Relation between Floer and Morse

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Contents

1 Statements of results 1

2 Linearization of the flow of pseudo-gradient field. Proof of 1.3 4
   2.1 Linearization of the equation . . . . . . . . . . . . . . . . . . . 5
   2.2 Exponential decay of solutions . . . . . . . . . . . . . . . . . . 6
   2.3 Fredholm property . . . . . . . . . . . . . . . . . . . . . . . 7
   2.4 Calculation of the index . . . . . . . . . . . . . . . . . . . . 10
   2.5 Smale Condition . . . . . . . . . . . . . . . . . . . . . . . . 12

3 Proof of regularity theorem 1.2 12
   3.1 Proof of 1.2 . . . . . . . . . . . . . . . . . . . . . . . . . . . . 13

4 The Morse and Floer trajectories coincide 17
   4.1 Trajectories that are independent of \( t \) are regular . . . . . . . . . 17
   4.2 Trajectories are independent of \( t \) . . . . . . . . . . . . . . . . 19

1 Statements of results

Let \( CF_\ast(H,J) \) be the Floer complex associated to a Hamiltonian \( H \) (\( H \) is autonomous for this chapter so the Morse complex can be defined) and an almost complex structure \( J \). Let \( CM_\ast(H,J) \) be the Morse complex associated to a Morse function \( H \) and the metric defined by \( J \) (and \( \omega \)). The main result is:
Theorem 1.1. There exists (see second remark below) a nondegenerate $H$ sufficiently small (in the $C^2$ sense) such that

$$CF_*(H,J) = CM_{*+n}(H,J).$$

- The theorem contains the fact that these complexes are well defined.
- If $H_0$ is a nondegenerate Hamiltonian, we can make $H$ of the form $\frac{H_0}{k}$ for large $k$.
- The fact that $H$ is $C^2$ small guarantees that all periodic orbits are constant and therefore $\text{Crit}(A_H) = \text{Crit}(H)$.
- We know that always nondegenerate as a periodic orbit implies nondegenerate as a critical point (for autonomous Hamiltonians). Thus, $H$ is Morse.
- We know (5.4.6) that $d_x\psi^1 = \text{Jac}_x\psi^1$ in local coordinates is $\exp(J_0\text{Hess}_x(H))$. Nondegeneracy says that $\text{Hess}_x(H)$ does not have eigenvalues in $2\pi\mathbb{Z}$ and hence by 7.2.1, we can calculate the Maslov index of $x$ is $\text{Ind}(\text{Hess}_x(H)) - n$, i.e.

$$\text{Ind}_H(x) = \mu(x) + n.$$

Therefore, the two complexes have the same groups up to a grading shift.

Thus, it remains to consider the differentials of the two complexes. i.e. we need to

- Define the differentials.
- Show they coincide.

To define the differential of the Morse complex, we need a vector field $X$ from the Morse function $H$ that satisfies the Smale condition.

To show they coincide we need to relate trajectories

$$\frac{du}{ds} + X(u) = 0$$

and solutions

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \text{grad}H(u) = 0.$$
Thus, we want $X$ to be the gradient of the function $H$ defined by the metric defined by the metric defined using the almost complex structure $J$ that is compatible with $\omega$. We prove:

**Theorem 1.2.** Let $H$ be a Morse function on a symplectic manifold $(W, \omega)$, then there exists a dense set $J_{\text{reg}}(H)$ of almost complex structures compatible with $\omega$ such that $(H, -JX_H)$ is Morse-Smale.

We break the proof of this theorem into two steps.

First, we study a general Morse function $f$ and a vector field $X$. We linearize the equation $\frac{du}{ds} + X(u) = 0$ along one of its solutions and study $L_uY = 0$. Then we prove:

**Theorem 1.3.** Let $f$ be a Morse function and $U$ be a trajectory joining critical points $x$ and $y$. Then $L_u$ is a Fredholm operator whose index is the difference of the Morse indices of $x$ and $y$.

**Corollary 1.4.** For a nondegenerate Hamiltonian $H$ and a trajectory $u$ of $-JX_H$, $(dF)_u$ and $L_u$ have the same index.

**Proof.** Due to the relation between Morse and Maslov indices mentioned above.

We also prove:

**Theorem 1.5.** The vector field $X$ is Morse-Smale if and only if the operators $L_u$ are surjective.

The second step is we now fix a specific Hamiltonian $H$ and prove an analog of the transversality of Chapter VIII. We will then compare trajectories and solution to Floer’s equation.

**Remark 1.6.** It is clear the Floer solutions that do not depend on $t$ are exactly trajectories of the vector field $X = -\text{grad}H$.

The remark above suggests the following (true) Proposition.

**Proposition 1.7.** If $H$ is $C^2$ small, then

$$\text{Ker}(dF)_u = \text{Ker}L_u$$

for all Morse trajectories $u$. 

3
This proposition implies that elements in the kernel of \((d\mathcal{F})u\) do not depend on \(t\) (because things that \(L_u\) takes in doesn’t depend on \(t\)). The equality of indices (from the theorem above) and the characterization of Morse-Smale vector fields (corollary above) clearly implies:

**Corollary 1.8.** The Fredholm operator \((d\mathcal{F})u\) is surjective along all trajectories of \(-\text{grad}H\).

Let \(H_k = \frac{H}{k}\). Then we get:

**Proposition 1.9.** If \(k\) is sufficiently large, then the solutions of Floer’s equation for \(H_k\) joining critical points \(x\) and \(y\) with indices satisfying

\[
\text{Ind}_{H_k}(x) - \text{Ind}_{H_k}(y) \leq 2
\]

are all independent of \(t\).

