Diffusions in random environment

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1. Random walks in space-time iid random environment
2. Integrability (Bethe ansatz)
3. Continuous limit and sticky Brownian motions
4. Stochastic PDEs and turbulence

1+2: arXiv:1503.04117 (with Ivan Corwin)

3+4: arXiv:1905.10280 (with Mark Rychnovsky)

Consider the simple random walk X_t on $\mathbb{Z}$, starting from 0.

$$\mathsf{P}\big(X_{t+1} = X_t + 1\big) = \frac{\alpha}{\alpha+\beta}, \quad \mathsf{P}\big(X_{t+1} = X_t - 1\big) = \frac{\beta}{\alpha+\beta}.$$

The CLT says that

$$\frac{X_t - t\frac{\alpha-\beta}{\alpha+\beta}}{\sigma\sqrt{t}} \Longrightarrow \mathcal{N}(0,1).$$

where $\sigma = 2\sqrt{\alpha\beta}/(\alpha+\beta)$.

Theorem ([Cramér 1938])

For $\frac{\alpha-\beta}{\alpha+\beta} < x < 1$,

$$\frac{\log\Big(\mathsf{P}(X_t > xt)\Big)}{t} \xrightarrow[t\to\infty]{} -I(x),$$

where $I(x) = \sup_{z\in\mathbb{R}}\big(zx - \lambda(z)\big)$ *is the Legendre transform of*

$$\lambda(z) := \log\Big(\mathbb{E}\big[e^{zX_1}\big]\Big),$$

e.g. for $\alpha = \beta$, $I(x) = \frac{1}{2}((1+x)\log(1+x) + (1-x)\log(1-x))$.

Random walk in random environment

Consider simple random walk on $\mathbb{Z}$ in space-time i.i.d. environment:

$$\mathsf{P}\big(X_{t+1} = x+1 \big| X_t = x\big) = B_{t,x}, \quad \mathsf{P}\big(X_{t+1} = x-1 \big| X_t = x\big) = 1 - B_{t,x},$$

where $(B_{t,x})_{t,x}$ are i.i.d.

Notation

- $\mathbb{P}, \mathbb{E}$: measure/expectation on the environment.
- P, E : measure/expectation on random walk paths, conditionally on the environment.

Averaging over $B_{t,x}$, then X_t is just the simple random walk. However, two random walks X_t^1, X_t^2 sampled in the same environment do not behave as two independent random walks.

General result: almost sure CLT

Theorem ([Rassoul-Agha and Seppäläinen, 2004])

Assume that $\mathbb{P}\big(0 < B_{t,x} < 1\big) > 0$. *Let* $v = 2\mathbb{E}[B_{t,x}] - 1$ *be the expected drift. For* $\mathbb{P}$*-almost every environment, when* $n \to \infty$, *we have the convergence in distribution of processes*

$$\frac{X_{\lfloor nt \rfloor} - \lfloor nt \rfloor v}{\sigma \sqrt{n}} \Rightarrow \textit{Brownian motion}(t),$$

where $\sigma = \sqrt{1 - v^2}$.

The results holds more generally (higer dimensions, unbounded steps, mixing environment...)

The limit no longer depends on the environment.

General result: almost sure large deviation principle

Theorem ([Rassoul-Agha, Seppäläinen and Yilmaz, 2013])

Assume that $\mathbb{E}[\log(B_{t,x})^3] < \infty$. *Then, the limit*

$$\lambda(z) := \lim_{t\to\infty} \frac{1}{t} \log\Big(\mathsf{E}\big[e^{zX_t}\big]\Big),$$

exists almost surely and

$$\frac{\log\Big(\mathsf{P}(X_t > xt)\Big)}{t} \xrightarrow[t\to\infty]{a.s.} -I(x).$$

where $I(x)$ *is the Legendre transform of* λ.

The result holds more generally.

- Finding an explicit formula for $\lambda(z)$ or $I(x)$ is generally not possible.
- RWRE $I(x) \geq$ SRW $I(x)$ (by Jensen's inequality).

