COLUMBIA UNIVERSITY

MATHV1201
Calculus III
Fall 2016

Final Exam (practice)

Instructor: Guillaume Barraquand

Time: December 22, 2016. Section 7: 9:00 am. Section 9: 16:00pm

Your name: __________________________________________________________

UNI: __________________________
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**Instructions:**

- Please write your UNI on every page.
- Unless stated otherwise, you must explain how you arrive to the result to get all the points.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, phones or any other electronic devices are **not** allowed.
Exercise 1

Determine whether the following statements are true (T) or false (F). You do not need to justify your answer for this question.

(a) (2 points) __ The function \( f(x, y) = x^2 - 4y^2 \) has a minimum value on \( \mathbb{R}^2 \).

(b) (2 points) __ For a differentiable function in \( \mathbb{R}^3 \), the gradient is always orthogonal to the level surfaces.

(c) (2 points) __ At the intersection of the surfaces \( f(x, y, z) = 0 \) and \( g(x, y, z) = 0 \), we have

\[ \nabla f(x, y, z) \cdot \nabla g(x, y, z) = 0 \]

(d) (2 points) __ The length of a curve does not depend on the parametrization of a curve. However, the curvature does.

(e) (2 points) __ Any bounded function of 2 variables defined on a closed and bounded set attains a maximum on this set.

(f) (2 points) __ Suppose \( f \) is twice continuously differentiable. At an inflexion point of the curve \( y = f(x) \), the curvature is zero.

(g) (2 points) __ If \( \kappa(t) = 0 \) for all \( t \), the curve is a straight line.

(h) (2 points) __ If \( f(x, y) = \ln(y) \), then \( \nabla f(x, y) = 1/y \).

(i) (2 points) __ If \( f(x, y) \to L \) when \((x, y)\) approaches \((0,0)\) along any straight line, then

\[ \lim_{(x,y) \to (0,0)} f(x, y) = L. \]

(j) (2 points) __ Let \( f \) be a differentiable function with continuous second order partial derivatives. If \( f(x, y) \) has two local maxima, then \( f \) must have a local minimum.
Exercise 2

Let \( f(x, y, z) = xy + yz + zx \).

(a) (3 points) In what direction does the function increase fastest at the point \( P = (1, 0, 2) \)?

(b) (3 points) What is the rate of fastest increase in part (a)?
(c) (2 points) What is the equation of the level surface of $f$ through the point $(1, 0, 2)$?

(d) (4 points) Find the equation of the tangent plane to the level surface of part (c) through the point $(1, 0, 2)$. 
Exercise 3

We define the functions \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \), \( g(x, y, z) = x^2 - y^2 - 2z \) and \( h(x, y, z) = 4x^2 + y^2 + z - 1 \).

(a) (2 points) What is the nature of the surface of equation \( g(x, y, z) = 0 \)?

- Elliptic paraboloid
- Cone
- Plane
- Ellipsoid
- Cylinder
- Hyperbolic paraboloid
- Hyperboloid of one sheet

(b) (2 points) What is the nature of the surface of equation \( h(x, y, z) = 0 \)?

- Elliptic paraboloid
- Cone
- Plane
- Ellipsoid
- Cylinder
- Hyperbolic paraboloid
- Hyperboloid of two sheets

(c) (2 points) Is \( f \) differentiable at 0?
Let $\mathcal{C}$ be the curve of intersection of the surfaces $g(x, y, z) = 0$ and $h(x, y, z) = 0$. We parametrize this curve by

$$\vec{r}(\theta) = (a(\theta) \cos(\theta), 2a(\theta) \sin(\theta), z(\theta)), \quad \theta \in [0, 2\pi),$$

where $a(\theta)$ and $z(\theta)$ are some functions to determine, with $a(\theta) > 0$.

(d) (2 points) Using $h(x, y, z) = 0$, express $z(\theta)$ in terms of $a(\theta)$.

(e) (2 points) Using $g(x, y, z) = 0$, express $a(\theta)$ and $z(\theta)$ as functions of $\theta$. 
(f) (4 points) Find the minimum of $f$ under the constraint that $h(x, y, z) \geq 0$ (assuming the minimum exists).

(g) (4 points) Find the maximum of $f$ under the constraint $g(x, y, z) = 0$ and $h(x, y, z) = 0$. 
(h) (4 points) Find an equation of the normal plane to the curve $C$ at the point where $f$ is maximal (i.e. the point corresponding to the maximum in (g)).
Exercise 4 ................................................................. 8 points

Find all local maxima, minima, and saddle points of the function

\[ f(x, y) = x^2 - y^3 + 3xy \]

and say what type they are.
Exercise 5

\[ \vec{r}(t) = \left( \cos(t), \frac{1}{2} \sin(2t), \sin(t) \right) \]

(a) (1 point) Compute \( \vec{r}'(t) \)

(b) (1 point) Compute \( \vec{r}''(t) \)

(c) (2 points) Let \( s(t) \) be the arc length function (length of the curve between 0 and \( t \)). Compute the derivative of \( s \) with respect to \( t \), that is \( s'(t) \).
(d) (4 points) Compute the curvature $\kappa(t)$.

(e) (4 points) Find the equation of the osculating circle at $t = 0$. 
Exercise 6

If a projectile is fired with angle $\alpha$ and initial speed $v$, then parametric equations for its trajectory are:

$$x(t) = v \cos(\alpha) t, \quad y(t) = v \sin(\alpha) t - \frac{1}{2} gt^2.$$ 

(a) (4 points) Show that the horizontal distance traveled $d$ is maximum when $\alpha = \pi/4$.

(b) (4 points) What value of $\alpha$ maximizes the total distance $D$ traveled by the projectile (length of the curve)?
Exercise 7

Find the points on the surface $xy^2z^3 = 2$ that are closest to the origin. Hint: one can minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $xy^2z^3 = 2$. 


Exercise 8

If \( \vec{r}(t) \) determines a parametric curve such that \( \vec{r}''(t) = \vec{c} \times \vec{r}'(t) \), where \( \vec{c} \) is a constant vector.

(a) (4 points) Show that \( \| \vec{r}(t) \| \) is constant.

(b) (4 points) Describe the curve.
Your UNI:

Extra space.
Extra space.