We conclude the for \(H = \frac{H}{k}\) for large \(k\) and for \(J \in \mathcal{J}_{\text{reg}}\) (from 1.2 the Floer trajectories associated to \((H, J)\) joining two critical points \(x\) and \(y\) (also critical points of \(H\)), with \(\text{Ind}(x) - \text{Ind}(y) \leq 2\) are exactly trajectories of the Morse-Smale vector field \(X = -JX_H\). Thus, the linearized Floer operator along trajectories is surjective. This regularity says that \(\mathcal{M}^{(H, J)}(x, y)\) is a manifold and this allows us to define the Floer complex (and prove transversality).

**Remark 1.10.** We only establish transversality for \(\text{Ind}(x) - \text{Ind}(y) \leq 2\) (not in general as in Chapter VIII). However, this is enough to construct the Floer complex \(CF_*(H, J)\). We see that it coincides with \(CM_{s+n}(H, J)\). Thus, it remains to prove the theorems stated in this section.

## 2 Linearization of the flow of pseudo-gradient field. Proof of 1.3

Consider a Morse function \(f\) on a manifold \(V\) and let \(X\) be a pseudo-gradient field. Trajectories of the vector field are solutions to

\[
\frac{du}{ds} + X(u(s)) = 0.
\]

For simplicity, suppose that \(V\) is embedded in \(\mathbb{R}^m\) and think of \(u \mapsto X(u)\) as a map with values in \(\mathbb{R}^m\).
Fix a metric \( g \) on \( V \) so that \( X \) is the gradient of \( f \) with respect to \( g \). Let

\[
\mathcal{M} = \{ u : \mathbb{R} \to V | \frac{du}{ds} + \text{grad}f = 0 \text{ and } \int_{\mathbb{R}} \left\| \frac{du}{ds} \right\|^2 ds < \infty \}.
\]

The space of finite energy solutions is compact (recall that if \( V \) is compact, all solutions have finite energy). \( \mathcal{M} = \bigcup_{x, y \in \text{Crit}(f)} \mathcal{M}(x, y) \).

### 2.1 Linearization of the equation

Fix a solution \( u \) joining critical points \( x \) and \( y \). Fix a trivialization \( (Z_1, \ldots, Z_n) \) (orthonormal with respect to \( g \)) of \( TV \) in a neighborhood of the image of \( u \). Using a Morse chart, pick \( Z_i \) such that \( Z_i(s) \) does not depend on \( s \) in a neighborhood of \( x \) and \( y \), i.e. \( \frac{dz_i}{ds}(s) = 0 \) for \( |s| \) large. For \( Y = (y_1, \ldots, y_n) \in \mathbb{R}^n \), let \( \tilde{Y}(s) = \sum_{i=1}^n y_i(s)Z_i(u(s)) \).

Recall that linearizing the Hamiltonian differential equation

\[
\dot{x}(t) = X(x(t))
\]
gives

\[
\frac{dY}{dt} = (dX)_x(t)Y.
\]

Therefore, linearizing the equation

\[
\frac{du}{ds} + X(u(s)) = 0
\]
gives

\[
\frac{d\tilde{Y}}{ds} + (dX)_{u(s)}(\tilde{Y}) = 0
\]

which we will write as

\[
\tilde{L}_u \tilde{Y} = 0
\]

where the unknown function \( s \mapsto \tilde{Y}(s) \) is a tangent vector to \( V \) along the solution \( u \).

In the given trivialization,

\[
\tilde{L}_u(\tilde{Y}) = \frac{d}{ds} \left( \sum y_i Z_i \right) + S_u(s) \left( \sum y_i Z_i \right) \quad \text{where} \quad S_u(s) = (dX)_{u(s)}
\]

\[
= \sum_{i=1}^n \left( \frac{dy_i}{ds} + \sum_{j=1}^n (a_{ji} + S_{ji})y_j \right) Z_i \quad \text{where} \quad \frac{dZ_i}{ds} = \sum a_{ij}Z_j \text{ and } S_u(s) \cdot Z_i = \sum S_{ij}Z_j
\]
Let $A_{ij} = a_{ij} + S_{ij}$ and let $L_u$ be the operator

$$L_u Y : W^{1,2}(\mathbb{R}; \mathbb{R}^n) \rightarrow L^2(\mathbb{R}; \mathbb{R}^n)$$

$$Y \mapsto \frac{dY}{ds} + A(s)Y$$

Thus, for $Y(s) \in \mathbb{R}^n$, we obtain the differential equation

$$L_u Y = \frac{dY}{ds} + AY = 0.$$  

By the hypothesis on $Z_i$. $a_{ij}(s) = 0$ for $|s|$ large. Thus, as $s \to \pm \infty$, $A$ tends to the Hessian of $f$ at $x$ or $y$. Specifically,

$$\lim_{s \to -\infty} A(S) = \text{Hess}_x(f), \quad \lim_{s \to \infty} A(S) = \text{Hess}_y(f).$$

**Remark 2.1.** The space of solutions to the above equation can interpreted as the tangent space to $\mathcal{M}(x, y)$ at $u$.

### 2.2 Exponential decay of solutions

First, note that if $Y$ is a solution of $L_u Y = 0$ in $W^{1,2}(\mathbb{R}; \mathbb{R}^n)$, then $Y$ is continuous and therefore, because it satisfies an (ordinary) differential equation, it is $C^1$ and so by (very elementary) bootstrapping, any solution in $W^{1,2}$ is $C^\infty$. In particular, solutions form a finite dimensional vector space.