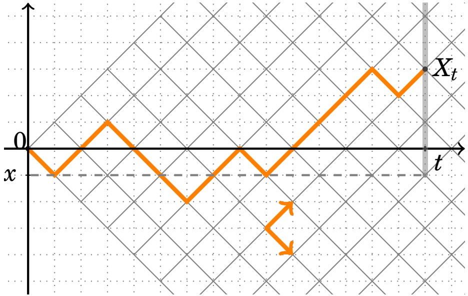
Integrable model: Beta RWRE

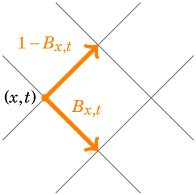
- Assume that $(B_{t,x})$ follow the $Beta(\alpha,\beta)$ distribution.

$$\mathbb{P}\big(B_{t,x} \in [y, y+\mathrm{d}y]\big) = y^{\alpha-1}(1-y)^{\beta-1}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\mathrm{d}y.$$

  The fact that this model is integrable comes from the work of [Povolotsky 2013] on Bethe ansatz solvable particle systems.
- To see how the random environment enters into play, we will compute more precisely the distribution of the random variable

$$\mathsf{P}(X_t > x).$$





Fluctuations around the large deviation principle

For simplicity, focus for the moment on the case $\alpha = \beta = 1$, i.e. $B_{t,x}$ are uniform in $[0,1]$.

Theorem ([B.-Corwin, 2015])

The large deviation principle rate function is

$$\lim_{t\to\infty} -\frac{\log \mathsf{P}(X_t > xt)}{t} = I(x) = 1 - \sqrt{1-x^2}.$$

We have the convergence in distribution as $t \to \infty$,

$$\frac{\log\Big(\mathsf{P}\big(X_t > xt\big)\Big) + I(x)t}{\sigma(x)\cdot t^{1/3}} \Longrightarrow \mathcal{L}_{GUE},$$

where $\mathcal{L}_{GUE}$ is the GUE Tracy-Widom distribution, and $\sigma(x)^3 = \frac{2I(x)^2}{1-I(x)}$.

Cube-root $\mathcal{L}_{\mathrm{GUE}}$ fluctuations are a hallmark of the Kardar-Parisi-Zhang (KPZ) universality class.

Extreme value theory

- Consider N independent **simple** random walks $X_t^{(1)}, \dots, X_t^{(N)}$.

$$\mathsf{P}\left(\max_{1\leqslant i\leqslant N} X_t^{(i)} \leqslant x\right) = \mathsf{P}(X_t \leqslant x)^N = \left(1 - \mathsf{P}(X_t > x)\right)^N \approx \exp(N\mathsf{P}(X_t > x))$$

- For $N = e^{ct}$ we get a non-trivial probability when $\mathsf{P}(X_t > x)$ is of order e^{-ct}, i.e. by scaling x close to $tI^{-1}(c)$.
- More precisely, for $N = e^{ct}$ the order of the maximum is $tI^{-1}(c)$, and [Fisher–Tippett–Gnedenko]

$$\max_{1\leqslant i\leqslant N} X_t^{(i)} \approx tI^{-1}(c) + c' \log(t) + dGumbel$$

where $dGumbel$ is a discrete variant of the Gumbel distribution.

The fluctuations of the extremes value of e^{ct} independent simple random walks are $\mathcal{O}(1)$.

Extreme values in random environment

Corollary ([B.-Corwin, 2015])

Let $X_t^{(1)}, \dots, X_t^{(N)}$ be random walks drawn independently in the same environment. Set $N = e^{ct}$. Then, for $\alpha = \beta = 1$,

$$\frac{\max_{i=1,\dots,e^{ct}} \left\{ X_t^{(i)} \right\} - t\sqrt{1-(1-c)^2}}{d(c) \cdot t^{1/3}} \Longrightarrow \mathcal{L}_{GUE},$$

where $d(c)$ is an explicit function.

The fluctuations of the extreme value of e^{ct} independent simple random walks in random environment are $\mathcal{O}(t^{1/3})$.

