COLUMBIA UNIVERSITY

MATHV1201
Calculus III
Fall 2016

Midterm 2 (section 7)

Instructor: Guillaume Barraquand

Time: November 10, 2016. 8:40am – 10:55am

Your name: ____________________________________________

UNI: WRITE THIS CLEARLY.
Your UNI:

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Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, Show your work in every question.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Exercise 1

Determine whether the following statements are true (T) or false (F). You do not need to justify your answer for this question.

(a) \( \square \) Let \( \vec{r}(t) \) be a differentiable vector function, defining some parametric curve. If \( \vec{r}''(t) \) is constant, then the curve is a straight line.

(b) \( \square \) If a curve in \( \mathbb{R}^3 \) admits two different parametrizations, the length of a curve does not depend on the parametrization. However, the tangent vectors at a given point of the curve in \( \mathbb{R}^3 \) do depend on the parametrization.

(c) \( \square \) Let \( f: \mathbb{R}^2 \to \mathbb{R} \) such that \( f_{xyz} \) and \( f_{yxy} \) exist and are continuous. Then we have that \( f_{xyz} = f_{yxy} \).

(d) \( \square \) If \( \vec{r}(t) \) is a differentiable vector function, then

\[
\frac{d}{dt} \| \vec{r}(t) \| = \| \vec{r}'(t) \|.
\]

(e) \( \square \) If \( f(x,y) \) and \( g(u,v) \) are both differentiable, and we assume that \( y = g(x,x) \) and \( z = f(x,y) \), then we can write

\[
\frac{dz}{dx} = f_x(x,y) + f_y(x,y) (g_u(x,x) + g_v(x,x)).
\]

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Train also to compute:

- equation of tangent plane
- tangent lines
- arc-length functions and reparametrization.
Exercise 2 ............................................................................... 4 points
Determine if the following limit exists and calculate it if it exists.
\[
\lim_{(x,y) \to (0,0)} \frac{x^3y}{x^6 + y^2}
\]

If the limit exists, it must be 0 because if \( f(x,y) = \frac{x^3y}{x^6 + y^2} \) then \( f(x,0) = 0 \).

\[
f(x,x^3) = \frac{x^6}{2x^6} = \frac{1}{2} \quad \text{for} \quad x \neq 0.
\]

\[
f(x,x^3) \xrightarrow{x \to 0} \frac{1}{2}
\]

so that the function is not \underline{continuous does not} have a limit at (0,0).
Exercise 3

Consider the vector function $\vec{r}(t) = (e^t, 2e^{-t}, 1 - 2t)$ for $t \in \mathbb{R}$.

(a) (2 points) Compute $\vec{r}'(t)$.

$$\vec{r}'(t) = (e^t, -2e^{-t}, -2)$$

(b) (2 points) Compute the unit tangent vector $\vec{T}(t)$.

$$\|\vec{r}'(t)\| = \sqrt{e^{2t} + 4e^{-2t} + 4} = e^t + 2e^{-t}$$

$$\vec{T}(t) = \left( \frac{e^t}{e^t + 2e^{-t}}, \frac{-2e^{-t}}{e^t + 2e^{-t}}, \frac{-2}{e^t + 2e^{-t}} \right)$$

(c) (2 points) Find $\lim_{t \to +\infty} \vec{T}(t)$.

$$\lim_{t \to \infty} \frac{e^t + 2e^{-t}}{e^t} = \lim_{t \to \infty} 1 + 2e^{-2t} = \infty$$

So

$$\lim_{t \to \infty} \frac{e^t}{e^t + 2e^{-t}} = 1 \quad \lim_{t \to \infty} \frac{-2e^{-t}}{e^t + 2e^{-t}} = 0 \quad \lim_{t \to \infty} \frac{-2}{e^t + 2e^{-t}} = 0$$

Conclusion $\lim_{t \to \infty} \vec{T}(t) = (1, 0, 0)$
(d) (2 points) Find the length of the curve for $t$ between 0 and 1.

\[ \text{the length is } \int_0^1 (e^t + 2e^{-t}) \, dt \]

\[ = \left[ e^t - 2e^{-t} \right]_0^1 = e - \frac{2}{e} + 1 \]

\[ \text{length} = e + 1 - \frac{2}{e} . \]

(e) (2 points) Write a parametric representation of the tangent line to the curve at the point \((\frac{5}{2}, \frac{1}{2}, 1)\).

