Compactness

Rob Castellano

Let \((W, \omega)\) be a compact symplectic manifold.

**Hypothesis 0.1.** \(\langle [\omega], \pi_2(W) \rangle = 0\)

Let \(\mathcal{M} = \{u : \mathbb{R} \times S^1 \to W \text{ satisfying Floer's equation and having finite energy}\}\)

**Theorem 0.2.** \(\mathcal{M}\) is compact in \(C^\infty_{\text{loc}}(\mathbb{R} \times S^1, W)\)

The key part of the proof of this theorem is 0.3

**Proposition 0.3.** There exists an \(A\) such that for all \(u \in \mathcal{M}\) and \((s, t) \in \mathbb{R} \times S^1, \|\nabla_{(s,t)} u\| \leq A\)

*Proof of 0.2 assuming 0.3.* Let \(\{u_n\}\) be a sequence in \(\mathcal{M}\). 0.3 implies equicontinuity of this sequence. Since \(W\) is compact, we can apply Arzela-Ascoli to conclude that \(\{u_n\}\) has a subsequence \(\{u_{n_k}\}\) that converges to some \(u_0\) in \(C^0_{\text{loc}}\). We can see that \(u_0\) is a weak solution to Floer’s equation. Elliptic regularity then implies that \(u_0\) is a real solution to Floer’s equation, is \(C^\infty\), and if fact the convergence \(u_n \to u_0\) is \(C^\infty_{\text{loc}}\). Therefore, \(\mathcal{M}\) is compact in the \(C^\infty_{\text{loc}}\) topology.

*Proof of 0.3.* Suppose to that contrary that we have a sequence \(\{u_k\} \subset \mathcal{M}\) and \(\{(s_k, t_k)\} \subset \mathbb{R} \times S^1\) such that \(\lim_{k \to \infty} \|\nabla_{(s_k, t_k)} u_k\| = \infty\). We will show that there must be bubbling, which contracts the assumption of symplectically aspherical. Let \(\{\tilde{\epsilon}_k\}\) be a sequence tending to zero such that \(\lim_{k \to \infty} \tilde{\epsilon}_k \|\nabla_{(s_k, t_k)} u_k\| = \infty\) (for example \(\tilde{\epsilon}_k = \sqrt{\|
abla_{(s_k, t_k)} u_k\|}\)). We will then repeatedly apply 0.4 to \(g = \|\nabla_{(s_k, t_k)} u_k\|, x_0 = (s_k, t_k), \epsilon_0 = \tilde{\epsilon}_k\)
Lemma 0.4. Let \( g : X \to \mathbb{R}^+ \) be a continuous function on a complete metric space. Let \( x_0 \in X \) and \( \varepsilon_0 > 0 \). Then there exists \( y \in X \) and \( \varepsilon \in (0, \varepsilon_0) \) such that

\[
\begin{align*}
d(x, y) &\leq 2\varepsilon \\
\varepsilon g(y) &\geq \varepsilon_0 g(x_0) \\
g(x) &\leq 2g(y), \forall x \in B(y, \varepsilon)
\end{align*}
\]

For a proof of this lemma see JHOL page 94.

Remark 0.5. We need this lemma because \( S^1 \times \mathbb{R} \) not compact, so we can’t just take maximums and get a uniform bound of 1 on the gradients of our rescalings. However, this lemma guarantees that we can find an “approximate” maximum. More precisely, we can find points where our function is “almost a maximum” on a neighborhood that is “big enough.”

Using this lemma we get new sequences \( \{(s_k, t_k)\}, \{\varepsilon_k\} \). Let \( R_k = \|\text{grad}_{(s_k, t_k)} u_k\| \).

Note that the lemma implies that \( \varepsilon_k \to 0, \varepsilon_k R_k \to 0 \) (and hence \( R_k \to \infty \)), and \( \|\text{grad}_{(s,t)} u_k\| \leq 2\|\text{grad}_{(s_k, t_k)} u_k\| \) for \( (s, t) \in B((s_k, t_k), \varepsilon_k) \).