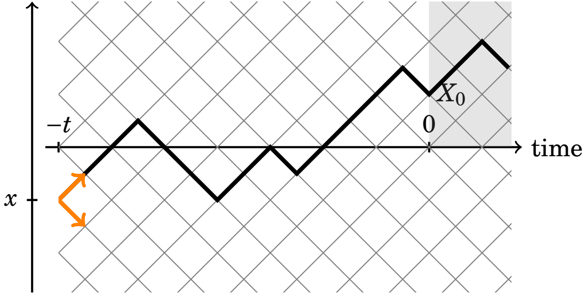
Recurrence

We have the equality in law

$$\mathsf{P}(X_t > x) \overset{(d)}{=} \mathsf{P}(X_0 > 0 | X_{-t} = -x) =: P(t, -x)$$

We have the recurrence

$$P(t,x) = B_{-t,x} P(t-1, x+1) + (1 - B_{-t,x}) P(t-1, x-1).$$



It is convenient to change variables and call $Z(t,n) = P(t,x)$ where $x = t - 2n$.

Recurrence for moments

$$Z(t,n) = B_{-t,n} \cdot Z(t-1,n) + (1 - B_{-t,n}) \cdot Z(t-1,n-1).$$

To compute the moments $\mathbb{E}[Z(t,n)^k]$, let

$$u(t,\vec{n}) := \mathbb{E}\big[Z(t,n_1)Z(t,n_2)\dots Z(t,n_k)\big].$$

$k=1$

$$u(t,n) = \frac{\alpha}{\alpha+\beta} u(t-1,n) + \frac{\beta}{\alpha+\beta} u(t-1,n-1)$$

General k

For $\vec{n} = (n,\dots,n)$,

$$\begin{aligned} u(t,\vec{n}) &= \sum_{j=0}^{k} \binom{k}{j} \mathbb{E}\left[B^j(1-B)^{k-j} Z(t-1,n)^j Z(t-1,n-1)^{k-j}\right] \\ &= \sum_{j=0}^{k} \binom{k}{j} \frac{(\alpha)_j(\beta)_{k-j}}{(\alpha+\beta)_k}\, u\Big(t-1,\big(n,\dots,n,n-1,\dots,n-1\big)\Big). \end{aligned}$$

where $(a)_k = a(a+1)\dots(a+k-1)$.

Non-commutative binomial expansion

The evolution of $u(t,\vec{n})$ can be written

$$u(t,\vec{n}) = \mathscr{L} u(t-1,\vec{n}),$$

where $\mathscr{L}$ is an operator on functions $\mathbb{W}^k \to \mathbb{C}$, and

$$\mathbb{W}^k = \left\{\vec{n} \in \mathbb{Z}^k : n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k\right\}.$$

In general, $\mathscr{L}$ acts as shown previously for each cluster of equal components in $\vec{n}$.

Lemma ([Rosengren 2000, Povolotsky 2013])

Let X, Y generate an associative algebra such that

$$0 = XX + (\alpha+\beta-1)XY - (1+\alpha+\beta)YX + YY.$$

Then we have the following non-commutative binomial identity:

$$\left(\frac{\alpha}{\alpha+\beta}X + \frac{\beta}{\alpha+\beta}Y\right)^k = \sum_{j=0}^{k} \binom{k}{j} \frac{(\alpha)_j(\beta)_{k-j}}{(\alpha+\beta)_k} X^j Y^{k-j}.$$

Bethe ansatz

Let us define $\tau^{(i)}$ the operator acting on $f(\vec{n})$ by replacing n_i by $n_i - 1$.
Define the operator L on functions $f : \mathbb{Z}^k \to \mathbb{C}$ by

$$\mathsf{L}f(\vec{n}) = \prod_{i=1}^{k} \left(\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\tau^{(i)} \right).$$

