Instructions:

- Please write your UNI on every page.

- Unless stated otherwise, your intermediate computations and reasoning must be readable and will be graded.

- Please write neatly, and put your final answer in a box.

- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Exercise 1 ................................. 10 points
Determine whether the following statements are true (T) or false (F). **You do not need to justify your answer for this question.**

(a) ___ For two invertible matrices $A$ and $B$ of size $n \times n$, we have

$$(ABA^{-1})^3 = AB^3 A^{-1}.$$ 

(b) ___ For two matrices $A$ and $B$ of size $n \times n$, we have

$$(A + B)^2 = A^2 + 2AB + B^2.$$ 

(c) ___ If $A^2 = I_2$, then $A$ must be either $I_2$ or $-I_2$. 

(d) ___

$$\text{rank } \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{pmatrix} = 3$$

(e) ___ There exists a $3 \times 4$ matrix with rank 4.
Exercise 2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 6 points

Find the inverse of

\[
\begin{pmatrix}
1 & 1 & 1 \\
2 & 3 & 0 \\
1 & 2 & 1
\end{pmatrix}.
\]
Exercise 3 ..............................................................

Let

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}.
\]

In this exercise, we will show that for a nonnegative integer \( n \),

\[
A^n = \begin{pmatrix}
1 & n & u_n \\
0 & 1 & n \\
0 & 0 & 1
\end{pmatrix},
\]

where \( u_n \) is a certain sequence to be determined. Let \( B \) be the matrix

\[
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

(a) (2 points) Find \( \ker(B) \).

(b) (1 point) Is \( B \) invertible?  ○ Yes  ○ No

(c) (2 points) Compute \( B^2 \).

(d) (2 points) Compute \( B^3 \).
(e) (2 points) Show that for all \( n \geq 4, \ B^n = 0. \)

(f) (2 points) Using the fact that \( A = I_3 + B, \) Compute \( A^n \) for a positive integer \( n. \)

    *Hint: You may use without justification that for two \( n \times n \) square matrices \( M \) and \( N \) such that \( MN = NM, \)

\[
(M + N)^n = \sum_{i=0}^{n} \binom{n}{i} M^i N^{n-i},
\]

with the understanding that \( M^0 = N^0 = I_n. \)

(g) (2 points) To help you, we suggest possible choices for the value of \( u_n \) (tick the correct one):

\[
\begin{align*}
\text{○ } &\frac{n(n^2 + 3n + 5)}{6} & \text{○ } n & \text{○ } \frac{n(n-1)}{2} & \text{○ } \frac{n(n+1)}{2} \\
\text{○ } &\frac{6 - 8n + 12n^2 - n^3}{6} & \text{○ } n^2 + 14n - 6 & \text{○ } n^2 - n + 1
\end{align*}
\]
(h) (3 points) Find the inverse of $A$.

(i) (2 points) Is the formula for $A^n$ also valid when $n$ is negative?  ○ Yes  ○ No

(j) (Bonus) Find a general formula for

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$
For $\theta \in \mathbb{R}$, let

$$
R_{\theta} = \begin{pmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$

(a) (4 points) For $\alpha, \beta \in \mathbb{R}$, compute and simplify the product $R_{\alpha}R_{\beta}$.

*Hint: You can do the computation using the usual product rule and simplify. Otherwise you can use block matrices and guess the product using an appropriate result from the course.*
(b) (2 points) Let $P$ be the subspace

$$P = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Interpret geometrically the linear transformation of $P$ defined by

$$\overrightarrow{x} \mapsto R_\theta \overrightarrow{x}$$

for $\overrightarrow{x} \in P$.

(c) (1 point) For any $z \in \mathbb{R}$, compute

$$R_\theta \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}.$$
(d) (3 points) Interpret geometrically the linear transformation of $\mathbb{R}^3$ defined by

$$\vec{x} \mapsto R_\theta \vec{x}.$$ 

*Hint:* any $\vec{x} \in \mathbb{R}^3$ can be decomposed as

$$\vec{x} = \vec{p} + \vec{p}^\perp,$$

where $\vec{p} \in P$ and $\vec{p}^\perp$ is of the form $\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}$ for a certain $z$. 


Exercise 5 ................................................................. 6 points
Can one find a matrix of size 10 × 10 with 92 ones among its entries that is invertible? If your answer is yes, give an example. If your answer is no, explain why and give an example of such a matrix with rank 9.
Your UNI:

Extra space.