Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, your intermediate computations and reasoning must be readable and will be graded.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Exercise 1

Determine whether the following statements are true (T) or false (F). You do not need to justify your answer for this question.

(a) (2 points) T The image of a $3 \times 4$ matrix is a subspace of $\mathbb{R}^4$.

(b) (2 points) T If $\text{ker } A = \{0\}$, then the columns of $A$ are linearly independent.

(c) (2 points) T The linear transformation $T(f) = f + f''$ is an isomorphism of the space of smooth functions defined on $\mathbb{R}$ (think about the kernel of $T$).

(d) (2 points) F Any four dimensional space has infinitely many linear subspaces of dimension three.

(e) (2 points) T If $A$ and $B$ are symmetric $n \times n$ matrices, then $AB$ must be symmetric as well.

(f) (4 points) T There exists a $2 \times 2$ matrix such that $A^2 \neq 0$ but $A^3 = 0$. 
Exercise 2 ................................................................. 6 points

For which value of the constant $k$ do the vectors below form a basis of $\mathbb{R}^4$?

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
2
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
1 \\
0 \\
3
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
1 \\
4
\end{pmatrix}, \quad \begin{pmatrix}
2 \\
3 \\
4 \\
k
\end{pmatrix}.
\]
Exercise 3

Let $V$ be the space of all upper triangular $2 \times 2$ matrices. Consider the linear transformation

$$T \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = aI_2 + bQ + cQ^2,$$

where $I_2$ is the identity matrix and

$$Q = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

(a) (2 points) Find the matrix $A$ of the transformation $T$ with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(b) (2 points) Find a base of the image of $T$. 
(c) (2 points) What is the rank of $T$?

(d) (2 points) Find a base of the kernel of $T$. 


Exercise 4

We consider in this exercise the space $\mathbb{R}_n[X]$ of all polynomials with degree at most $n$.

(a) (1 point) What is the dimension of $\mathbb{R}_n[X]$?

(b) (2 points) Is the transformation $P(X) \mapsto P'(X)$ invertible?

(c) (2 points) Consider the basis $\{1, X, X^2, etc.\}$. Write the matrix $M$ of the linear transformation

$$P(X) \mapsto P(X) - P'(X)$$

in this basis.
(d) (3 points) Compute the inverse of $M$, if it exist.
Exercise 5

Consider the set $\mathbb{H}$ of all matrices $M$ of size $4 \times 4$ of the form

$$M = \begin{pmatrix} A & -B^T \\ B & A^T \end{pmatrix}$$

where $A$ and $B$ are $2 \times 2$ matrices representing a rotation combined with a scaling, which means that there exist real numbers $p, q, r, s$ such that

$$A = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} r & -s \\ s & r \end{pmatrix}.$$ 

(a) (1 point) Is $\mathbb{H}$ a linear space?  
- $\bigcirc$ yes \quad $\bigcirc$ no

(b) (1 point) What is the dimension of $\mathbb{H}$?

(c) (2 points) A matrix in the set $\mathbb{H}$ is  
- $\bigcirc$ symmetric  \quad $\bigcirc$ skew-symmetric  \quad $\bigcirc$ neither symmetric or antisymmetric

(d) (2 points) If $M$ and $N$ are in $\mathbb{H}$ is $MN$ in $\mathbb{H}$ as well?  
- $\bigcirc$ yes \quad $\bigcirc$ no

(e) (3 points) Which matrices in $\mathbb{H}$ are invertible?
(f) (3 points) For a matrix $M$ in $\mathbb{H}$, write the inverse of $M$ when it exists.
(g) (2 points) Under which conditions on $p, q, r, s$ the matrix

\[
\begin{pmatrix}
p & -q & -r & -s \\
q & p & s & -r \\
r & -s & p & q \\
s & r & -q & p \\
\end{pmatrix}
\]

is orthogonal?
Extra space.