \dots, G(n_k,m_k)\right)$$

where the family of random variables $G(n,m)$ satisfies the recursion

$$\begin{cases} G(n,m) = \max\left\{G(n-1,m), G(n,m-1)\right\} + \mathrm{Geom}(q) \text{ for } n > m \\ G(n,n) = G(n,n-1) + \mathrm{Geom}(c\sqrt{q}). \end{cases}$$

As the geometric distribution converges to the exponential,

Proposition

If we set $c = \sqrt{q}(1 + (\alpha-1)(q-1))$, then as $q \to 1$,

$$\left\{(1-q)G(n_i,m_i)\right\}_{i=1}^k \Longrightarrow \left\{H(n_i,m_i)\right\}_{i=1}^k$$

where $H(n,m)$ are the passage times in LPP with exponential weights on a half quadrant (and parameter α on the diagonal).

Pfaffian Point process

A random configuration $X \subset \mathbb{X}$ (state space) is a **Pfaffian point process** if one can write the correlation function as

$$\rho(Y) = \mathbb{P}(Y \subset X) = \mathrm{Pf}\left[K(x,y)\right]_{x,y \in Y},$$

where

$$K(x,y) = \begin{pmatrix} K_{11}(x,y) & K_{12}(x,y) \\ K_{21}(x,y) & K_{22}(x,y) \end{pmatrix}$$

is a skew-symmetric matrix indexed by elements in $\mathbb{X}$; called the correlation kernel.
The gap probabilities are given by Fredholm Pfaffians

$$\mathbb{P}\big(\text{no point in } Y\big) = \mathrm{Pf}\big(J - K\big)_{\mathbb{L}^2(Y)}$$

where

$$\mathrm{Pf}\big(J - K\big)_{\mathbb{L}^2(Y)} := 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_Y \mathrm{d}x_1 \dots \int_Y \mathrm{d}x_k \mathrm{Pf}\left[K(x_i, x_j)\right]_{i,j=1}^{k}$$

The Pfaffian Schur process is Pfaffian

Theorem (Borodin-Rains 2005)

For $0 < n_1 \leqslant \cdots \leqslant n_k$, $m_1 \geqslant \cdots \geqslant m_k$, with $n_i \geqslant m_i$, the Pfaffian Schur process is Pfaffian in the sense that

$$\left(1, \lambda_i^{(n_1,m_1)} - i\right)_{i\geqslant 1} \cup \cdots \cup \left(k, \lambda_i^{(n_k,m_k)} - i\right)_{i\geqslant 1} \in \mathbb{X} = \{1,\ldots,k\} \times \mathbb{Z}$$

is a Pfaffian point process with an explicit correlation kernel K.

The variables $G(n_i,m_i) \overset{(d)}{=} \lambda_1^{(n_i,m_i)}$ are extremal points in the Pfaffian point process, so that

$$\mathbb{P}\Big(G(n_1,m_1) \leqslant h_1, \ldots, G(n_k,m_k) \leqslant h_k\Big) = \mathrm{Pf}(J-K)_{\mathbb{L}^2(\ldots)}.$$

Finally, sending $q \to 1$ yields the probability distribution of passage times in exponential LPP on the half-quadrant.

In the limit, the state space becomes $\{1,\dots,k\}\times\mathbb{R}$.

Proposition (Baik-B.-Corwin-Suidan)

For $0<n_1<\dots<n_k$, $m_1>\dots>m_k$ *with* $n_i>m_i$, $h_1,\dots,h_k>0$

$$\mathbb{P}\Big(H(n_1,m_1)\leqslant h_1,\dots,H(n_k,m_k)\leqslant h_k\Big)=\mathrm{Pf}\big(J-K^{\exp}\big)_{\mathbb{L}^2\left(\Delta_k(h_1,\dots,h_k)\right)}.$$

where

$$\Delta_k(h_1,\dots,h_k)=\Big\{(i,x)\in\mathbb{Z}\times\mathbb{R}\Big|x>h_i\Big\},$$

and the kernel K *is given by*

$$K_{11}^{\exp}(i,x;j,y)=\frac{1}{(2\mathbf{i}\pi)^2}\int_{\infty e^{-\mathbf{i}\pi/3}}^{\infty e^{\mathbf{i}\pi/3}}\mathrm{d}z\int_{\infty e^{-\mathbf{i}\pi/3}}^{\infty e^{\mathbf{i}\pi/3}}\mathrm{d}w\,\frac{z-w}{4zw(z+w)}e^{-xz-yw}$$
$$\frac{(1+2z)^{n_i}(1+2w)^{n_j}}{(1-2z)^{m_i}(1-2w)^{m_j}}(2z+2\alpha-1)(2w+2\alpha-1),$$

where the contours pass to the right of 0, *and we have formulas of a similar taste for* K_{12} *and* K_{22}.

Since the GSE/GOE/GUE distribution functions can be written as a Fredholm Pfaffian, one concludes by asymptotic analysis of the above formula.

