COLUMBIA UNIVERSITY

MATHV1201
Calculus III
Fall 2016

Midterm 2 (section 7)

Instructor: Guillaume Barraquand

Time: November 10, 2016. 8:40am – 10:55am

Your name: ________________________________

UNI: ________________________________
<table>
<thead>
<tr>
<th>Exercise:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points:</td>
<td>10</td>
<td>12</td>
<td>6</td>
<td>10</td>
<td>8</td>
<td>4</td>
<td>50</td>
</tr>
<tr>
<td>Score:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, Show your work in every question.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Exercise 1. Determine whether the following statements are true (T) or false (F). You do not need to justify your answer for this question.

(a) \( T \) Assume that \( f : \mathbb{R}^3 \to \mathbb{R} \) has continuous partial derivatives of all orders. Then
\[
f_{xyz} = f_{yxz}.
\]

(b) \( T \) Let \( a, b \) and \( c \) be continuous functions from \( \mathbb{R}^2 \) to \( \mathbb{R} \). Then
\[
f(x, y, z) = a(b(x, y), c(x, z))
\]
is a continuous function from \( \mathbb{R}^3 \) to \( \mathbb{R} \).

(c) \( T \) If \( f : \mathbb{R}^2 \to \mathbb{R} \) is such that \( f_x \) and \( f_y \) are continuous functions, then \( f \) is continuous.

(d) \( T \) For two vector functions \( \vec{u}(s) \) ans \( \vec{v}(t) \), we consider the function
\[
f(s, t) = \vec{u}(s) \times \vec{v}(t).
\]
Then we have
\[
\frac{\partial f}{\partial t}(s, t) = \vec{u}'(s) \times \vec{v}'(t).
\]

(e) \( F \) If \( \vec{r}(s) \) is a vector function representing a curve parametrized by arc length, then \( \vec{r}'(s) \) and \( \vec{r}''(s) \) are orthogonal for all \( s \).

(a) Since \( f \) has continuous partial derivatives of every order, one can exchange the order of partial derivatives,
\[
f_{xy} = f_{yx} \quad \text{so that} \quad f_{xyz} = f_{yxz}.
\]
\( f_y \) has continuous partial derivatives so that
\[
f_{yxz} = f_{yzx}.
\]
Finally, \( f_{xyz} = f_{yxz} \).
Exercise 2

Consider the function of two variables $f$ defined by

$$f(x, y) = y^2 + \sin(x/y).$$

(a) (2 points) Precise the domain of $f$.

Set of points such that $y \neq 0$

$$\text{domain} = \{(x, y) \in \mathbb{R}^2 : y \neq 0\}$$

(b) (2 points) Does $f(x, y)$ have a limit at $(0,0)$?  ○ Yes  ☑ No

(c) (4 points) For $(x, y)$ in the domain, compute $f_x(x, y)$ and $f_y(x, y)$.

$$f_x(x, y) = \frac{1}{y} \cos \left(\frac{x}{y}\right),$$

$$f_y(x, y) = 2y + \frac{-x}{y^2} \cos \left(\frac{x}{y}\right).$$
(d) (4 points) Consider the surface associated to $f$, that is the surface with equation $z = f(x, y)$. Find an equation of the tangent plane to the surface at the point $(0, -1, 1)$.

The equation is

$$z - 1 = \frac{\partial f}{\partial x}(0, -1)(x - 0) + \frac{\partial f}{\partial y}(0, -1)(y + 1)$$

$$z - 1 = (-1)(x) + (-2)(y + 1)$$

That is

$$x + 2y + z + 1 = 0$$
Exercise 3

Let \( a \) and \( b \) be fixed real parameters. Consider for \((t, x) \in \mathbb{R}^2\) the function

\[
f(t, x) = e^{-at} \sin(bx).
\]

(a) (2 points) Compute \( f_t(t, x) \).

\[
f_t(t, x) = -a e^{-at} \sin(bx)
\]

(b) (2 points) Compute \( f_{xx}(t, x) \).

\[
f_x(t, x) = b e^{-at} \cos(bx)
\]

\[
f_{xx}(t, x) = -b^2 e^{-at} \sin(bx)
\]
(c) (2 points) Assume $b 
eq 0$. Determine the value of the constant $c$ so that $f$ satisfies the partial differential equation
\[ f_t = c f_{xx}. \]

\[ f_t(t,x) = -a e^{-at} \sin(bx) \]
\[ f_x(t,x) = -b^2 e^{-at} \sin(bx) \]

\[ f_t = c f_{xx} \implies a = c b^2 \]
\[ \implies c = \frac{a}{b^2}. \]

Indeed, when $c = \frac{a}{b^2}$ we have
\[ f_t(t,x) = c f_{xx}(t,x) \text{ for all } (t,x) \in \mathbb{R}^2. \]
Exercise 4

Consider the parametric curve defined by the vector function

\[ \vec{r}(t) = \left( \frac{1 - \cos(t)}{t}, \frac{\sin(t)}{t}, t \right) \quad \text{for } t > 0. \]