We want to study what type of behavior we need at $\infty$ in order to be in $W^{1,2}$. Near a critical point, say $x$, we have a Morse chart. We can use this chart to choose a trivialization of $TV$ so that in this chart, the differential equation simplifies to

$$\frac{dY}{ds} = -AY$$

where $A$ is constant and we can further suppose that it is diagonal. So our system is

$$\frac{dy_i}{ds} = -\lambda_i y_i, 1 \leq i \leq n.$$  

This gives solutions of the form

$$y_i(s) = y_i e^{-\lambda_i s}.$$  

It is clear that for a vector to be in $W^{1,2}(\mathbb{R}; \mathbb{R}^n)$, it must tend to 0 as $s \to$. This calculation show that the converse is true.
2.3 Fredholm property

**Proposition 2.2.** If $u$ is a gradient trajectory joining two critical points $x$ and $y$, then $L_u$ is Fredholm.

To prove this, we want to get an estimate like

$$
\| Y \| \leq C (\| L_u Y \| + \| K Y \|)
$$

like we did in §8.7. In this situation, we prove:

**Proposition 2.3.** For $T > 0$ sufficiently large, there exists $C > 0$ such that for a solution $u$, we have

$$
\| Y \|_{W^{1,2}} \leq C \left( \| L_u Y \|_{L^2} + \| Y \|_{L^2([-T,T])} \right)
$$

Thus proposition is exactly what we want because the inclusion/restriction

$$
W^{1,2}(\mathbb{R}; \mathbb{R}^n) \hookrightarrow L^2([-M,M]; \mathbb{R}^n)
$$

is compact.

To prove this proposition, we need two lemmas.

**Lemma 2.4.** Let $B$ be an invertible matrix. Then there exists $C_1 > 0$ such that all $Y \in W^{1,2}(\mathbb{R}; \mathbb{R}^n)$ satisfy

$$
\| Y \|_{W^{1,2}} \leq C_1 \left\| \frac{dY}{ds} + BY \right\|_{L^2}
$$

• This is the analog what we prove in §8.7 (8.7.3) where we proved

$$
D = \bar{\partial} + S(t) : W^{1,p}(\mathbb{R} \times S^1; \mathbb{R}^{2n}) \to L^p(\mathbb{R} \times S^1; \mathbb{R}^{2n})
$$

is bijective for $p > 1$.

• We will apply this lemma for $B$ the symmetric matrix $\text{Hess}_x(f), \text{Hess}_y(f)$ (invertible because our function is Morse).

• This property is clearly stable under small perturbations, so we obtain that if $Y(s) = 0$ for $-M \leq s \leq M$ and $M$ is sufficiently large,

$$
\| Y \|_{W^{1,2}}^2 = \int_{-\infty}^{\infty} \left( \| Y \|^2 + \left\| \frac{dY}{ds} \right\|^2 \right) ds \leq C_2 \int_{-\infty}^{\infty} \left\| \frac{dY}{ds} + AY \right\|^2 ds
$$

(perturb $B$ to $A$)

Thus, we can calculate this integral for arbitrary $Y$ on $(-\infty, M] \cup [M, \infty)$. The next lemma deals with it on $[-M, M]$. 

7
Lemma 2.5. There exists a $C_3 > 0$ such that

$$
\int_{-M}^{M} \left( \|Y\|^2 + \left\| \frac{dY}{ds} \right\|^2 \right) ds \leq C_2 \int_{-M}^{M} \left( \|Y\|^2 + \left\| \frac{dY}{ds} + AY \right\|^2 \right) ds
$$

Proof of 2.3 assuming lemmas. Pretty straightforward from here. Use a bump function that is 1 on $[-M, M]$ and 0 outside $[-M - 1, M + 1]$. Then use the two lemmas to deal with each piece. Very similar to what we did at the end of §8.7.

Proof of 2.4. We study the differential equation

$$
\frac{dY}{ds} + BY = Z
$$

via the method of variation of parameters. We look for a solution of the form

$$
Y(s) = e^{-Bs}Y_0(0) + e^{-Bs} \int_0^s e^{B\sigma} Z(\sigma) d\sigma.
$$

We want to show that when $Z \in L^2$, we have $Y \in W^{1,2}$. Specifically,

$$
\|Y\|_{W^{1,2}} \leq C_1 \|Z\|_{L^2}.
$$

We use the Fourier transform

$$
\hat{Y}(v) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ivs} Y(s) ds.
$$

If $Z = \frac{dY}{ds} + BY$, then by integrating by parts,

$$
\hat{Z}(v) = (vI + B)\hat{Y}(v).
$$

$B$ is real, so if we choose a purely imaginary $v = iu$ for $u \in \mathbb{R}$, then

$$
\|vI + B\|^2 = u^2 + \|B\|^2
$$

Thus,

$$
\|\hat{Z}(iu)\|^2 = \|iuI + B\|^2 \|\hat{Y}(iu)\|^2 \geq (u^2 + C_0^2) \|\hat{Y}(iu)\|^2 \text{ because } B \text{ is invertible.}
$$
This implies

\[(1 + u^2)\|\hat{Y}(iu)\|^2 \leq C_1(C_0^2 + u^2)\|\hat{Y}(iu)\|^2 \text{ where } C_1 = \max(1, \frac{1}{C_0^2}) \]

\[\leq C_1\|\hat{Z}(iu)\|^2\]

Integrating with respect to \(\mathbb{R}\) gives the desired inequality. \(\square\)

Proof of 2.5. We first develop an easy inequality. Consider

\[(u + 2v)^2 = u^2 + 4uv + 4v^2 \geq 0\]

\[\Rightarrow \frac{1}{2}(u + 2v)^2 = \frac{1}{2}u^2 + 2uv + 2v^2 \geq 0\]

\[\Rightarrow u^2 + 2uv + v^2 \geq \frac{1}{2}u^2 - v^2\]

\[\Rightarrow (u + v)^2 \geq \frac{1}{2}u^2 - v^2\]

We then conclude that

\[\|u + v\|^2 \geq \frac{1}{2}\|u\|^2 - \|v\|^2\]

