Columbia University

MATHUN2030
INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATIONS
FALL 2017

Practice Midterm II

Instructor: Guillaume Barraquand

Time: Thursday 2 November, 2017. 11:40am – 12:55am

Your name: ________________________________

UNI: __________________________
Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, your intermediate computations and reasoning must be readable and will be graded.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Problem 1. Short questions. For this problem you do not need to justify anything.

(a) (2 points) The functions $\sin, \cos$ and $\tan$ are linearly independent on the domain $(-\pi/2, \pi/2)$.
   ○ True  ○ False

(b) (4 points) How many solutions the following initial value problem have?
\[
\begin{cases}
y'' + 2y' + y = 0. \\
y(0) = 1 - y'(0)
\end{cases}
\]
   ○ one  ○ two  ○ infinitely many
   Find one solution
   \[y = \ldots\]

(c) (2 points) Let $L$ be a linear $n-$th order differential operator and $g$ be a continuous function. If $\phi$ satisfies $L(\phi) = g$, $\psi$ satisfies $L(\psi) = 0$ and $W(\phi, \psi) \neq 0$, then any solution of $L(y) = g$ can is a linear combination of $\psi$ and $\phi$.
   ○ True  ○ False

(d) (2 points) The operator defined by
   \[
   L(f) = f'f - f
   \]
is linear.
   ○ True  ○ False

(e) (3 points (bonus)) Consider an ODE of the form
\[
y'''' + ay'' + by' + cy = 0,
\]
for some real constants $a, b, c$. Then there exist $\gamma \in \mathbb{R}$ such that $e^{\gamma t}$ is a solution.
   ○ True,  ○ False.
Problem 2

Solve

\[
\begin{cases}
y'' - 2y' + y = \sin(t) \\
y(0) = 1 \\
y'(0) = 0.
\end{cases}
\]

This problem is a fairly standard problem worth 15 points over a total of 50 points. You should make sure your answer is fully correct by taking the time to check it carefully in the end.
Problem 3. ........................................................................................................................................

Acoustic Beats

You do not need to read the following paragraph since all questions will be phrased in mathematical terms, but it may be helpful to.

“In acoustics, a beat is an interference pattern between two sounds of slightly different frequencies, perceived as a periodic variation in volume whose rate is the difference of the two frequencies. When tuning instruments that can produce sustained tones, beats can be readily recognized. Tuning two tones to a unison will present a peculiar effect: when the two tones are close in pitch but not identical, the difference in frequency generates the beating. The volume varies like in a tremolo as the sounds alternately interfere constructively and destructively. As the two tones gradually approach unison, the beating slows down and may become so slow as to be imperceptible.”

Wikipedia page Beat (acoustics)

(a) (10 points) Find the general solution of

\[ mu'' + ku = F \cos(\omega t) \]  

(\text{{\textit{(i)}}})

As usual, \( m \) and \( k \) are real positive constants here. We assume that \( \omega_0 = \sqrt{k/m} \) is different than \( \omega \).
(b) (2 points) Find a constant $C$ such that the following identity is true

$$\cos(x) - \cos(y) = C \sin \left( \frac{x + y}{2} \right) \sin \left( \frac{x - y}{2} \right).$$

$C =$________$

(c) (2 points) We consider the initial problem $u(0) = u'(0) = 0$. Find the solution of the ODE above that satisfy this initial value problem. Denote this solution $\phi_{\omega_0}(t)$. 
(d) (2 points) Factor your answer using the trigonometric identity (I).

(e) (1 point) Check that your solution satisfies the ODE (τ) and the initial data. You do not need to write calculations below. If your answers above are correct, you will get automatically 1 point for this question. Otherwise, you will get 1 point only if you answer correctly below.

   - I have checked my answer and it is correct.
   - I have checked my answer and unfortunately, it is not correct.
   - I did not find time to check my answer (wrong answer).

(f) (2 points) Sketch precisely the graph of the solution $\phi(t)$ for the following values of parameters:

$$m = 1, \quad \omega_0 = \frac{2\pi + \pi/3}{2}, \quad \omega = \frac{2\pi - \pi/3}{2} \quad \text{and} \quad F = \frac{\pi^2}{6} = \frac{(\omega_0^2 - \omega^2)}{4}.$$
The dotted curves are here to help you but you have to choose the correct ones.

(g) (4 points) For each fixed $t$, compute the limit of the solution $\phi_{\omega_0}(t)$ when $\omega_0$ goes to $\omega$. Denote

$$\phi_{\text{tuned}}(t) = \lim_{\omega_0 \to \omega} \phi_{\omega_0}(t).$$
(h) (2 points) Does $\phi_{\text{tuned}}(t)$ is a solution of the ODE ($\square$) when $\omega = \omega_0$. You do not need to justify your answer.

- Yes
- No
- It depends
Problem 4. Consider the initial value problem
\[
\begin{aligned}
y'' - x^2 y &= 1 + x, \\
y(0) &= 1, \\
y'(0) &= 1.
\end{aligned}
\]
Assume that there exist a solution of the form
\[
\phi(x) = \sum_{n=0}^{+\infty} a_n x^n.
\]
Determine \(a_2, a_3, a_4, a_5, a_6\).
Problem 5. Let $f$ and $g$ be two continuous functions on $\mathbb{R}_+$ and assume that for some $s > 0$, their Laplace transforms $F(s)$ and $G(s)$ exist. We define a third function $h(t)$ related to $f$ and $g$ via the relation

$$h(t) = \int_0^{+\infty} f(s)g(t-s)ds.$$ 

Express the Laplace transform $H(s)$ of $h$ in terms of $F$ and $G$. 

(More difficult)