Columbia University

MATHUN2010
LINEAR ALGEBRA
SPRING 2017

Practice Midterm II

Instructor: Guillaume Barraquand

Time: March 29, 2017. 10:10am – 11:25am

Your name: ________________________________

UNI: ________________________________
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Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, your intermediate computations and reasoning must be readable and will be graded.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are **not** allowed.
Exercise 1

Determine whether the following statements are true (T) or false (F). You do not need to justify your answer for this question.

(a) (2 points) T If the image of an \( n \times n \) matrix \( A \) is all \( \mathbb{R}^n \), then \( A \) must be invertible.

(b) (2 points) F The columns of the change of basis matrix \( S_{A \rightarrow B} \) (from a basis \( A \) to a basis \( B \)) are the elements of \( B \) written in the basis \( A \).

(c) (2 points) T Let \( V \) be any linear space. If \( T \) is a linear transformation from \( V \) to \( V \), then the set

\[ \{ v \in V \text{ such that } T(v) = v \} \]

is a linear subspace of \( V \).

(d) (2 points) T If square matrices \( A \) and \( S \) are orthogonal, then \( S^{-1}AS \) must be orthogonal as well.

(e) (2 points) F If \( AA^T = A^TA \) for a square matrix \( A \), then \( A \) is orthogonal.

(b) \( S_{A \rightarrow B} \) has columns which are the elements of \( A \) written in the basis \( B \).

(d) A matrix \( M \) is orthogonal if and only if \( M M^T = I \).

\[
(S^{-1}AS)(S^{-1}AS)^T = S^{-1}AS S^TA^T (S^{-1})^T
\]

\[ = S^{-1}A A^T S \quad \text{because } S \text{ is orthogonal} \]

\[ = S^{-1}S \quad \text{because } A \text{ is orthogonal} \]

\[ = I \]

(e) Counter example: \( A = 0 \).
Exercise 2

Consider the matrix

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \end{pmatrix} \]

(a) (4 points) Compute \( \text{rref}(A) \).

\[ \text{rref}(A) = \text{rref} \left( \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) = \text{rref} \left( \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \]

\[ = \text{rref} \left( \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right) \]
(b) (2 points) Find a basis of the image of $A$.

The leading ones of $\text{rref}(A)$ are in columns 1 and 2. Thus, a basis of the image of $A$ can be chosen as the two first columns of $A$.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

is a basis of $\text{Im}(A)$. 


Exercise 3

In the plane $V$ defined by the equation

$$2x_1 + x_2 - 2x_3,$$

Consider the bases

$$\mathcal{A} = \{\vec{a}_1, \vec{a}_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}$$

and

$$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \right\}.$$

(a) (2 points) Write an equation relating the matrices $A = (\vec{a}_1, \vec{a}_2)$, $B = (\vec{b}_1, \vec{b}_2)$ and $S = S_{B \rightarrow A}$ (that is the change of basis matrix from $B$ to $A$). You do not need to justify your answer.

$$\mathcal{B} = \mathcal{A}S$$

(b) (2 points) Complete the diagram below explaining your answer to the previous question. You must precise the direction of the arrows and what is the matrix of the transformation corresponding to each arrow.

Recall that $(\vec{x})_A$ is the vector formed by the coordinates of $\vec{x}$ in the basis $\mathcal{A}$, and $(\vec{x})_B$ is the vector formed by the coordinates of $\vec{x}$ in the basis $\mathcal{B}$.
(c) (2 points) Is $A$ an orthonormal basis of $V$?

No because $\|a_1\| \neq 1$.

(d) (4 points) Write the matrix of the orthogonal projection onto $V$ in $\mathbb{R}^3$.

$\vec{a}_1$ and $\vec{a}_2$ are orthogonal and $\|\vec{a}_1\| = \|\vec{a}_2\| = 3$

So \[
\begin{bmatrix}
\frac{1}{3} & \frac{2}{3} \\
\frac{1}{3} & -\frac{2}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\] is an orthonormal basis of $V$.

The matrix of the orthogonal projection onto $V$ is hence

\[
\begin{pmatrix}
\frac{1}{3} & \frac{2}{3} \\
\frac{1}{3} & -\frac{2}{3} \\
\frac{1}{3} & \frac{1}{3}
\end{pmatrix} \begin{pmatrix}
\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix} = \frac{1}{9} \begin{pmatrix}
1 & 2 \\
2 & -2 \\
2 & 2
\end{pmatrix}
\begin{pmatrix}
1 & 2 & 2 \\
2 & -2 & 1 \\
2 & 1 & 1
\end{pmatrix}
\]

\[
= \frac{1}{9} \begin{pmatrix}
5 & -2 & 4 \\
-2 & 8 & 2 \\
4 & 2 & 5
\end{pmatrix}
\]

The matrix of the orthogonal projection is

\[
\begin{pmatrix}
\frac{5}{9} & -\frac{2}{9} & \frac{4}{9} \\
-\frac{2}{9} & \frac{8}{9} & \frac{2}{9} \\
\frac{4}{9} & \frac{2}{9} & \frac{5}{9}
\end{pmatrix}
\]
Exercise 4 ................................................................. 10 points

Find the least squares solution \( \vec{x}^* \) of the system \( A\vec{x} = b \) where

\[
A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

The least squares solution is the solution of the system \( A^TA \vec{x}^* = A^T\vec{b} \).

\[
A^TA = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}
\]

\[
A^T\vec{b} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]

We consider the augmented matrix of the system:

\[
\begin{pmatrix} 3 & 2 & -1 \\ 2 & 6 & 1 \end{pmatrix} \quad \text{which reduces to} \quad \begin{pmatrix} 1 & 3 & \frac{1}{2} \\ 0 & -7 & -\frac{5}{2} \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 3 & \frac{1}{2} \\ 0 & 1 & \frac{5}{4} \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -\frac{8}{14} \\ 0 & 1 & \frac{5}{14} \end{pmatrix}.
\]