\[ (1, 2, 1) \text{ correspond to } t = 0 . \]

\[ R'(0) = (1, -2, -2) \]

\[ \text{parametric representation: } \begin{cases} x = 1 + s \\ y = 2 - 2s \\ z = 1 - 2s \end{cases}, \quad s \in \mathbb{R} . \]
Exercise 4

Let \( f(x, y, z) = e^{5x} \sin(4y) \cos(3z) \). The Laplace equation in \( \mathbb{R}^3 \) is the partial differential equation

\[
    u_{xx} + u_{yy} + u_{zz} = 0.
\]

(a) (2 points) Compute \( f_{xx} \).

\[
    f_{xx}(x, y, z) = 5 e^{5x} \sin(4y) \cos(3z)
\]

(b) (2 points) Compute \( f_{yy} \).

\[
    f_{yy}(x, y, z) = 25 e^{5x} \sin(4y) \cos(3z)
\]

(c) (2 points) Compute \( f_{zz} \).

\[
    f_{zz}(x, y, z) = 25 e^{5x} \sin(4y) \cos(3z)
\]

(d) (2 points) Does \( f \) solve the Laplace equation?

\[
    \text{YES}, \quad f_{yy} = -16 f, \quad f_{zz} = -9 f, \quad f_{xx} = 25 f
\]

so that

\[
    f_{xx} + f_{yy} + f_{zz} = 0.
\]
(c) (2 points) Compute \( f_{xy} \) for \( x > 0 \) and \( y < 0 \).

\[
f_{xy}(x,y) = \frac{3x^2}{2} \times \frac{-1}{2 \sqrt{3^3 - y}} \times -1
\]

\[
= \frac{3x^2}{4 \sqrt{x^3 - y}} (x^3 - y)
\]

(d) (2 points) Show that \( f_y(0,0) \) does not exist.

(Use the definition \( f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} \))

\[
\frac{f(0,h) - f(0,0)}{h} = \frac{\sqrt{-h} - 0}{h} = \frac{\sqrt{-h}}{h}
\]

does not have a limit as \( h \to 0 \) because \( t \to \sqrt{t} \) is not differentiable at 0.

(on just because \( \frac{\sqrt{-h}}{h} \to -\infty \) as \( h \to 0 \) for \( h < 0 \))
Exercise 5

Let \( f(x, y) = \sqrt{x^3 - y} \).

(a) (2 points) Find the domain of \( f \) and sketch it.

\[
\text{domain} = \{ (x, y) \text{ such that } y \leq x^3 \}
\]

(b) (4 points) Compute \( f_x \) and \( f_{xx} \) for \( x > 0 \) and \( y < 0 \).

\[
f_x(x, y) = \frac{3x^2}{2\sqrt{x^3 - y}}
\]

\[
f_{xx}(x, y) = \frac{12x\sqrt{x^3 y}}{4(x^3 - y)} + \frac{3x^2}{4\sqrt{x^3 - y}} \cdot \frac{1}{(x^3 - y)}
\]

\[
to\ simplify...
\]
Exercise 6

Consider the curve with vector equation

\[ \vec{r}(t) = (a_1t^2 + b_1t + c_1, a_2t^2 + b_2t + c_2, a_3t^3 + b_3t + c_3), \quad t \in \mathbb{R}. \]

Let \( \vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3) \) and \( \vec{c} = (c_1, c_2, c_3) \).

(a) (4 points) Find a vector \( \vec{n} \) such that for all \( t \in \mathbb{R} \),

\[ (\vec{r}(t) - \vec{c}) \cdot \vec{n} = 0. \]

\[ \vec{r}(t) = t^2 \vec{a} + t \vec{b} + \vec{c} \]

it is enough to take \( \vec{n} = \vec{a} \times \vec{b} \).

\[ \vec{n} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \]

(b) (4 points) Deduce that the curve \( \vec{r}(t) \) lies on a plane and find the equation of that plane.

the components of \( \vec{r}(t) \) satisfy the equation \[ k x (x - a_1, y - a_2, z - a_3) \cdot \vec{n} = 0 \]

it gives an equation replacing \( \vec{n} \) by its value...
That is

\[(x-c_1)(a_2b_3-a_3b_2) + (y-c_2)(a_3b_1-a_1b_3) + (z-c_3)(a_1b_2-a_2b_1) = 0.\]

This shows that \( \vec{r}(t) \) stays on a plane and above is the equation of that plane.