It coincides with $\mathscr{L}$ on the interior of $\mathbb{W}^k$.
Define the boundary operator ($X \to 1, Y \to \tau$)

$$B^{(i,i+1)} :\ 1 + (\alpha+\beta-1)\tau^{(i+1)} - (1+\alpha+\beta)\tau^{(i)} + \tau^{(i)}\tau^{(i+1)}$$

Corollary

Any function $u : \mathbb{Z}^k \to \mathbb{C}$ *which satisfies for all* $1 \leqslant i \leqslant k-1$

$$B^{(i,i+1)} u(\vec{n}) \Big|_{n_i = n_{i+1}} = 0,$$

is such that for all $\vec{n} \in \mathbb{W}^k$,

$$\mathscr{L}u(\vec{n}) = \mathsf{L}u(\vec{n}).$$

Moment formula

The solution of the evolution equation can be written as a contour integral using Bethe ansatz.

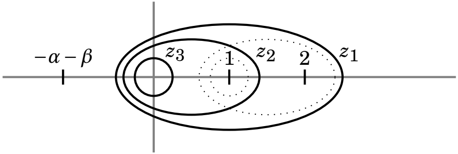
Proposition

For $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k \geqslant 1$,

$$\mathbb{E}\Big[\prod_{i=1}^{k} \mathsf{P}(X_t > t - 2n_i)\Big] =$$

$$\frac{1}{(2i\pi)^k}\int \cdots \int \prod_{1\leqslant A<B\leqslant k} \frac{z_A - z_B}{z_A - z_B - 1} \underbrace{\prod_{j=1}^{k}\left(\frac{\alpha+\beta+z_j}{z_j}\right)^{n_j}\left(\frac{\alpha+z_j}{\alpha+\beta+z_j}\right)^{t}}_{\text{solution of } u(t,\vec{n})=\mathsf{L}u(t-1,\vec{n})} \frac{\mathrm{d}z_j}{\alpha+\beta+z_j}$$

where the contour for z_k are nested around zero so that the contour for z_i contains $z_j + 1$ for $i < j$.



Proof of the Theorem

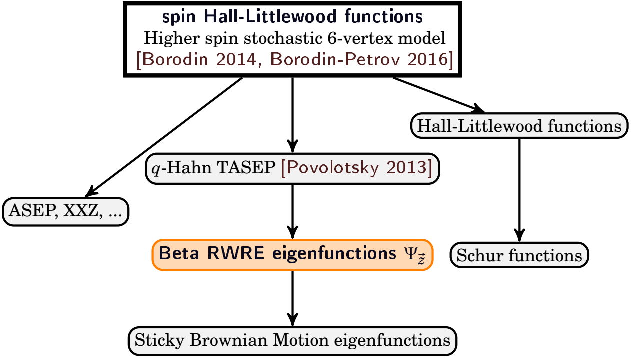
1. Rewrite the integral formula deforming contours so that all variable are integrated along a small circle through zero. The combinatorics of residues involved has been analyzed in the context of Macdonald processes [Borodin-Corwin 2014].
2. Compute the sum
$$\sum_{k=1}^{\infty} \frac{z^k}{k!} \mathbb{E}\left[\mathsf{P}(X_t > xt)^k\right]$$
that is the Laplace transform $\mathbb{E}\left[e^{z\mathsf{P}(X_t > x)}\right]$ as a series of integrals (Fredholm determinant) which is amenable to asymptotic analysis.
3. Asymptotic analysis using saddle point method (open problem for general α, β, relatively easy for $\alpha = \beta = 1$).

Hierarchy of Bethe ansatz eigenfunctions

The family of symmetric rational functions

$$\Psi_{\vec{z}}(\vec{n}) = \sum_{\sigma\in\mathcal{S}_n} \sigma\left(\prod_{A>B}\frac{z_A - z_B - 1}{z_1 - z_B}\prod_{j=1}^{k}\left(\frac{\alpha+\beta+z_j}{z_j}\right)^{n_j}\right)$$

are eigenfunctions of $\mathscr{L}$, they form a Fourier-like basis [Borodin-Corwin-Petrov Sasamoto 2015] and satisfy beautiful identities. These fit in a more general context:



Diffusions in space-time iid environment

- For fixed α, β rescaling diffusively n Beta RWRE yields n independent Brownian motions.
- Question: Is there a nontrivial Brownian version of the model?

