Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, Show your work in every question.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Exercise 1 ............................................................. 12 points

Determine whether the following statements are true (T) or false (F). You do not need to justify your answer for this question.

(a)  F  In spherical coordinates, the equation \( \rho = 1 \) determines a cylinder whose horizontal traces are circles with radius 1.

(b)  F  For any three vectors \( \vec{a}, \vec{b}, \) and \( \vec{c} \) in \( \mathbb{R}^3 \), we have

\[
(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c}).
\]

(c)  F  In \( \mathbb{R}^3 \), one can always find a plane containing two given lines.

(d)  F  The complex number \( i \) is the only complex number \( z \) such that \( z^2 = -1 \).

(e)  F  In \( \mathbb{R}^2 \) the vector \( (a, b) \) is parallel to the line with equation \( ax + by = 0 \).

(f)  F  For any two vectors \( \vec{a}, \vec{b} \) in \( \mathbb{R}^3 \), we have

\[
\text{comp}_a(\vec{b}) = \|\text{proj}_a(\vec{b})\|.
\]

General advice: Always check your answers when possible. In True-False questions, be careful with words like only, always... When you don’t know how to start an exercise, draw a picture. You might even get points for that even if you do not find the final answer.
Exercise 2

Consider points in \( \mathbb{R}^3 \) with coordinates \( A(1,0,1), B(-1,2,-4), P(-1,1,0), Q(1,0,3) \) and \( R(1,0,-2) \).

(a) (3 points) Find a parameteric representation of the line going through \( A \) and \( B \).

Solution: We have \( \overrightarrow{AB} = (-2, 2, -5) \), so that a parametric representation is

\[
\begin{align*}
    x &= 1 - 2t \\
    y &= 2t \\
    z &= 1 - 5t
\end{align*}
\]

, \( t \in \mathbb{R} \)

(b) (4 points) Find an equation of the plane passing through \( P, Q, \) and \( R \).

Solution: We have \( \overrightarrow{PQ} = (2, -1, 3), \overrightarrow{PR} = (2, -1, -2) \), so that

\[
\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 3 \\ 2 & -1 & -2 \end{vmatrix} = (5, 10, 0).
\]

An equation of the plane is

\[
5(x + 1) + 10(y - 1) = 0
\]

or, after simplification,

\[
x + 2y - 1 = 0.
\]

(c) (3 points) Find the distance from the point \( A \) to the plane passing through \( P, Q, \) and \( R \).

Solution: Applying the formula from the course, this distance, say \( d \), is given by

\[
d = \frac{|1 \times 1 + 2 \times 0 + 0 \times 1 - 1|}{\sqrt{5^2 + 10^2}} = 0.
\]

The point \( A \) belongs to the plane (this could be guessed from the fact that \( A \) is the middle of the segment \( QR \)).
Exercise 3................................................................. 5 points
(a) (4 points) Find an equation for the set of points equidistant to the plane \( z = -2 \) and the point \( A(2, 3, 2) \).
(b) (1 point) What type of surface is it?

**Solution:** Let \( S \) be the set of points we want to determine. We actually know from the course that \( S \) is a **circular paraboloid**, but this will be confirmed by the equation. A point \( P(x, y, z) \) belongs to \( S \) iff

\[
|z + 2| = \sqrt{(x - 2)^2 + (y - 3)^2 + (z - 2)^2} \iff (z + 2)^2 = (x - 2)^2 + (y - 3)^2 + (z - 2)^2 \iff 8z = (x - 2)^2 + (y - 3)^2,
\]

which is the equation of an elliptic (actually circular) paraboloid as expected.

Exercise 4................................................................. 5 points
Assume that a parallelepiped has edges parallel to the axes (the three directions determined by the edges are the directions of the coordinate axes) and is such that the points \( (4, 2, 0) \) and \( (5, -2, 1) \) are some of its vertices. Find the volume of that parallelepiped.

**Solution:** *Here, it is a good idea to draw a figure.* The parallelepiped is rectangular, so that its volume is the product of its edge lengths along the three dimensions. The length along \( x \) is 1, the length along \( y \) is 4, and the length along \( z \) is 1, so **that the volume is 4**.
Exercise 5

(a) (4 points) Determine the type of surface and sketch the graph of

\[ 4x^2 - y^2 + z^2 - 8x + 4y + 8 = 0 \]

**Solution:** The equation is equivalent to

\[ 4(x - 1)^2 - (y - 2)^2 + z^2 + 8 = 0 \iff 4(x - 1)^2 - (y - 2)^2 + z^2 + 8 = 0. \]

Traces \( z = \text{cst} \) are hyperbolas, traces \( x = \text{cst} \) are hyperbolas as well, while traces along \( y = \text{cst} \) are ellipses or the empty set, which means that the surface is a hyperboloid with two sheets.

(b) (2 points) Transform the equation

\[ r^2 + r^2 \sin^2(\theta) = 1 \]

from cylindrical coordinates to rectangular coordinates.

**Solution:** In cylindrical coordinates, \( r^2 = x^2 + y^2 \) and \( \sin(\theta) = y/r \), so that the equation becomes

\[ x^2 + 2y^2 = 1. \]

(c) (1 point) What type of surface is it?

(c) \underline{Cylinder}
Exercise 6

Find the complex conjugate and the modulus of the complex numbers

(a) (2 points)

\[(1 - i)^3\]

Solution:

\[(1 - i)^3 = 1 - i^3 = 1 + i\]

and \(|(1 - i)^3| = 2\sqrt{2}|.

(b) (2 points)

\[\frac{1 - \sqrt{3}i}{1 + \sqrt{3}i}\]

Solution:

\[\frac{1 - \sqrt{3}i}{1 + \sqrt{3}i} = \frac{(1 - \sqrt{3}i)^2}{1 + 3} = \frac{-1}{2} - \frac{\sqrt{3}}{2}i,

which has modulus 1 and complex conjugate \(\frac{-1}{2} + \frac{\sqrt{3}}{2}i|.

(c) (2 points)

\[(2 - 3i)(2 + 3i)\]

Solution: \((2 - 3i)(2 + 3i) = 4 + 9 = 13\) which has modulus 13 and complex conjugate 13.
Your UNI:

In case you need more space