Now viewing \( u : \mathbb{R}^2 \to W \), define

\[
v_k(s, t) = u_k \left( \frac{(s, t)}{R_k} + (s_k, t_k) \right).
\]

So

\[
\text{grad}_{(s,t)} v_k = \frac{1}{R_k} \text{grad}_{(s_k, t_k)} u_k.
\]

In particular,

\[
\text{grad}_{(0,0)} v_k = \frac{1}{R_k} \text{grad}_{(s_k, t_k)} u_k.
\]

Also on \( B(0, \varepsilon_k R_k) \)

\[
\|\text{grad}_{(s,t)} v_k\| \leq \frac{1}{R_k} \|\text{grad}_{(s_k, t_k)} u_k\|
\]

\[
\leq \frac{2}{R_k} \|\text{grad}_{(s_k, t_k)} u_k\|
\]

\[
= 2
\]

Thus, we can apply Arzela-Ascoli to conclude that a subsequence of \( \{v_k\} \) converges in \( C^0_{\text{loc}}(\mathbb{R}^2, W) \) to some \( v \). Now note that \( v_k \) satisfies

\[
\frac{\partial v_k}{\partial s}(s, t) + J(v_k) \frac{\partial v_k}{\partial t}(s, t) + \frac{1}{R_k} \text{grad}_{v_k(s,t)} H_{t_k} + \frac{1}{R_k} \text{grad}_{v_k(s,t)} (v_k(s, t)) = 0
\]
This is a Floer equation, so we apply elliptic regularity to conclude that $v$ is $C^\infty$ (there is some work to do here). Moreover, we have

\[
\begin{cases}
\|\nabla_{(0,0)} v\| = 1 \quad \text{(so } v \text{ is nonconstant)} \\
\|\nabla_{(s,t)} v\| \leq 2 \quad \forall (s, t) \in \mathbb{R}^2 \\
\frac{\partial v}{\partial s} + J(v) \frac{\partial v}{\partial t} = 0 \quad \text{(so } v \text{ is J-holomorphic)}
\end{cases}
\]

Next we want to show that $v$ has finite energy. Let $B_k = B((s_k, t_k), \varepsilon_k)$

\[
E(v|_{B(0, \varepsilon_k R_k)}) = \int_{B(0, \varepsilon_k R_k)} \|\nabla v_k(s, t)\|^2
\]

\[
= \int_{B(0, \varepsilon_k R_k)} \|\nabla u_k \left(\frac{(s, t)}{R_k} + (s_k, t_k)\right)\|^2
\]

\[
= \int_{B_k} \|\nabla u_k(s, t)\|^2
\]

\[
\leq \int_{B_k} \left(\frac{\partial u_k}{\partial s}\right)^2 + \left(\frac{\partial u_k}{\partial t}\right)^2 \right) ds dt
\]

\[
= \int_{B_k} \left(\frac{\partial u_k}{\partial s}\right)^2 + \left(\frac{\partial u_k}{\partial t} - X_t(u_k)\right)^2 \right) ds dt
\]

\[
+ \int_{B_k} \left(\|X_t(u_k)\|^2 + 2 \left\|\frac{\partial u_k}{\partial t} - X_t(u_k)\right\| \|X_t(u_k)\|\right) ds dt
\]

\[
\leq \int_{B_k} \left(\frac{\partial u_k}{\partial s}\right)^2 + 2 \left\|\frac{\partial u_k}{\partial t} - X_t(u_k)\right\|^2 + 2\|X_t(u_k)\|^2 \right) ds dt
\]

\[
= 3E(u_k) + 2 \int_{B_k} \|X_t(u_k)\|^2 ds dt
\]

(The last inequality follows from the inequality $2fg \leq f^2 + g^2$) The first term of the result is bounded by some $C$ because energy is universally bounded on $\mathbb{M}$ and the second term goes to zero because $\varepsilon_k \to 0$. Finally, $B(0, \varepsilon_k R_k)$ tends to $\mathbb{R}^2$, so $v$ has finite energy by Fatou’s lemma.