(I) Brownian motion with a random drift

Physicists [Le Doussal-Thiery 2017] write

$$dX_t = dB_t + \xi(t,x)dt$$

where ξ is a space-time white noise.

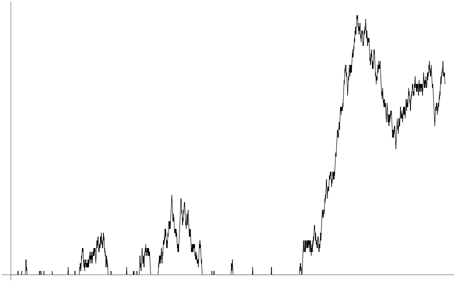
(II) Law of n paths in the same environment

Rescaling $\alpha = \beta = \lambda\epsilon$, the weights $B_{t,x}$ become very close to 0 or 1. The law of n Beta RWRE converges to a family of Brownian motions with attractive interaction called sticky Brownian motions [Le Jan-Lemaire 2004, Howitt-Warren 2009].

The advantage of approach (II) is that it yields an integrable model.

Reflected Brownian motion sticky at 0

- The heat equation
$$\partial_t u(t,x) = \partial_{xx} u(t,x)$$
on $\mathbb{R}_+$ with a boundary condition at 0 of Dirichlet type $u|_{x=0} = 0$ or Neumann type $\partial_x u|_{x=0} = 0$ is related to Brownian motion killed at zero or reflected at zero.
- [Feller 1952] investigated more general boundary conditions involving the second derivative at zero:
$$\partial_{xx} u|_{x=0} = \lambda \partial_x u|_{x=0}$$

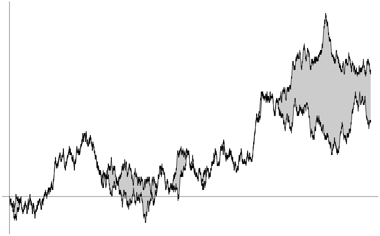


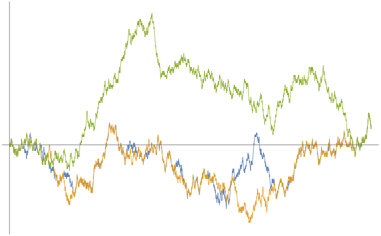
- Simulation: simple random walk X_t above zero, which stays at zero with probability $1-\lambda\epsilon$, rescaled diffusively $Y_t := \lim \epsilon X_{t\epsilon^{-2}}$.
- Weak solution of
$$dY_t = \lambda \mathbb{1}_{Y_t=0} dt + \mathbb{1}_{Y_t>0} dB_t.$$

Brownian motions with sticky interaction

There exists many ways to define a family of Brownian motions such that the distance between any pair of them is a sticky BM.

- A pair of sticky BM depends on a parameter λ, the “rate” at which the two particles split when they are stuck together.
- For 3 sticky BM one needs one more parameter which tells us how three particles split.
- In general, one needs a family of parameters $\theta(k,l)$ which govern how $k+l$ particles stuck together split into groups of k and l.





Definition of n-point sticky Brownian motion

The n-point motion can be defined through

- A scaling limit of various discrete models,
- A system of SDEs,
- A martingale problem [Howitt-Warren 2009],
- Characterizing the transition probabilities of paths using Dirichlet forms [Le Jan-Raimond 2004, Le Jan-Lemaire 2004].