Summary

We have seen that

- The fluctuations of the first particle in exclusion processes are not universal.
- For the FTASEP, we find the **GSE Tracy-Widom distribution**.
- This is proved via a coupling with Last Passage Percolation in a half-quadrant.
- Which can be studied exhaustively via **Pfaffian Schur Processes**, when the weights are geometric or exponential.

Outlook

Further directions

- One can play with parameters in LPP, proving phase transitions and studying crossover distributions.
- There are other marginals of the Pfaffian Schur process (other particle dynamics, symmetric plane partitions...).
- Pfaffian Schur processes can be leveraged to Pfaffian Macdonald processes, leading to positive temperature models.

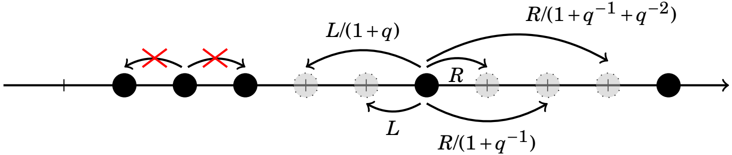
Questions

- In presence of a jump discontinuity, can one prove the $t^{1/3}$ behaviour in general?
- Can one understand the geometric behaviour of the geodesic in LPP ? give a probabilistic interpretation of the phase transition ? Compare to the slow bond problem.

Thank you

Proofs for MADM

MADM



The limit theorem follows from

- A **Markov duality** between the MADM exclusion process and a zero range analogue, so that for $\vec{n} = n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k$, the function

$$(t, \vec{n}) \longmapsto \mathbb{E}\Big[\prod_{i=1}^{k} q^{x_{n_i}(t)} \Big],$$

  satisfies a closed system of differential equations (Kolmogorov equation for the dual system).
- This system of ODEs is solvable via **Bethe ansatz**. It leads to contour integral formulas for the moments of $q^{x_n(t)}$.

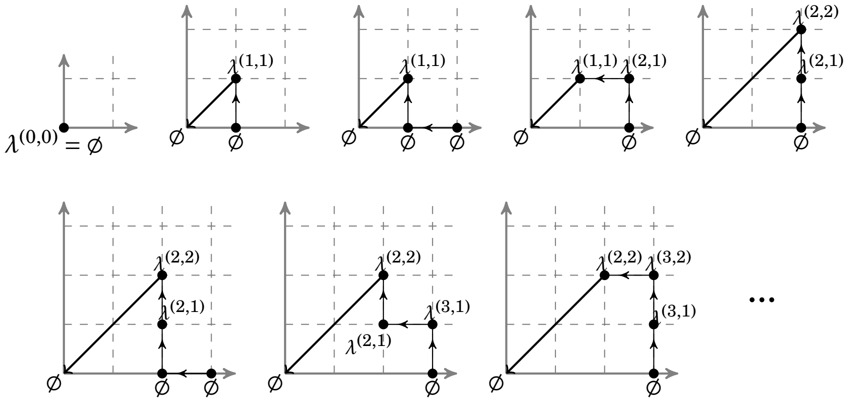
$$\mathbb{E}\left[\prod_{i=1}^{k} q^{x_{n_i}(t)+n_i}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint_{\gamma_1} \cdots \oint_{\gamma_k} \prod_{1\leq A<B\leq k} \frac{z_A - z_B}{z_A - q z_B}$$
$$\times \prod_{j=1}^{k} \left(\frac{1-qz_j}{1-z_j}\right)^{n_j} \exp\left((q-1)t\left(\frac{Rz_j}{1-qz_j} - \frac{Lz_j}{1-z_j}\right)\right) \frac{dz_j}{z_j(1-qz_j)},$$

where the integration contours $\gamma_1, \ldots, \gamma_k$ are nested in order to enclose all poles except 0 and $1/q$.

- The moments do characterize the distribution of $x_n(t)$. One can take the moment generating function and form the (q-deformed) Laplace transform of $q^{x_n(t)}$.
- Rearranging terms as in a Fredholm determinant expansion, a saddle-point asymptotic analysis yields the GUE limit theorem.

Dynamics on the Pfaffian Schur Process

We define dynamics preserving the Pfaffian Schur processes that correspond to LPP in a half quadrant. We make a path γ grow as follows



At each stage we consider a Pfaffian Schur process indexed by the path. We update the partitions where the path has changed according to Markov transition kernels.