One way to complete from here is to say that because $v$ has finite energy, we can apply the removable singularity theorem for J-holomorphic maps to
conclude that $v$ extends to J-holomorphic sphere. $E(v) > 0$ because $v$ is nonconstant, but for J-holomorphic spheres

$$E(v) = \int_{S^2} v^* \omega$$

and this is zero by our hypothesis.

We will present an alternative (complete) proof. The removable singularity theorem can be proved using the results we will prove. We now need the following lemma.

**Lemma 0.6.** There exists a sequence $r_k \to \infty$ such that the length of $v(\partial B(0, r_k)) \to 0$.

We can now finish the proof of 0.3 assuming 0.6. For large $k$, $v(\partial B(0, r_k))$ lies in a Darboux chart $U$ containing a disk $D_{r_k}$ with boundary $v(\partial B(0, r_k))$. Let $\omega = d\lambda$ on this chart.

Form a sphere $S^2_{r_k}$ from $v(B(0, r_k))$ and $D_{r_k}$. Then

$$0 = \int_{S^2_{r_k}} \omega = \int_{D_{r_k}} \omega + \int_{v(B(0, r_k))} \omega$$

$$\int_{D_{r_k}} \omega = \int_{v(\partial B(0, r_k))} \lambda$$

Thus, $|\int_{D_{r_k}} \omega| \leq \ell(v(\partial B(0, r_k))) \sup_{U} \|\lambda\| \to 0$ as $k \to \infty$. Thus, $\int_{v(B_{r_k})} \omega \to 0$. On the other hand, $\int_{v(B_{r_k})} \omega \to \int_{R^2} v^* \omega = E(v) > 0$.

**Proof of 0.6.** Since $v$ is J-holomorphic, $v^* \omega$ is a symplectic form on $\mathbb{R}^2$ (look at $v^*(\partial_x, \partial_t)$ to see this). Write $v^* \omega_{(\rho, \theta)} = f(\rho, \theta) drd\theta$ where $f > 0$. Since $v$ is J-holomorphic with respect to the usual almost complex structure, the pullback Riemannian metric is $f(\rho, \theta)(d\rho^2 + \rho^2 d\theta^2)$ (this is a small calculation). Thus,

$$\ell(r) := \ell(v(\partial B_r)) = \int_{\partial B_r} \sqrt{g \left( \frac{\partial v}{\partial \theta}, \frac{\partial v}{\partial \theta} \right)} d\theta$$

$$= \int_0^{2\pi} \sqrt{v^* g_{(r,\theta)} \left( \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta} \right)} d\theta$$

$$= \int_0^{2\pi} \sqrt{f(r, \theta)} r^2 = r \int_0^{2\pi} \sqrt{f(r, \theta)} d\theta$$

4
Similarly, $A(r) := A(v(B_r)) = \int_{B_r} v^* \omega = \int_0^{2\pi} \int_0^r f(\rho, \theta) \rho d\rho d\theta$. So $A'(r) = r \int_0^{2\pi} f(r, \theta) d\theta$. Now

$$\ell(r) = \int_0^{2\pi} \sqrt{f(r, \theta)} d\theta$$

$$\leq r \sqrt{\int_0^{2\pi} d\theta \int_0^{2\pi} f(r, \theta) d\theta} \quad \text{(by Cauchy-Schwarz)}$$

$$= r \sqrt{2\pi A'(r)}$$

So $\ell(r)^2 \leq 2\pi r A'(r)$.

We want to show that there is a sequence $\{r_k\}$ such that $r_k A'(r_k) \to 0$. Well $\lim_{k \to \infty} \frac{A(k^2) - A(k)}{\log k} = 0$ because the top is bounded (W is compact). Also

$$\frac{A(k^2) - A(k)}{\log k} = \frac{A(k^2) - A(k)}{\log k^2 - \log k}.$$ Change variables to $t = \log k$ and apply the mean value theorem to the function $f(t) = A(e^t)$ on the interval $[t, 2t]$ to conclude that

$$\frac{A(k^2) - A(k)}{\log k^2 - \log k} = r_k A'(r_k)$$

for some $r_k \in (k, k^2)$

This completes the proof. \(\square\)