Using this, inequality,

\[\int_{-M}^{M} \|L_u Y\|^2 ds = \int_{-M}^{M} \left(\frac{dY}{ds} + AY\right)^2 ds\]

\[\geq \int_{-M}^{M} \left(\frac{1}{2}\left\|\frac{dY}{ds}\right\|^2 - \|AY\|^2\right) ds\]

\[\geq \frac{1}{2}\int_{-M}^{M} \left\|\frac{dY}{ds}\right\|^2 ds - C\int_{-M}^{M} \|Y\|^2 ds\]

This implies that

\[\frac{1}{2}\int_{-M}^{M} \left\|\frac{dY}{ds}\right\|^2 ds \leq C\int_{-M}^{M} \|Y\|^2 ds + \int_{-M}^{M} \|L_u Y\|^2 ds\]

Adding \(\int_{-M}^{M} \|Y\|^2 ds\) to both sides clearly gives the desired inequality. \(\square\)

Therefore, we have prove 2.3. Thus, we conclude the \(L_u\) has a finite dimensional kernel and a closed image. We now need to deal with the cokernel. This is accomplished by the next lemma.
Lemma 2.6. Let

\[ L^*_u : W^{1,2}(\mathbb{R}; \mathbb{R}^n) \to L^2(\mathbb{R}; \mathbb{R}^n) \]

\[ L^*_u Z \mapsto -\frac{dZ}{ds} + A^T Z \]

Then the cokernel of \( L_u \) is the kernel of \( L^*_u \).

Proof. It is obvious that

\[ \int_{-\infty}^{\infty} \langle L_u Y, Z \rangle ds = \int_{-\infty}^{\infty} \langle Y, L^*_u(Z) \rangle ds. \]

If \( Z \in \text{Coker}L_u \), then \( L^*_u Z = 0 \) in the sense of distributions. Therefore, by the domain of definition of \( L^*_u \), \( \frac{dZ}{ds} \in L^2 \) and hence \( Z \in W^{1,2} \). Therefore, \( Z \in \text{Ker}L_u \), so \( \text{Coker}L_u \subset \text{Ker}L^*_u \). The inverse inclusion is obvious.

Proof of 2.2. We can now apply 2.3 to \( L^*_u \) to get that \( \text{Coker}L_u \) is finite dimensional. This proves 2.2.

2.4 Calculation of the index

Proposition 2.7. The Fredholm index of \( L_u \) is \( \text{Ind}(x) - \text{Ind}(y) \).

Proof. Recall that solutions of \( L_u Y = 0 \) in \( W^{1,2} \) tend to 0 exponentially as \( s \to \pm \infty \) (this is key in this proof). Define a map

\[ \Phi(\sigma,s) : T_{u(\sigma)}V \to T_{u(s)}V \]

where \( \Phi(\sigma,s) \) tends \( \tilde{Y} \in T_{u(\sigma)}V \) to \( Y(s) \) where \( Y(\tau) \in T_{u(\tau)}V \) is the unique solution to \( L_u Y(\tau) = 0 \) such that \( Y(\sigma) = \tilde{Y} \). Define

\[ E^u(\sigma) := \left\{ \tilde{Y} \in T_{u(\sigma)}V \mid \lim_{s\to-\infty} \Phi(\sigma,s)\tilde{Y} = 0 \right\} \]

\[ E^s(\sigma) := \left\{ \tilde{Y} \in T_{u(\sigma)}V \mid \lim_{s\to\infty} \Phi(\sigma,s)\tilde{Y} = 0 \right\} \]

Note that

\[ E^u(\sigma) = T_{u(\sigma)}W^u(x) \]

\[ E^s(\sigma) = T_{u(\sigma)}W^s(y) \]
If \( \tilde{Y} \in E^u(\sigma) \cap E^s(\sigma) \), then the solution to \( L_u Y(\tau) = 0 \) such that \( Y(\sigma) = \tilde{Y} \) is in \( W^{1,2}(\mathbb{R}; \mathbb{R}^n) \) (by exponential decay). Conversely,

\[
\text{Ker} L_u \cong E^u(\sigma) \cap E^s(\sigma)
\]

so

\[
\text{Ker} L_u \cong T_u(\sigma) W^u(\sigma) \cap T_u(\sigma) W^s(\sigma).
\]

Now we study \( \text{Coker} L_u \). Define

\[
\Psi_{(\sigma,s)} = \Phi_{(s,\sigma)}^T : T_u(\sigma)V \to T_u(\sigma)V
\]

Choose \( Z_0 \in T_u(\sigma)V \) and define \( Z_0(s) = \Psi_{(\sigma,s)} Z_0 \). Note that since \( \Psi_{(\sigma,s)} = \text{Id} \), we have \( Z_0(\sigma) = Z_0 \). Do this for \( Y_0 \in T_u(\sigma)V \). Then we have

\[
\langle \Psi_{(\sigma,s)} Z_0, \Psi_{(\sigma,s)} Y_0 \rangle = \langle Z_0, \Phi_{(s,\sigma)} \Phi_{(\sigma,s)} Y_0 \rangle
\]

by definition of \( \Psi = \langle Z_0, Y_0 \rangle \) by definition of \( \Phi \).