When the path grows by one box from a corner formed by partitions κ, μ and ν, we update according to some transition kernel

$$\begin{tikzcd} \nu & \pi \arrow[l, "\rho_2"'] & & \nu & \pi \arrow[l, "\rho_2"'] \\ \mu \arrow[u, "\rho_1"] & \kappa \arrow[l, "\rho_2"] \arrow[u, "\rho_1"'] \arrow[rr, "\mathscr{U}^{\llcorner}_{\rho_1,\rho_2}"] & & \mu \arrow[u, "\rho_1"] & \kappa \arrow[l, "\rho_2"] \arrow[u, "\rho_1"'] \end{tikzcd}$$

where we need that

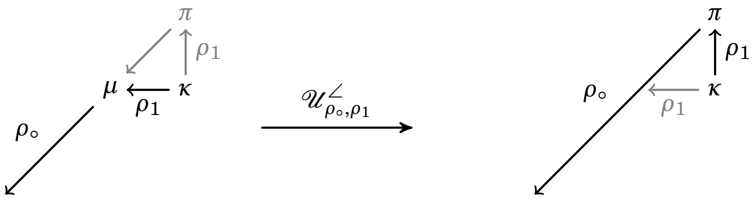
$$\sum_{\mu} s_{\kappa/\mu}(\rho_2) s_{\nu/\mu}(\rho_1) \mathscr{U}^{\llcorner}_{\rho_1,\rho_2}(\pi|\nu,\mu,\kappa) = const.\ s_{\pi/\kappa}(\rho_1) s_{\pi/\nu}(\rho_2)$$

so that the Pfaffian Schur structure is preserved. *const* is a normalization constant depending only on the specializations ρ_1, ρ_2. We choose

$$\mathscr{U}^{\llcorner}_{\rho_1,\rho_2}(\pi|\nu,\mu,\kappa) = \mathscr{U}^{\llcorner}_{\rho_1,\rho_2}(\pi|\nu,\kappa) = \frac{s_{\pi/\nu}(\rho_2) s_{\pi/\kappa}(\rho_1)}{\sum_\lambda s_{\lambda/\nu}(\rho_2) s_{\lambda/\kappa}(\rho_1)}.$$

This corresponds to so-called "push-block" dynamics in the usual (determinantal) Schur process.

Similarly, when the path grows by a half-box along the diagonal, we update according to



where we need that

$$\sum_{\mu} s_{\kappa/\mu}(\rho_1)\tau_{\mu}(\rho_\circ)\mathscr{U}^{\angle}_{\rho_\circ,\rho_1}(\pi|\kappa,\mu) = const.\ s_{\pi/\kappa}(\rho_1)\tau_{\pi}(\rho_\circ)$$

so that the Pfaffian Schur structure is preserved.
We choose

$$\mathscr{U}^{\angle}_{\rho_\circ,\rho_1}(\pi|\kappa,\mu) = \mathscr{U}^{\angle}_{\rho_\circ,\rho_1}(\pi|\kappa) = const\,\frac{\tau_{\pi}(\rho_\circ)s_{\pi/\kappa}(\rho_1)}{\tau_{\kappa}(\rho_\circ,\rho_1)}.$$

First coordinate marginal

Assume that all ρ_e are specializations into a single variable $\rho_e = \sqrt{q}$, and $\rho_\circ = c$. Then we have that

$$s_{\lambda/\mu}(\rho_e) = \mathbb{1}_{\mu \prec \lambda} \left(\sqrt{q}\right)^{\sum \lambda_i - \sum \mu_i}.$$

where

$$\mu \prec \lambda \iff \lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant\geqslant \mu_2 \geqslant \ldots,$$

and

$$\tau_\lambda(\rho_\circ) = c^{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \ldots}.$$

- Under the transition operator $\mathscr{U}^{\llcorner}(\pi|\nu,\kappa)$,

$$\pi_1 = \max\{\nu_1, \kappa_1\} + \mathrm{Geom}(q).$$

- Under the transition operator $\mathscr{U}^{\angle}(\pi|\kappa)$,

$$\pi_1 = \kappa_1 + \mathrm{Geom}(q).$$

Geometric last passage percolation

It implies that for $0 < n_1 \leqslant \cdots \leqslant n_k$, $m_1 \geqslant \cdots \geqslant m_k$, with $n_i \geqslant m_i$,

$$\left(\lambda_1^{(n_1,m_1)}, \dots, \lambda_1^{(n_k,m_k)}\right) \overset{(d)}{=} \left(G(n_1,m_1), \dots, G(n_k,m_k)\right)$$

where the family of random variables $G(n,m)$ satisfies the recursion

$$\begin{cases} G(n,m) = \max\left\{G(n-1,m), G(n,m-1)\right\} + \mathrm{Geom}(q) \text{ for } n > m \\ G(n,n) = G(n,n-1) + \mathrm{Geom}(q). \end{cases}$$

As the geometric distribution converges to the exponential,

Proposition

If we set $\rho_\circ = c = \sqrt{q}(1+(\alpha-1)(q-1))$*, then as* $q \to 1$*,*

$$\left\{(1-q)G(n_i,m_i)\right\}_{i=1}^{k} \Longrightarrow \left\{H(n_i,m_i)\right\}_{i=1}^{k}$$

where $H(n,m)$ *are the passage times in LPP with exponential weights on a half quadrant (and parameter* α *on the diagonal).*