(a) (2 points) Explain why \( \vec{r}(t) \) has a limit as \( t \) goes to zero, and find the limit. Let us call

\[ \vec{r}(0) = \lim_{t \to 0} \vec{r}(t), \]

so that the curve is continuous at 0.

\[ \cos \text{ is differentiable at } 0 \text{ with derivative } 0, \text{ i.e.} \]

\[ \lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0 \]

\[ \sin \text{ is differentiable at } 0 \text{ with derivative } 1, \text{ i.e.} \]

\[ \lim_{h \to 0} \frac{\sin(h)}{h} = 1 \]

We deduce that \( \lim_{t \to 0} \vec{r}(t) = (0, 1, 0) \).
(b) (2 points) Compute \( \vec{r}'(t) \) for \( t > 0 \).

\[
\vec{r}'(t) = \left( \frac{t \sin(t) - 1 + \cos(t)}{t^2}, \frac{t \cos(t) - \sin(t)}{t^2}, 1 \right)
\]

(c) (2 points) Find a parametric representation of the tangent line to the curve at the point \( (2/\pi, 0, \pi) \).

The point \( \left( \frac{2}{\pi}, 0, \pi \right) \) correspond to \( t = \pi \).

\[
\vec{r}'(\pi) = \left( -\frac{2}{\pi^2}, -\frac{\pi}{\pi^2}, 1 \right)
\]

Thus, a parametric representation is

\[
\begin{align*}
x &= \frac{2}{\pi} - \frac{2}{\pi^2} t + t \\
y &= -\frac{t}{\pi} \\
z &= \pi + t
\end{align*}
+ \epsilon \mathbf{R}
\]
(d) (4 points) Is the curve differentiable for \( t = 0 \)? If yes, find \( \vec{r}'(0) \), if not explain why. If you have not found the answer to question (a), you can assume that \( \vec{r}'(0) \) is \((0, 1, 0)\).

We need to find if the limit \( \lim_{t \to 0} \frac{\vec{r}(t) - \vec{r}(0)}{t} \) exist when \( t \to 0 \).

\[
\frac{\vec{r}(t) - \vec{r}(0)}{t} = \left( \frac{1 - \cos(t)}{t^2}, \frac{\sin(t) - t}{t^2}, 1 \right).
\]

For \( t \) close to \( 0 \), \( \cos(t) \approx 1 - \frac{t^2}{2} \)

\( \sin(t) \approx t - \frac{t^3}{6} \)

so that \( \frac{1 - \cos(t)}{t^2} \to \frac{1}{2} \)

\( \frac{\sin(t) - t}{t^2} \to 0 \)

It implies that the curve is differentiable at \( 0 \) and

\[
\vec{r}'(0) = \lim_{t \to 0} \frac{\vec{r}(t) - \vec{r}(0)}{t} = \left( \frac{1}{2}, 0, 1 \right).
\]
Exercise 5

Let \( \vec{r}(t) = (\cos(t^3), \sin(t^3), 1 + t^3) \) for \( t \in \mathbb{R}_+ \).

(a) (2 points) Compute \( \vec{r}'(t) \).

\[
\vec{r}'(t) = \left( -2t \sin(t^3), 2t \cos(t^3), 2t \right)
\]

(b) (3 points) Compute the arc length function \( s(t) \) (length of the curve between \( t = 0 \) and \( t \)).

\[
\| \vec{r}'(t) \| = 2t \sqrt{\sin^2(t^3) + \cos^2(t^3) + 1} = 2\sqrt{2} \, t.
\]

\[
\Delta(t) = \int_0^t \frac{1}{2} \, u \, du = \sqrt{2} \, t^2.
\]

\[
s(t) = \sqrt{2} \, t^2
\]
(c) (3 points) Reparametrize the curve by arc-length. This means find a vector function \( \vec{r}_2(s) \) parametrizing the same curve such that the parameter \( s \) corresponds to arc length.

\[
\vec{r}(t) = \sqrt{2} \ t^2 \quad \Rightarrow \quad t(s) = \sqrt{\frac{s}{\sqrt{2}}} \quad \text{(because } t \in \mathbb{R}_+) \]

The curve can be reparametrized by

\[
\vec{r}_2(s) = \vec{r}(t(s))
\]

\[
\vec{r}_2(s) = \left( \cos \left( \frac{s}{\sqrt{2}} \right), \sin \left( \frac{s}{\sqrt{2}} \right), 1 + \frac{s}{\sqrt{2}} \right)
\]
Exercise 6: ................................................................. 4 points
Is the function

\[ f(x,y) = \frac{x^4y}{x^2 + y^2} \]

continuous at \((0,0)\)?

\[ \text{cannot be continuous at } \, 0. \]

\[ \lim_{y \to 0} f(0, y) = 0 \]

\[ \lim_{x \to 0} f(x, x^4) = \lim_{x \to 0} \frac{x^8}{2x^8} = \frac{1}{2}. \]

We find two different limits when approaching the point \((0,0)\) along two different curves, so that the function cannot be continuous.
Extra space.