Chapter 2  (End of chapter problems)

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\[ t^2 y'' + 2t y' - 1 = 0 \quad (t > 0), \]

Let \( v = y' \) the ODE becomes

\[ t^2 v' + 2t v = 1 \]

\[ \Rightarrow (t^2 v)' = 1 \]

If we integrate, we get

\[ t^2 v(t) = t + C_1 \]

which yields the general solution

\[ v(t) = \frac{1}{t} + \frac{C_1}{t} \]

Since \( v(t) = y'(t) \), we integrate again and find

\[ y(t) = \log(t) + \frac{C_2}{t^2} + C_3 \]

where \( C_2 \) and \( C_3 \) are arbitrary constants.
\[ 42 \]
\[ y y'' + (y')^2 = 0 \]

1st method

recognize that \((yy')' = yy'' + (y')^2\)

so the equation becomes

\((yy')' = 0\).

Moreover \(yy' = \frac{d}{dt}\left(\frac{y^2}{2}\right)\), so the equation becomes

\[ \frac{d^2}{dt^2} \left(\frac{y^2}{2}\right) = 0 \]

\[ \Rightarrow \quad y^2 = C_1 t + C_2 \quad \text{for two constant} \quad C_1, C_2 \in \mathbb{R} \]

\[ y(t) = \pm \sqrt{C_1 t + C_2} \]
2nd method to solve \( y(t) y''(t) + (y'(t))^2 = 0 \)

We want to find \( y \) as a function of \( t \).

Let \( v = \frac{dy}{dt} \).

Then the equation can be written

\[
y \frac{dv}{dt} + v^2 = 0
\]

\[
\frac{dv}{dt} = \frac{dv}{dy} \times \frac{dy}{dt} = \frac{dv}{dy} \times v
\]

The equation becomes

\[
y v \frac{dv}{dy} + v^2 = 0
\]

We may first solve \( v \) as a function of \( y \), and then using that \( v = \frac{dy}{dt} \).
we will get a new ODE for $y$ which will be a 1st order ODE.

\[ y \frac{dv}{dy} + v^2 = 0 \]

\[ \Rightarrow v = 0 \quad \text{or} \quad y \frac{dv}{dy} + v = 0 \]

\[ \Rightarrow v y = C \]

Now recall that $v = \frac{dy}{dt}$. We get

\[ \frac{dy}{dt} = 0 \quad \text{or} \quad y \frac{dy}{dt} = C \]

\[ y = C_1 \quad \text{or} \quad \frac{d}{dt} (y^2) = 2C \]

\[ y^2 = 2C_2 t + C_3 \]

\[ \Rightarrow y = \pm \sqrt{C_2 t + C_3} \]
Conclusion: Any solution can be written as

\[ y(t) = \pm \sqrt{C_2 t + C_3} \]

(the case \( y(t) = C \) is a particular case of the above).
\[ \frac{dy}{dx} = \frac{x^3 - 2y}{x} \]

This is a 1st order linear ODE.

Let us first consider the homogeneous ODE

\[ y' + \frac{2y}{x} = 0 \]

The general sol. is given by

\[ y(x) = C_1 e^{\int \frac{2}{x} \, dx} = C_1 e^{-2 \log |x|} = \frac{C_1}{x^2} \]

Let us find a solution of the original ODE of the form

\[ y(x) = \frac{f(x)}{x^2} \]

Then \( f \)

must satisfy

\[ \frac{f'(x)}{x^2} = x^2 \]
Which implies

\[ f(x) = \frac{x^5}{5} + C \]

Thus

\[
\begin{aligned}
y(x) &= \frac{x^3}{5} + \frac{C}{x^2} \\
\end{aligned}
\]

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\[
\frac{dy}{dx} = \frac{1 + \cos(x)}{1 - \sin(y)}
\]

(separable equation)

\[
\begin{array}{l}
dy - \sin(y) \, dy = dx + \cos(x) \, dx \\
\end{array}
\]

\[
\Rightarrow \quad y + \cos(y) = x + \sin(x) + C
\]

One cannot go further
3. see review session.

the trick is to use

\[(xy)' = y + xy'.\]

\[
\frac{dy}{dx} = -\frac{2xy + y^2 + 1}{x^2 + 2xy}
\]

\[
x^2\,dy + 2xy\,dy + 2xy\,dx + y^2\,dx + dx = 0
\]

the equation can be rewritten

\[d(x^2y) + d(xy^2) + dx = 0\]

\[\Rightarrow x^2y + xy^2 + x = C.\]
Another way of writing it:

\[
\frac{dy}{dx} = \frac{-2xy + y^2 + 1}{x^2 + 2xy}
\]

can be rewritten

\[
x^2y' + 2xxy' + 2xy + y^2 + 1 = 0
\]

so the equation can be written

\[
(x^2y)' + (xy^2)' + 1 = 0
\]

integrating gives

\[
x^2y + xy^2 + x = C.
\]
This may be solved.

\[ ax^2 + x^2y + x - C = 0 \]

\[ \Delta = x^4 - 4x^2(x-C) = x^4 - 4x^2 + 4xC. \]

If \( \Delta \geq 0 \),

\[ y(x) = \frac{-x^2 \pm \sqrt{x^4 - 4x^2 + 4xC}}{2x} \]

[8] \[ x \frac{dy}{dx} + 2y = \frac{\sin(x)}{x} \]

First solve the homogeneous equation

\[ xe^{y'} + 2y = 0 \]

yields \( y = Cxe^{-2} \) as in [1]
we look for a general solution of the form
\[ y(x) = \frac{f(x)}{x^2}. \]

This implies \[ x \frac{f'(x)}{x^2} = \frac{\sin(x)}{x}, \]
\[ f'(x) = \frac{\sin(x)}{x}. \]
\[ f(x) = C - \cos(x). \]

So \[ y(x) = \frac{C - \cos(x)}{x^2}, \]

since \[ y(2) = 1, \quad \frac{C - \cos(2)}{4} = 1 \]
so that \[ C = 4 + \cos(2) \]
and
\[ y(x) = \frac{4 + \cos(2) - \cos(x)}{x^2}. \]