Another way to write this is

\[
\langle Z_0(s), Y_0(s) \rangle = \langle Z_0, Y_0 \rangle.
\]

Differentiating this equation yields

\[
\left\langle \frac{dZ_0}{ds}(s), Y_0(s) \right\rangle + \left\langle Z_0(s), \frac{dY_0}{ds}(s) \right\rangle = 0.
\]

Now recall that \( Y_0(s) = \Phi_{(\sigma,s)} Y_0 \) is a solution to \( L_u Y = 0 \), so \( \frac{dY_0}{ds}(s) = -AY_0(s) \). This implies

\[
\left\langle \frac{dZ_0}{ds}(s), Y_0(s) \right\rangle - \left\langle Z_0(s), AY_0(s) \right\rangle = 0
\]

\[
\Rightarrow \left\langle \frac{dZ_0}{ds}(s) - A^T Z_0(s), Y_0(s) \right\rangle = 0
\]

\[
\Rightarrow - \frac{dZ_0}{ds} + A^T Z_0 = 0
\]

Therefore,

\[
L_u^* Z_0(s) = 0 \text{ and } Z_0(\sigma) = Z_0.
\]

Since \( Z_0(s) \) was defined using \( \Psi \), we conclude that \( \Psi \) is to \( L_u^* \) as \( \Phi \) is to \( L_u \).
Now by the definition of adjoint, we conclude that if $Z(s)$ is a solution of $L^*uZ = 0$, then $\lim_{s \to -\infty} Z(s) = 0$ if and only if $Z(\sigma) \perp E^u(\sigma)$. Similarly, $\lim_{s \to \infty} Z(s) = 0$ if and only if $Z(\sigma) \perp E^s(\sigma)$. Therefore, due to exponential decay,

$$\text{Ker} L^*_u = ((T_{u(\sigma)}W^u(x) + T_{u(\sigma)}W^s(y))^\perp$$

Therefore,

$$\text{Ind}(L_u) = \dim \text{Ker} L_u - \dim \text{Ker} L^*_u$$

$$= (\dim(T_{u(\sigma)}W^u(x) + T_{u(\sigma)}W^s(y))) - (\dim((T_{u(\sigma)}W^u(x) + T_{u(\sigma)}W^s(y)))$$

$$= (\dim W^u(x) + \dim W^s(y) - \dim(T_{u(\sigma)}W^u(x) + T_{u(\sigma)}W^s(y))$$

$$= (\dim W^u(x) + \dim W^s(y) - n)$$

$$= (\text{Ind}(x)) + (n - \text{Ind}(y)) - n$$

$$= \text{Ind}(x) - \text{Ind}(y)$$

\[ \square \]

2.5 Smale Condition

We want to prove 1.5, i.e. that $X$ satisfies the Smale condition if and only if $L_u$ is surjective.

**Proof.** By the previous subsection, $L_u$ is surjective if and only if $L^*_u$ is injective if and only if

$$T_{u(\sigma)}W^u(x) + T_{u(\sigma)}W^s(y) = T_{u(\sigma)}V$$

if and only if

$$W^u(x) \pitchfork W^s(y).$$

\[ \square \]

3 Proof of regularity theorem 1.2

We now fix an $H$ and consider two critical points $x, y$. Define

$$\mathcal{Z}(x, y) := \{(u, J)| J \text{ is } \omega\text{-compatible and } u \text{ is a trajectory of } -JX_H \text{ joining } x \text{ and } y\}$$

Let

$$\mathcal{J}_c(\omega) = \{\omega - \text{compatible almost complex structures}\}$$
Note that
\[ T_J \mathcal{J}_c(\omega) = \{ S \in \text{End}(TW) | JS + SJ = 0 \text{ and } \omega(S\xi, \eta) + \omega(\xi, S\eta) = 0 \}. \]
Fix \( S \in T_J \mathcal{J}_c(\omega) \) and define
\[ J_t = J \exp(-tJS) \]

**Lemma 3.1.** \( J_t = \exp(tJS)J \), i.e.
\[ J \exp(-tJS) = \exp(tJS)J \]

**Proof.** Do this locally, so that the exponential is just the usual exponential of matrices. Thus, it suffices to prove that
\[ J(-tJS)^n = (tJS)^n J \]
This is easily do by induction and remembering that \( J \) and \( S \) anticommute.

**Lemma 3.2.** For small \( t \), \( J_t \in \mathcal{J}_c(\omega) \)

**Proof.** We first prove that \( J_t \) is an almost complex structure. By the lemma above,
\[ J_t = \exp(tJS)JJ \exp(-tJS) = -\text{Id}. \]

Next, we show that \( J_t \) is \( \omega \)-compatible. To do this, it suffices to show that \( \frac{d}{dt} \omega(J_t \xi, J_t \eta) = 0 \)

\[
\frac{d}{dt} \omega(J_t \xi, J_t \eta) = \frac{d}{dt} \omega(\exp(-tJS)\xi, \exp(-tJS)\eta) \\
= \omega(-JS \exp(-tJS)\xi, \exp(-tJS)\eta) + \omega(\exp(-tJS)\xi, -JS \exp(-tJS)\eta) \\
= 0 \text{ by the skew-symmetry of } \omega
\]

**3.1 Proof of 1.2**

As in §8.3, we use the \( C_\varepsilon^\infty \) norm on the space of perturbations of \( J \).

\[ \|S\|_\varepsilon = \sum_{k=0}^{\infty} \varepsilon_k \|S\|_{C^k} \]
(for a sequence \( \varepsilon = (\varepsilon_k) \) sufficiently decreasing). As in §8.3, we have
Proposition 3.3. For a well chosen $\varepsilon$, the space $C_\varepsilon^\infty(J)$ of sections $S$ of $\text{End}(TW)$ such that

\[
\begin{align*}
JS + JS &= 0 \\
\omega(S\xi, \eta) + \omega(\xi, S\eta) &= 0 \quad \text{and } \|S\|_\varepsilon < \infty
\end{align*}
\]

is a separable Banach space and is dense in the space of $L^2$ sections with fibre in $TJ_\varepsilon(\omega)$.

The proof of this is the same as before.

Fix an almost complex structure $J_0 \in J_\varepsilon(\omega)$ and $\delta > 0$. Define the ball

\[
J_0(\delta) := \{J_0 \exp(-J_0 S)|S \in C_\varepsilon^\infty \text{ and } \|S\|_\varepsilon < \delta\}
\]

and

\[
Z_0(x, y) := \{(u, J) \in Z(x, y) | J \in J_0(\delta)\}
\]

(this depends on $\delta$, but this is suppressed from the notation).

