Columbia University

MATHUN2030
Introduction to Ordinary Differential Equations
Fall 2017

Practice Final Exam

Instructor: Guillaume Barraquand

Time: Thursday 21 December, 2017. 4:10pm – 7:00pm

Your name: ________________________________________________________________

UNI: ___________________________
Instructions:

- Please write your UNI on every page.
- Unless stated otherwise, your intermediate computations and reasoning must be readable and will be graded.
- Please write neatly, and put your final answer in a box.
- Books, notes, calculators, smartphones or any other electronic devices are not allowed.
Problem 1

Short questions. For this problem you do not need to justify anything. If you do not know the answer, you can sometimes reply “random” instead of choosing an answer at random, this will give you half of the points for the corresponding question.

(a) (2 points) Let $F$ be the Laplace transform of $f$. Then the Laplace transform of $f'$ is $sF(s) - f(0)$  ○ True,  ○ False,  ○ Random.

(b) (2 points) The series
$$
\sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!}
$$
solves the ODE
$$
y'' + y = 0.
$$
○ True,  ○ False,  ○ Random.

(c) (2 points) The solution $y = K$ of the logistic equation
$$
\frac{dy}{dt} = r y \left( 1 - \frac{y}{K} \right)
$$
is asymptotically stable.
○ True,  ○ False,  ○ Random.

(d) (2 points) Consider the equation
$$
\begin{cases}
    y^{(4)} + 4y^{(3)} + 6y^{(2)} + 4y^{(1)} + y = 0, \\
y(0) = 0, \\
y'(0) = 0.
\end{cases}
$$
How many solutions solutions can you find
○ one,  ○ two,  ○ four,  ○ infinitely many?

(e) (2 points) Consider a solution $\phi(t)$ of $y'' + y' + y = g(t)$, where $g(t)$ is function discontinuous at $t = 1$. Then the solution $\phi(t)$ must be discontinuous as well at $t = 1$.
○ True,  ○ False,  ○ Random.

(f) (2 points) Assume $\frac{d}{dt} \vec{x} = A \vec{x}$, and $A$ has three distinct real eigenvalues. Then
$$
\lim_{t \to +\infty} \|\vec{x}(t)\|
$$
is either 0 or $+\infty$.  ○ True,  ○ False,  ○ Random.

(g) (2 points) A non-linear first order ODE may have 2 linearly independent solutions.
○ True,  ○ False,  ○ Random.

(h) (2 points) For a second order ODE with constant coefficients, the Wronkian of two solutions is always a constant.
○ True,  ○ False,  ○ Random.
(i) (2 points) For positive parameters $m, k, \gamma$, the solution of

$$mu'' + \gamma u' + ku$$

always goes to zero as $t$ goes to $+\infty$.

○ True, ○ False, ○ Random.

(j) (2 points) If vectors $\vec{x}^{(0)}, \ldots, \vec{x}^{(k)}$ are such that for all $t \in (0, 1)$

$$\vec{x}^{(0)} + t \vec{x}^{(1)} + \frac{t^2}{2} \vec{x}^{(2)} + \cdots + \frac{t^k}{k!} \vec{x}^{(k)} = \vec{0},$$

then

$$\vec{x}^{(0)} = \vec{x}^{(1)} = \cdots = \vec{x}^{(k)}.$$

○ True, ○ False, ○ Random.
Problem 2 ................................................................. 25 points

Let

\[ A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Solve the following system of ODEs

\[
\begin{cases}
\frac{d}{dt} \tilde{x}(t) = A \tilde{x}(t), \\
\tilde{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\end{cases}
\]
Problem 3 .................................................. 10 points
Solve for $t > 0$ the ODE

$$\begin{cases}
y' + \frac{2}{t} y = \sin(t), \\
y(\pi/2) = 1.
\end{cases}$$
Your UNI: 
Problem 4 ................................. 12 points

Consider a system of ODE of the form of the form

$$\frac{d}{dt}\vec{x}(t) = \mathbf{A} \vec{x}(t),$$

where $\mathbf{A}$ is chosen as

1. $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$,

2. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

3. $\mathbf{A} = \begin{pmatrix} 0 & 1/3 \\ -1 & 0 \end{pmatrix}$,

4. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$,

5. $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$,

6. $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

Match each case with one of the 6 possible direction fields below.

<table>
<thead>
<tr>
<th>$\mathbf{A}$</th>
<th>1.</th>
<th>2.</th>
<th>3.</th>
<th>4.</th>
<th>5.</th>
<th>6.</th>
</tr>
</thead>
<tbody>
<tr>
<td>direction field</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) 

(b)
Problem 5

We consider a planet whose position with respect to the sun is given by coordinates $x, y, z$, which satisfy

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}. $$

How long is a year on this planet? Mathematically, this amounts to finding the smallest $T > 0$ such that

$$\begin{pmatrix} x(T) \\ y(T) \\ z(T) \end{pmatrix} = \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}. $$
Problem 6

Consider the second order linear ODE

\[
\begin{cases}
y'' + y' - 2y = t, \\
y(0) = 1, \\
y'(0) = 0.
\end{cases}
\]

(a) (6 points) Solve this ODE using the method of undetermined coefficients. (This means, find the general solution of the homogeneous ODE, find a particular solution of the inhomogeneous ODE, write the general solution and adjust the coefficients to match with the initial value data.)
(b) (4 points) Find a matrix $A$ such that the ODE $(\star)$ can be rewritten as

$$\begin{align*}
\frac{d}{dt} \begin{pmatrix} y \\ y' \end{pmatrix} &= A \begin{pmatrix} y \\ y' \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix} \\
\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{align*}$$

(c) (6 points) Compute $\exp(At)$.

*Hint: Recall the decomposition $A = TDT^{-1}$ and use that

$$\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}^{-1} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}.$$*

*(For the exam you may not be given the inverse. If the matrix $A$ is symmetric, it is always possible to find the eigenvectors orthogonal to each other with norm one, and then in that case the inverse of $T$ is the transpose of $T$.)
(d) (2 points) Let $\vec{g}(t) = \begin{pmatrix} 0 \\ t \end{pmatrix}$. Find the first coordinate of the vector

$$\int_0^t \exp((t-s)A)\vec{g}(s) + \exp(tA) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. $$

*Hint: You do not need to do any computation!*
Problem 7

(This is not a typical bonus problem but an extra practice problem. Typical bonus problems are harder (less intermediate questions) and they are assigned a number of points such that it is always a bad strategy to spend time on them unless you have finished the rest.)

Let

\[ J = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \]

where \( a \in \mathbb{R} \) is some parameter.

(a) (1 point (bonus)) Compute \( J^2 \).

(b) (2 points (bonus)) Compute \( J^3 \) and \( J^4 \).

(c) (2 points (bonus)) Let \( n \) be a positive integer. Find the missing coefficients

\[ J^k = \left( \begin{array}{c} a^k \\ \frac{a^k}{k!} \end{array} \right) \]

(d) (2 points (bonus)) Simplify

\[ \sum_{k=0}^{+\infty} k x^{k-1} \frac{t^k}{k!} = \]

where we recall that by convention, \( 0! = 1 \). Hint: Recognize that this is the derivative with respect to \( x \) of something that you know.
(e) (2 points (bonus)) Determine \( \exp(\mathbf{J}t) \).

(f) (2 points (bonus)) Solve

\[
\begin{cases}
\frac{d}{dt} \vec{x}(t) = \mathbf{J} \vec{x}(t), \\
\vec{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\end{cases}
\]
Extra space.