Definition ([Howitt-Warren 2009])

$\vec{X}(t) \in \mathbb{R}^n$ is a n-point sticky Brownian motion if it is a continuous square integrable martingale with covariance

$$\langle X_i(t), X_j(t) \rangle = \int_0^t \mathbb{1}_{X_i(s)=X_j(s)} ds,$$

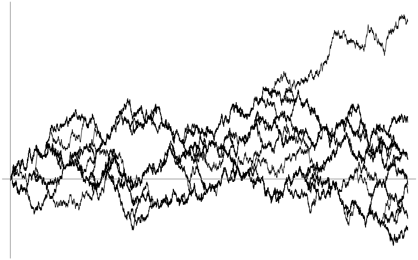
such that

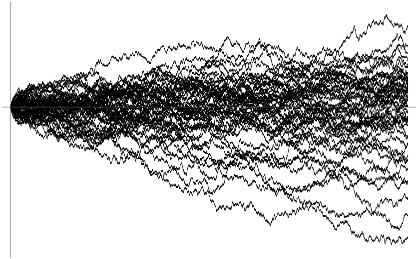
$$\max_i \{X_i(t)\} - 2\int_0^t \sum_{i=1}^n \theta(1,i) \mathbb{1}_{X_i(s)=\max} ds$$

is a martingale. Consistency: Any subset of k coordinates is a k-point sticky Brownian motion.

n-point sticky Brownian motion

For $n = 50$, we see that sticky Brownian motions are very different from independent BMs.





Sticky BM viewed as random motion in random environment

- Define for any $s \leq t$, $x \in \mathbb{R}$, Borel set A, the random "heat kernel"

$$\mathsf{K}_{s,t}(x,A) = \lim_{\epsilon \to 0} \mathsf{P}(\epsilon X_{t\epsilon^{-2}} \in A | \epsilon X_{s\epsilon^{-2}} = x),$$

  where X_t is the Beta RWRE with parameters $\alpha = \beta = \lambda\epsilon$.

- This is a stochastic flow of kernel [Le Jan-Raimond 2004] in the sense it defines a flow of random probability measures satisfying
  1. (Semigroup property) $\mathsf{K}_{s,t}\mathsf{K}_{t,u} = \mathsf{K}_{s,t}$,
  2. (Independent increments) For $s < t < u$, $\mathsf{K}_{s,t}$ and $\mathsf{K}_{t,u}$ are indep.
  3. (Stationarity) $\mathsf{K}_{s,t} \overset{(d)}{=} \mathsf{K}_{s+u,t+u}$.

- n paths sampled using the same kernels $\mathsf{K}_{s,t}(x,A)$ are sticky Brownian motions with splitting rates [Howitt-Warren 2009]

$$\theta(k,l) = \lambda \frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l)}.$$

Explicit construction of the environment

- Consistency of the definition of sticky Brownian motions implies

$$\theta(k,l) = \theta(k+1,l) + \theta(k,l+1).$$

  Then the sequence $\theta(k,l)$ can be encoded by a measure ν on $[0,1]$ such that

$$\theta(k,l) = \int_0^1 x^{k-1}(1-x)^{l-1}\nu(dx).$$

- For the stochastic flows of kernels corresponding to the sticky BM martingale problem, one can construct explicitly the environment using the Brownian web/net [Sun-Schertzer-Swart 2014].
- The measure ν encodes how paths branch on special points of the Brownian web.
- The integrable choice corresponds to chosing ν uniform, i.e. $\nu(dx) = \lambda\ dx$.

Bethe ansatz and sticky BM

Let $\Phi(t,\vec{x}) = \mathbb{E}[\mathsf{K}_{-t,0}(x_1,\mathbb{R}_+)\dots\mathsf{K}_{-t,0}(x_k,\mathbb{R}_+)]$.

Proposition ([B.-Rychnovsky 2019])

For $x_1 \geqslant \dots \geqslant x_k$,

$$\Phi(t,\vec{x}) = \int_{\alpha_1+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}w_1}{2\mathbf{i}\pi} \cdots \int_{\alpha_k+\mathbf{i}\mathbb{R}} \frac{\mathrm{d}w_k}{2\mathbf{i}\pi}$$

$$\prod_{1\leqslant A<B\leqslant k} \frac{w_B-w_A}{w_B-w_A-w_Aw_B} \prod_{j=1}^{k} \exp\left(\frac{t\lambda^2 w_j^2}{2} + \lambda x_j w_j\right) \frac{1}{w_j},$$

where for $i<j$, $0<\alpha_i<\frac{\alpha_j}{1+\alpha_j}$.