Proposition 3.4. The space $Z_0(x, y)$ is a Banach manifold.

The proof of this proposition is the heart of this section. As in §8.5, we exhibit $Z_0(x, y)$ as the zero set of a section of a bundle. However, the analysis in this case in much easier because we only have one variable and we can stay in $W^{1,2}$ and $L^2$. Define

\[
P^{1,2}(x, y) := \{u \in W^{1,2}(\mathbb{R}; W) | \lim_{s \to -\infty} u(s) = x \text{ and } \lim_{s \to \infty} u(s) = y\}
\]

(here the limits make sense because elements of $W^{1,2}$ are continuous). Next, define

\[
E := \{(u, J, Y) | u \in P^{1,2}(x, y), J \in J_0(\delta), Y \text{ is an } L^2 \text{ vector field along } u\}
\]

$E$ fibres over $P^{1,2}(x) \times J_0(\delta)$ with fibre over $(u, J)$ equal to $L^2$ vector fields along $u$. We will identify the set of $L^2$ vector fields along $u$ with $L^2(\mathbb{R}; \mathbb{R}^{2n})$ via our trivialization.

Consider

\[
G : P^{1,2}(x, y) \times J_0(\delta) \to E
\]

\[
(u, J) \mapsto \left( u, \frac{du}{ds} + \text{grad}_u H \right) = \left( u, \frac{du}{ds} - JX_H \right)
\]
$J_0(x,y)$ is exactly the zero set of $G$. Thus, we want to show that $G$ is transverse to the zero section. So we must calculate

$$(dG)_{(u,J_0)} : W^{1,2}(\mathbb{R}; \mathbb{R}^{2n}) \to L^2(\mathbb{R}; \mathbb{R}^{2n})$$

We have already linearized $G$ for a constant $J$ is §2.1. We also linearized $F$, which was the same thing, but with an extra $t$ variable. A similar calculation shows that this linearization composed with projection onto the fibre is

$$\Gamma = \pi \circ (dG)_{(u,J_0)}(Y, S) = \left( \frac{dY}{ds} - J_0[(X_H)u, Y] \right) - S(X_H)u$$

$$= L_u(Y) - S(X_H)u$$

where $L_u$ is the operator we have previous defined.

We proved that $L_u$ is Fredholm, so it has a closed image and a finite dimensional cokernel. It is easy to show (we essentially did it in the first part of lemma 8.5.1) that $\text{Im} \Gamma$ is also closed of finite codimension. We want to show that $\Gamma$ is surjective at points of $Z_0$. To do this, it suffices to prove the following lemma (reasoning after the statement).

**Lemma 3.5.** Only 0 is orthogonal to the image of $\Gamma$ in $L^2(\mathbb{R}; \mathbb{R}^{2n})$.

It suffices to prove this because things orthogonal to $\text{Im} \Gamma \subset L^2$ are in $L^2$ (because the dual of $L^2$ is $L^2$).

**Proof.** Let $Z$ be an orthogonal element, i.e. $Z \in L^2(\mathbb{R}; \mathbb{R}^{2n})$ and

$$\int_{-\infty}^{\infty} \langle Z(s), \Gamma(Y, S) \rangle ds = 0$$

for all $Y, S$.

This implies

$$\int_{-\infty}^{\infty} \left\langle Z(s), \frac{dy}{ds} - J_0[(X_H,Y) \right] ds - \int_{-\infty}^{\infty} \langle Z(s), S(X_H) \rangle ds = 0$$

for all $Y, S$. So

$$\int_{-\infty}^{\infty} \left\langle Z(s), \frac{dy}{ds} - J_0[(X_H,Y) \right] ds = 0$$

for all vector fields $Y$ along $u$ and

$$\int_{-\infty}^{\infty} \langle Z(s), S(X_H) \rangle ds = 0$$
for all sections $S$.

The first equality says that $\langle Z, L_u(Y) \rangle = 0$ for all $Y$ and hence $L_u^* Z = 0$. Thus,

$$-\frac{dZ}{ds} + B(s)Z = 0.$$  

A solution to a differential equation of this type is guaranteed to be $C^\infty$. Thus, we can work with $C^\infty Z$. Next,

$$\langle Z, S(X_H) \rangle = \langle Z, -SJ du ds \rangle = -\langle SZ, J du ds \rangle$$

since $S$ is symmetric. Thus,

$$\int_{-\infty}^{\infty} \langle SZ, J du ds \rangle = 0 \quad (3.1)$$

We now show that $Z$ is 0. Suppose for contradiction that $Z(s_0) \neq 0$ for some $s_0$. Pick a unitary trivialization near $x_0 = u(s_0)$ in which $J_0$ is standard. The $S$ is symmetric (depending on the point) such that $SJ_0 = -J_0 S$. Consider the vector $J_0 du ds (s_0)$. We want to construct a matrix $S_0$ such that

$$\langle Z(s_0), S_0 J_0 du ds (s_0) \rangle \neq 0$$

(this would contradict 3.1). We use an easy linear algebra lemma:

**Lemma 3.6.** Let $U, V$ be nonzero vectors in $\mathbb{C}^n = \mathbb{R}^{2n}$. Then there exists a $\mathbb{C}$-linear, symmetric, real matrix $S$ such that the Euclidean inner product $\langle U, SV \rangle$ is nonzero.

Using this lemma, there is a map in $C^\infty$ with support in a neighborhood that takes value $s_0$ at $x_0$. This contradicts 3.1. That completes the proof of 3.5 and hence 3.4. □

**Proof of lemma.** $U$ and $V$ generate a $\mathbb{C}$-vector space of dimension 2, so it suffices to prove the lemma in $\mathbb{C}^2$. We can assume that $U$ is the first vector in a basis. Then $S$ should have the form

$$S = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

where $A$ is symmetric and $B$ is antisymmetric

so the first column of $S$ has no constraints, so we make it $V$. Thus, $S(U) = V$ and

$$\langle U, SV \rangle = \langle SU, V \rangle = \langle V, V \rangle = \|V\|^2 \neq 0$$

□
We now complete the proof of transversality. Consider the projection
\[ \pi : Z_0(x, y) \to J_0(\delta) \]
\( (u, J) \mapsto J \)

By Sard-Smale, there is \( J_{\text{reg}} \subset J_0(\delta) \) open and dense regular values of \( \pi \).