The least squares solution is \( \begin{pmatrix} -\frac{8}{14} \\ \frac{5}{14} \end{pmatrix} \).
Your UNI:

Extra space
Exercise 5

In the space $\mathcal{C}([0,1])$ of continuous functions defined on $[0,1]$, we introduce the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t) \, dt.$$ 

For $i \in \{0, 1, 2, \ldots \}$ define $f_i(t) = t^i$. Let

$$V_n = \text{span}\{f_0(t), \ldots, f_n(t)\},$$

that is the subspace of polynomials defined on $[0,1]$ of degree at most $n$. It is well known that

$$\mathcal{F}_n = \{f_0(t), \ldots, f_n(t)\} = \{1, \ldots, t^n\}$$

forms a basis of $V_n$.

(a) (2 points) Let $n, m$ be positive integers. Find the inner product of $f_n(t)$ and $f_m(t)$, that is $\langle t^n, t^m \rangle$.

$$\int_0^1 t^n t^m \, dt = \left[ \frac{t^{n+m+1}}{n+m+1} \right]_0^1 = \frac{1}{n+m+1}$$

so

$$\langle t^n, t^m \rangle = \frac{1}{n+m+1}$$

(b) (2 points) Find $\|f_n\|$ the norm of $f_n(t) = t^n$.

$$\|f_n\| = \sqrt{\langle f_n, f_n \rangle} = \sqrt{\frac{1}{2n+1}}.$$
Now we consider \( V_2 \), the subspace of polynomials defined on \([0,1]\) with degree at most 2. We want to apply the Gram-Schmidt process on the basis \( \mathcal{F}_2 = \{1, t, t^2\} \). We start with \( g_0(t) = \frac{f_0(t)}{\|f_0\|} = 1 \).

(c) (2 points) Compute \( \text{proj}_{V_0}(f_1) \). Recall that \( \text{proj}_{V_0}(f_1) \) is the projection of the function \( f_1 \) onto the subspace \( V_0 \) spanned by \( f_0 \).

\[
\{g_0\} \text{ is a basis of } V_0 \text{ so } \\
\text{proj}_{V_0}(f_1) = \left< f_1, g_0 \right> g_0 \\
= \left( \int_0^1 t \cdot 1 \, dt \right) \cdot 1 = \frac{1}{2}.
\]

\[
\text{proj}_{V_0}(f_1) = \frac{1}{2} \text{ (the constant function)}.
\]

(d) (2 points) Compute

\[
g_1(t) = \frac{f_1 - \text{proj}_{V_0}(f_1)}{\|f_1 - \text{proj}_{V_0}(f_1)\|}.
\]

\[
\|f_1 - \text{proj}_{V_0}(f_1)\|^2 = \int_0^1 (t - \frac{1}{2})^2 \, dt \\
= \int_0^1 (t^2 - t + \frac{1}{4}) \, dt \\
= \frac{1}{3} - \frac{1}{2} + \frac{1}{4} \\
= \frac{1}{12}.
\]

\[
\frac{1}{2\sqrt{3}}
\]

\[
\text{and } \left| g_1(t) \right| = \sqrt{3} (2t - 1)
\]
(e) (6 points) We continue applying the Gram-Schmidt process to the basis $F_2 = \{1, t, t^2\}$ of $V_2$ to construct an orthonormal basis $G_2 = \{g_0(t), g_1(t), g_2(t)\}$ of $V_2$. Find $g_2(t)$.

\[ \text{proj}_{V_1}(f_2) = \langle f_2, g_0 \rangle g_0 + \langle f_2, g_1 \rangle g_1, \]
\[ \langle f_2, g_0 \rangle = \int_0^1 t^2 \, dt = \frac{1}{3} \]
\[ \langle f_2, g_1 \rangle = \int_0^1 t^2 (\sqrt{3}(2t-1)) \, dt \]
\[ = \sqrt{3} \int_0^1 2t^3 - t^2 = \sqrt{3} \left( \frac{1}{2} - \frac{1}{3} \right) = \frac{\sqrt{3}}{6} \]
\[ \text{proj}_{V_1}(f_2) = \frac{1}{3} + \frac{\sqrt{3}}{6} \sqrt{3}(2t-1) \]
\[ = t - \frac{1}{6} . \]
\[ f_2 - \text{proj}_{V_1}(f_2) = t^2 - t + \frac{1}{6} . \]
\[ \| f_2 - \text{proj}_{V_1}(f_2) \|^2 = \int_0^1 (t^2 - t + \frac{1}{6})(t^2 - t + \frac{1}{6}) \, dt \]
\[ = \int_0^1 \left( t^4 + \frac{1}{36} - 2t^3 + \frac{t^2}{3} - \frac{t}{3} \right) \, dt \]
\[ = \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{2}{4} + \frac{1}{9} - \frac{1}{6} . \]
\[ \| f_2 - \text{proj}_{u_1}(f_2) \|^2 = \frac{1}{5} + \frac{12+1-18+4-6}{36} \]

\[ = \frac{1}{36 \times 5} \]

so

\[ \| f_2 - \text{proj}_{u_1}(f_2) \| = \frac{1}{6 \sqrt{5}} \]

and

\[ g_2(t) = \frac{f_2(t) - \text{proj}_{u_1}(f_2)}{\| f_2 - \text{proj}_{u_1}(f_2) \|} \]

so that

\[ g_2(t) = \sqrt{5} \left( 6t^2 - 6t + 1 \right) \]
Extra space.