The function $\Phi(t,\vec{x})$ solves

$$\begin{cases} \partial_t \Phi(t,\vec{x}) = \frac{1}{2}\Delta\Phi(t,\vec{x}), \\ (\partial_i\partial_{i+1} + \lambda(\partial_i - \partial_{i+1}))\Phi(t,\vec{x})|_{x_i=x_{i+1}} = 0. \end{cases}$$

Asymptotic results

Theorem ([B.-Rychnovsky 2019])

The large deviation principle rate function is given by

$$\lim_{t\to\infty} \frac{\log \mathsf{K}_{0,t}(0, xt+\mathbb{R}_+)}{t} = -\lambda^2 J(x/\lambda), \quad \textit{in probability}$$

where

$$J(x) = \max_{\theta>0}\left\{\frac{1}{2}\psi_2(\theta) + x\psi_1(\theta)\right\}$$

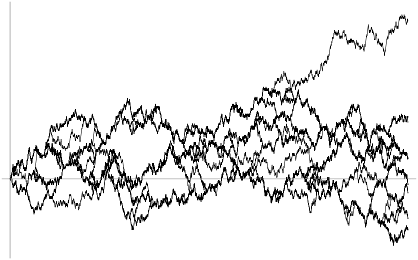
where $\psi_k(\theta)$ is the polygamma function $\psi_k(\theta) = \partial_\theta^{k-1}\log\Gamma(\theta)$. We have the convergence in distribution as $t\to\infty$,

$$\frac{\log\left(\mathsf{K}_{0,t}(0, xt+\mathbb{R}_+)\right) + \lambda J(x/\lambda)t}{\sigma(x)\cdot t^{1/3}} \Longrightarrow \mathscr{L}_{GUE},$$

where $\mathscr{L}_{GUE}$ is the GUE Tracy-Widom distribution, and $\sigma(x)$ is some explicit function.

Proof: Follows parallel arguments as in the discrete case, except that the asymptotic analysis is significantly more difficult here.

Various models of interacting (often repulsive) Brownian motions, starting with [Dyson 1962], are related to random matrix theory. Here we consider a model with attractive interaction.



Corollary ([B.-Rychnovsky 2019])

Let $\vec{X}(t)$ be the n-point sticky Brownian motion for $\theta(k,l) = \lambda \frac{\Gamma(k+l)}{\Gamma(k)\Gamma(l)}$. Set $n = e^{ct}$. Then,

$$\frac{\max_{i=1,\dots,N}\left\{X_i(t)\right\} - t \cdot \lambda J^{-1}(c/\lambda^2)}{\sigma(c) \cdot t^{1/3}} \underset{t\to\infty}{\Longrightarrow} \mathscr{L}_{GUE},$$

where $\sigma(c)$ is some explicit function.

Kardar-Parisi-Zhang equation

The KPZ equation

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}(\partial_x h)^2 + \xi \qquad (\xi \text{ is a space-time white noise})$$

is a paradigmatic model for stochastic growth of interfaces. We say that h solves the KPZ equation when $Z = e^h$ solves the heat equation with multiplicative noise

$$\partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + Z(t,x)\xi(t,x).$$

Proposition ([B.-Rychnovsky 2019])

Under the scalings

$$T = \lambda^2 t,\ X_i = \lambda^2 t + \lambda x_i,$$

The moments of $\mathsf{K}_{0,t}$ *converge to those of* $Z(t,x)$*:*

$$\mathbb{E}\left[\prod_{i=1}^{k} \lambda e^{t\lambda^2/2+\lambda x_i} \mathsf{K}_{0,T}(0, X_i + \mathbb{R}_+)\right] \xrightarrow[\lambda\to\infty]{} \mathbb{E}\left[\prod_{i=1}^{k} Z(t,x_i)\right].$$