**Lemma 3.7.** If \( J \in J_{\text{reg}} \) and \( u \) is a trajectory of \(-JX_H\) joining \( x \) and \( y \), then \( L_u \) is surjective.

**Proof.** Suppose for contradiction that \( L_u : W^{1,2}(\mathbb{R}; \mathbb{R}^{2n}) \to L^2(\mathbb{R}; \mathbb{R}^{2n}) \) is not surjective. Then there is a \( Z \in L^2(\mathbb{R}; \mathbb{R}^{2n}) \) orthogonal to \( \text{Im}L_u \). By hypothesis, \( (d\pi)_{(u, J)} \) is surjective.

The tangent space to \( Z_0(x, y) \) is exactly the kernel of \( \Gamma \). So
\[ T_{(u, J)}Z_0(x, y) = \{ (Y, S) \in W^{1,2}(\mathbb{R}; \mathbb{R}^{2n}) \times T_JJ_0(\delta)|L_u(Y) - S(X_H)u = 0 \}. \]

Since \( \langle L_uY, Z \rangle = 0 \) for all \( Y \in W^{1,2}(\mathbb{R}; \mathbb{R}^{2n}) \), surjectivity of \( (d\pi)_{(u, J)} \) implies that
\[ \langle S(X_H)u, Z \rangle = 0 \]
for all \( S \in T_JJ_0(\delta) \). We proves the the proof of 3.5 that this implies that \( Z = 0 \). This is a contradiction. \( \square \)

Therefore, we can take the intersections of the set \( J_{\text{reg}} \) for different pairs of critical points of \( H \) and we get the set \( J_{\text{reg}}(H) \). This prove 1.2.

4 The Morse and Floer trajectories coincide

4.1 Trajectories that are independent of \( t \) are regular

In this section we will prove 1.7 and hence 1.8.

**Proof of 1.7.** Clearly \( \text{Ker}L_u \subset \text{Ker}(d\mathcal{F})_u \) because if \( Y(s) \) satisfies
\[ \frac{dY}{ds} + S(s)Y = 0, \]
then it clearly satisfies
\[ (d\mathcal{F})_u(Y(s)) = \frac{\partial Y}{\partial s} + J_0 \frac{\partial Y}{\partial t} + S(s)Y = 0. \]
Now we show that the reverse inclusion is true. Suppose that \( Y \in \text{Ker}((dF)_u) \).

\[
\int_0^1 J_0 \frac{\partial Y}{\partial t} dt = 0
\]

because \( Y(s, t) \) is periodic with period 1 in \( t \). Thus,

\[
\int_0^1 Y(s, t) \in \text{Ker}(L_u).
\]

Let \( W(s) = \int_0^1 Y(s, t) \). Now we can assume that \( \int_0^1 Y(s, t) dt = 0 \) because otherwise you can replace \( Y(s, t) \) with \( Y(s, t) - W(s) \) (which has mean value 0). This is because \( Y - W \in \text{Ker}(dF)_u \) and \( W \in \text{Ker}(L_u) \), so if we can prove that \( Y - W \in \text{Ker}(L_u) \), then this implies that \( Y \in \text{Ker}(L_u) \).

We now need the following lemma.

**Lemma 4.1.** Let \( f : [0, 1] \to \mathbb{R} \) be differentiable and have mean value 0. Then for any \( p \),

\[
\int_0^1 \|f(t)\|^p dt \leq \int_0^1 \|f'(t)\|^p dt
\]

**Proof.**

\[
f(t_1) = f(t) + \int_t^{t_1} f'(\tau) d\tau, \text{ for } t, t_1 \in [0, 1]
\]

\[
\Rightarrow \int_0^1 f(t_1) dt = \int_0^1 f(t) dt + \int_0^1 \left( \int_t^{t_1} f'(\tau) d\tau \right) dt
\]

\[
\Rightarrow f(t_1) = \int_0^1 \int_t^{t_1} f'(\tau) d\tau dt
\]

\[
\Rightarrow \int_0^1 \|f(t_1)\|^p dt_1 \leq \int_0^1 \left( \int_0^1 \int_t^{t_1} \|f'(\tau)\|^p d\tau dt \right) dt_1
\]

\[
\leq \int_0^1 \int_0^1 \int_0^1 \|f'(\tau)\|^p d\tau dt_1 d_1 = \int_0^1 \|f'(\tau)\|^p d\tau
\]

\( \Box \)
We now apply this lemma to $f(t) = Y(s, t)$ and integrate with respect to $s$ to get:

$$\|Y\|_{L^2}^2 = \int_{-\infty}^{\infty} \int_0^1 \|Y(s, t)\|^2 dt ds \leq \int_{-\infty}^{\infty} \int_0^1 \left\| \frac{\partial Y}{\partial t}(s, t) \right\|^2 dt ds = \left\| \frac{\partial Y}{\partial t} \right\|_{L^2}^2$$

We always have:

$$\|\text{grad} Y\|_{L^2}^2 = \left\| \frac{\partial Y}{\partial s} \right\|_{L^2}^2 + \left\| \frac{\partial Y}{\partial t} \right\|_{L^2}^2$$

$$= \langle \frac{\partial Y}{\partial s}, \frac{\partial Y}{\partial s} \rangle + \langle \frac{\partial Y}{\partial t}, \frac{\partial Y}{\partial t} \rangle$$

$$= -\langle Y, \Delta Y \rangle$$

$$- \langle Y, \left( \frac{\partial}{\partial s} - J \frac{\partial}{\partial t} \right) \left( \frac{\partial}{\partial s} + J \frac{\partial}{\partial t} \right) Y \rangle$$

$$= \left\| \frac{\partial Y}{\partial s} + J \frac{\partial Y}{\partial t} \right\|_{L^2}^2$$

$$= \|S(s)Y\|_{L^2}^2$$ because $Y$ is a solution

$$\leq \sup_s \|S(s)\|^2 \|Y\|_{L^2}^2$$ (this is the operator norm for $S$).

