Columbia University

MATHUN 2010
LINEAR ALGEBRA
SPRING 2017

Practice Midterm II

Instructor: Guillaume Barraquand

Time: March 29, 2017, 10:10am – 11:25am

Your name: ____________________________

Solutions

UNI: ________________________________
Your UNI:

<table>
<thead>
<tr>
<th>Exercise</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Points</td>
<td>14</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>14</td>
<td>50</td>
</tr>
<tr>
<td>Score</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, your intermediate computations and reasoning must be readable and will be graded.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Exercise 1

Determine whether the following statements are true (T) or false (F). You do not need to justify your answer for this question.

(a) (2 points) F The image of a 3 \times 4 matrix is a subspace of \( \mathbb{R}^4 \).

(b) (2 points) T If \( \ker A = \{ \overrightarrow{0} \} \), then the columns of \( A \) are linearly independent.

(c) (2 points) F The linear transformation \( T(f) = f + f'' \) is an isomorphism of the space of smooth functions defined on \( \mathbb{R} \) (think about the kernel of \( T \)).

(d) (2 points) T Any four dimensional space has infinitely many linear subspaces of dimension three.

(e) (2 points) F If \( A \) and \( B \) are symmetric \( n \times n \) matrices, then \( AB \) must be symmetric as well.

(f) (4 points) F There exists a 2 \times 2 matrix such that \( A^2 \neq 0 \) but \( A^3 = 0 \).

(c) The functions \( \cos \) and \( \sin \) belong to \( \ker T \), so that \( T \) cannot be invertible.

(e) \( AB \) is symmetric if and only if \( AB = (AB)^T \) but \( (AB)^T = BA^T = BA \), so \( AB \) is symmetric if and only if \( AB = BA \) which is not always true for symmetric matrices.

\[
F_{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}
\]

(f) \( A \) has necessarily rank 1. Let \( \overrightarrow{x} \in \ker A \) and \( y \notin \ker A \). In the basis \( \{ \overrightarrow{x}, \overrightarrow{y} \} \), \( A \) has matrix of the form \( \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \). One can prove that \( b \neq 0 \) because \( A^2 \neq 0 \), and then \( A^3 \) cannot be 0.
Exercise 2. For which value of the constant $k$ do the vectors below form a basis of $\mathbb{R}^4$?

$$
\begin{pmatrix}
1 \\
0 \\
0 \\
2
\end{pmatrix},
\begin{pmatrix}
0 \\
1 \\
0 \\
3
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1 \\
4
\end{pmatrix},
\begin{pmatrix}
2 \\
3 \\
4 \\
k
\end{pmatrix}
$$

The vectors form a basis of $\mathbb{R}^4$ if the matrix

$$
\Pi = 
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
2 & 3 & 4 & k
\end{pmatrix}
$$

is invertible.

$$
\text{rank}(\Pi) = \text{rank}
\begin{pmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 3 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & k-4-9-16
\end{pmatrix}
$$

so $\Pi$ is invertible if and only if

$$
k \neq 29
$$

The vectors above form a basis of $\mathbb{R}^4$ for all real numbers $k$ such that $k \neq 29$. 

Page 4
Exercise 3

Let $V$ be the space of all upper triangular $2 \times 2$ matrices. Consider the linear transformation

$$T \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = aI_2 + bQ + cQ^2,$$

where $I_2$ is the identity matrix and

$$Q = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} 1 & 8 \\ 0 & 9 \end{pmatrix}.$$

(a) (2 points) Find the matrix $A$ of the transformation $T$ with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & 9 \end{pmatrix}$$

(b) (2 points) Find a base of the image of $T$.

$$\text{span}(A) = \text{span} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 2 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

so

$$\left\{ I_2, Q \right\} \text{ forms a basis of } \text{Im}(T).$$

(We have $Q^2 = 4Q - 3I_2$)
(c) (2 points) What is the rank of $T$?

$$\text{rank}(A) = 2 \quad \text{so that} \quad \text{rank}(T) = 2$$

(d) (2 points) Find a base of the kernel of $T$.

Since $Q^2 = 4Q - 3I_2$, the matrix

$$\begin{pmatrix} 3 & -4 \\ 0 & 1 \end{pmatrix} \in \text{ker } T$$

By the rank nullity theorem,

$$\dim(\text{ker } T) = 3 - \text{rank } T = 1$$

so $\text{ker } T = \text{span}\left\{ \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. 
Exercise 4

We consider in this exercise the space $\mathbb{R}_n[X]$ of all polynomials with degree at most $n$.

(a) (1 point) What is the dimension of $\mathbb{R}_n[X]$?

$$\dim(\mathbb{R}_n[X]) = n + 1.$$  

(b) (2 points) Is the transformation $P(X) \rightarrow P'(X)$ invertible?

No, because the kernel of the transformation contains all constant polynomials, so

$$\ker \neq \{0\}.$$  

(c) (2 points) Consider the basis $\{1, X, X^2, \text{etc.}\}$. Write the matrix $M$ of the linear transformation

$$P(X) \rightarrow P(X) - P'(X)$$

in this basis.