Physics: Motion of particles in turbulent flows

- [Kolmogorov 1941]'s theory of turbulence predicted how to scale the velocity field of a turbulent flow according to parameters of the fluid.
- There has been evidence in physics that this scaling behaviour is violated in some cases. One of the simplest models which theoretically explains such anomalous scaling is [Kraichnan 1968] an advection-diffusion equation

$$\begin{cases} \partial_t q = \frac{1}{2}\partial_{xx} q + v(t,x)\partial_x q + f(t,x), \\ q(t=0,x) = q_0(x) \end{cases}$$

  where $v(t,x)$ is a random velocity field that can be modeled by a Gaussian field with short range correlations and $f(t,x)$ is a source term that we take equal zero.
- Under some hypotheses on v and $f \equiv 0$, $q(t,x)$ corresponds to the expectation of $\mathsf{E}[q_0(X_t)]$ where X_t is a Brownian particle in the velocity field v, i.e.

$$\begin{cases} dX_t = dB_t + v(t,x)dt. \\ X_0 = x. \end{cases}$$

Moments of Kraichnan's equation

In one dimension, assuming that the velocity field is Gaussian with covariance

$$\mathbb{E}[v(t,x)v(s,y)] = \delta(t-s)R(x-y),$$

a formal computation [Bernard-Gawedski-Kupiainen 1998] shows that the mixed moments of $q(t,x)$, i.e.

$$\Phi^R(t,\vec{x}) = \mathbb{E}[q(t,x_1)\dots q(t,x_k)]$$

solve the PDE

$$\partial_t \Phi^R(t,\vec{x}) = \left(\Delta + \sum_{i\neq j} R(x_i - x_j)\partial_{x_j}\partial_{x_i}\right)\Phi^R(t,\vec{x}).$$

When $R(x-y) \to \frac{1}{\lambda}\delta(x-y)$, the velocity field becomes a Gaussian space-time white noise and the PDE becomes

$$\partial_t \Phi^R(t,\vec{x}) = \left(\Delta + \frac{1}{2\lambda}\sum_{i\neq j} \delta(x_i - x_j)\partial_{x_j}\partial_{x_i}\right)\Phi^R(t,\vec{x}).$$

Sticky BM are diffusions with a white noise drift

One can then show that Φ^R solves

$$\begin{cases} \partial_t \Phi^R(t,\vec{x}) = \frac{1}{2}\Delta\Phi^R(t,\vec{x}), \\ (\partial_i\partial_{i+1} + \lambda(\partial_i - \partial_{i+1}))\Phi^R(t,\vec{x})|_{x_i=x_{i+1}} = 0, \end{cases}$$

that is the same system as the moments of stochastic flows.
This suggests to identify

$$\zeta(t,x) = \int dy \mathsf{K}_{-t,0}(-x,dy)\zeta_0(y)dy$$

with the solution $q(t,x)$ of the stochastic PDE

$$\partial_t q(t,x) = \frac{1}{2}\partial_{xx} q(t,x) + \frac{1}{\sqrt{\lambda}}\xi(t,x)\partial_x q(t,x).$$

Using formally Kolmogorov backwards equation, this makes sense of the a priori ill-posed diffusion with white noise drift

$$dX_t = dB_t + \frac{1}{\sqrt{\lambda}}\xi(t,x)dt.$$

which can now be interpreted as the random diffusion corresponding to the stochastic flow.

Outlook

Summary

1. Random walks/diffusions in random environment has the same typical behaviour as without random environment, but the random environment drastically changes extreme value scalings/statistics to KPZ type.
2. Families of Brownian particles with attractive interaction also display RMT statistics.
3. Connections with turbulent flows and exactly solvable SPDE.

Open questions

- What about the fluctuations of extremes of other branching/coalescing structures?
- 1D Random walk in space-time random environment can be seen as 2D random walks in fixed random environment, i.e. Sinai's walk. Can one show TW type asymptotics for 2D ballistic Sinaï's random walk? Can one show convergence to the KPZ equation?

Thank you