Let $\; T : P \rightarrow P - P'.$

$$T(1) = 1, \quad T(X) = X - 1, \quad T(X^2) = X^2 - 2X, \quad \ldots$$

$$T(X^i) = X^i - iX^{i-1},$$

so the matrix of $T$ in the basis $\{1, X, X^2, \ldots, X^n\}$ is

$$
\begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & \cdots & -i & \cdots & 0 \\
& & & & \\
0 & \cdots & 1 & \cdots & 0 \\
& & & & \\
0 & \cdots & 0 & \cdots & 1
\end{pmatrix}.
$$

Page 7
(d) (3 points) Compute the inverse of $M$, if it exist.

If $Q = P - P'$ then

$$P = Q + Q' + Q'' + Q''' + \ldots + Q^{(n*)}.$$ 

So $T^{-1}(P) = P + P' + \ldots + P^{(n)}$.

$T^{-1}(1) = 1$

$T^{-1}(X) = X + 1$

$T^{-1}(X^2) = X^2 + 2X + 2$

More generally, $T^{-1}(X^i) = X^i + iX^{i-1} + \frac{i(i-1)}{2!}X^{i-2} + \ldots \quad \ldots + i! \cdot X + i!$.

where $i! = i(i-1)(i-2)\ldots \times 3 \times 2 \times 1$. Thus the matrix of $T^{-1}$ is column $(i+1)$

$$
\begin{pmatrix}
1 & 1 & 2 & 6 & \cdots & \xi \\
0 & 1 & 2 & 6 & \cdots & i! \\
0 & 0 & 1 & 3 & \cdots & \frac{i(i-1)(i-2)}{2!} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & i! \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 1
\end{pmatrix}
$$

Page 8
Exercise 5

Consider the set $\mathbb{H}$ of all matrices $M$ of size $4 \times 4$ of the form

$$M = \begin{pmatrix} A & -B^T \\ B & A^T \end{pmatrix}$$

where $A$ and $B$ are $2 \times 2$ matrices representing a rotation combined with a scaling, which means that there exist real numbers $p, q, r, s$ such that

$$A = \begin{pmatrix} p & -q \\ q & p \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} r & -s \\ s & r \end{pmatrix}.$$

(a) (1 point) Is $\mathbb{H}$ a linear space?

\[ \boxed{\text{yes}} \quad \boxed{\text{no}} \]

(b) (1 point) What is the dimension of $\mathbb{H}$?

\[ \text{dim}(\mathbb{H}) = 4 \]

(c) (2 points) A matrix in $\mathbb{H}$ is

\[ \boxed{\text{neither symmetric or antisymmetric}} \quad \boxed{\text{symmetric}} \quad \boxed{\text{antisymmetric}} \]

(d) (2 points) If $M$ and $N$ are in $\mathbb{H}$ is $MN$ in $\mathbb{H}$ as well?

\[ \boxed{\text{yes}} \quad \boxed{\text{no}} \]

(e) (3 points) Which matrices in $\mathbb{H}$ are invertible?

All matrices in $\mathbb{H}$ are invertible except the zero matrix.

Indeed, unless $A$ and $B$ are zero, one can find the inverse as the next question shows.
(f) (3 points) For a matrix $M$ in $H$, write the inverse of $M$ when it exists.

One notices that if $A = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$ and $B = \begin{pmatrix} r & -\alpha \\ -\beta & r \end{pmatrix}$,

then

$$\begin{pmatrix} A - B^T \\ B \\ A^T \end{pmatrix} \begin{pmatrix} A^T & B^T \\ B & A \end{pmatrix} = \begin{pmatrix} AA^T + B^TB & AB^T - B^TA \\ BAT - A^TB & BB^T + A^TA \end{pmatrix}$$

$$= \begin{pmatrix} p^2 + q^2 + r^2 + s^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So if $\Pi = \begin{pmatrix} A - B^T \\ B \\ A^T \end{pmatrix}$ then

$$\Pi^{-1} = \frac{1}{p^2 + q^2 + r^2 + s^2} \begin{pmatrix} A^T & B^T \\ -B & A \end{pmatrix}$$

And the inverse exists if $p^2 + q^2 + r^2 + s^2 \neq 0$, that is if and only if $\Pi \neq 0$.
(g) (2 points) Under which conditions on \( p, q, r, s \) the matrix
\[
\begin{pmatrix}
p & -q & -r & -s \\
q & p & s & -r \\
r & -s & p & q \\
s & r & -q & p \\
\end{pmatrix}
\]
is orthogonal?

It is easy to check that the dot product of any two columns is zero. Thus, the matrix is orthogonal if and only if \( p^2 + q^2 + r^2 + s^2 = 1 \) (that is when the columns have norm 1).

One can also argue that \( \mathbf{M} \) is orthogonal if and only if \( \mathbf{M}^{-1} = \mathbf{M}^T \), which is the case if and only if \( p^2 + q^2 + r^2 + s^2 = 1 \).
About Exercise 4 (a): To be very precise, the coefficient of $X^i$ in $T(X^j)$ (that is the coefficient in position $(i+1, j+1)$ in the inverse matrix) is in general for $j > i$.

$$j (j-1) (j-2) \ldots (i+1)$$

which can be written

$$\frac{j!}{i!}.$$

The coefficient in position $(i, j)$ for $i, j \in \{1, \ldots, n+1\}$ is thus

$$
\begin{cases}
\frac{(j-1)!}{(i-1)!} & \text{if } j \geq i \\
0 & \text{if } j < i
\end{cases}
$$

with the convention that $0! = 1$. 

Page 12