Finally,

$$\left\| \frac{\partial Y}{\partial t} \right\|_{L^2}^2 \leq \|\text{grad} Y\|_{L^2}^2 \leq \sup_s \|S(s)\|^2 \|Y\|_{L^2}^2 \leq \sup_s \|S(s)\|^2 \left\| \frac{\partial Y}{\partial t} \right\|_{L^2}^2$$

If $H$ is $C^2$ small, then $\sup_s \|S(s)\|^2$ is small. This implies that $\frac{\partial Y}{\partial t} = 0$.

Therefore, $Y$ is independent of $t$ and hence $Y \in \text{Ker}L_u$.  

\[\square\]

### 4.2 Trajectories are independent of $t$

**Proof of 1.3** Suppose not for contradiction. Then there exists a sequence $(n_k)$ tending to $\infty$ and solutions $(u_{n_k})$ (that do depend on $t$) for the Hamiltonian $H_{n_k} = \frac{H_{n_k}}{n_k}$. Thus, $u_{n_k}$ is a solution to the equation

$$\frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} + \frac{1}{n_k} \text{grad} H = 0.$$  

We can assume that $u_{n_k}$ connects critical points $x$ and $y$. Let

$$v_{n_k}(s, t) = u_{n_k}(n_k s, n_k t).$$
Then
\[
\frac{\partial v_{nk}}{\partial s} + J \frac{\partial v_{nk}}{\partial s} + \text{grad}H = n_{k} \frac{\partial u_{nk}}{\partial s} + n_{k} J \frac{\partial u_{nk}}{\partial t} + \text{grad}H = 0.
\]

Thus, \(v_{nk}\) is a solution of Floer's equation for the original \(H\) and \(v_{nk}\) is periodic with period \(\frac{1}{n_{k}}\) (because \(u_{nk}\) is periodic with period 1). Thus, \(v_{nk}\) can also be considered to have period 1 and hence \(v_{nk} \in M(x, y, H)\).

**Case I:** \(\text{Ind}(x) - \text{Ind}(y) = 1 = \mu(x) - \mu(y)\)

Due to theorem 9.16 (convergence to broken trajectories), we can extract a subsequence of \((v_{nk})\) that tends to some \(v \in M(x, y, H)\) (in general you will have to act by a sequence \(\sigma_{nk}\), but we can just assume that this is already done).

**Claim 4.2.** \(v\) does not depend on \(t\).

**Finishing proof assuming claim:** Then \((dF)_{u}\) is surjective by 1.7. \(v\) must therefore be in a component of dimension 1 of \(M(x, y, H)\), i.e., it is an isolated point in \(L(x, y)\). Therefore, for large \(k\), we have
\[
v_{nk}(s, t) = v(s + \sigma_{k}, t) \quad \text{(it must be the isolated point for large } k) \\
v_{nk}(s, t) = v(s + \sigma_{k})
\]

Thus, \(v_{nk}\) does not depend on \(t\), which is a contradiction.

**Proof of claim.** We use the fact that \(v\) is the limit of a sequence with period tending to 0. Let \(r \in \mathbb{R}\) and fix \(s \in \mathbb{R}, t \in \mathbb{R}\). Since \(v_{nk}\) is periodic with period \(\frac{1}{n_{k}}\),
\[
v_{nk}(s, t) = v_{nk}\left(s, t + \frac{\lfloor rn_{k}\rfloor}{n_{k}}\right).
\]
Letting \(k \to \infty\), we see that \(\frac{\lfloor rn_{k}\rfloor}{n_{k}}\) and \(v_{nk}(s, t) \to v(s, t)\). Thus, we get \(v(s, t) = v(s, t + r)\). \(r\) was arbitrary, so we get our contradiction.

This completes the proof of Case I.

**Case II:** \(\text{Ind}(x) - \text{Ind}(y) = 2 = \mu(x) - \mu(y)\).

Here, theorem 9.16 says that there is a subsequence of \((v_{nk}(s + s_{k}, t))\) that tends to either a limit \(v \in M(x, y, H)\) or possibly a broken trajectory \((v, w) \in M(x, z) \times M(z, y)\).

**Case (a):** It tends to a limit \(v\).
\( v \) is independent of \( t \) using the same proof as in Case I. Thus, by \([1.7]\), \((dF)_v\) is surjective, and so by lemma 8.5.8 (last lemma in the proof of transversality), \( M(x, y, H) \) is a submanifold of dimension 2 of \( Z(x, y, J) \) in a neighborhood of \((v, H)\). Moreover, since \(-JX_H = \text{grad}H\) is Morse-Smale, the trajectories joining \( x \) and \( y \) form a manifold of dimension 2. But these trajectories are also in \( M(x, y, H) \) and do not depend on \( t \). Therefore, on a connected component, either all trajectories depend on \( t \) or none do. Thus, since \( v_{nk}(s + s_k) \in M(x, y, H) \) and since this sequence tends to \( v \) (not depending on \( t \)), it follows that \( v_{nk}(s + s_k, t) \) is independent of \( t \) for large \( k \). This is a contradiction.

**Case (b):** It tends to a broken trajectory \((v, w)\). i.e there exists sequences \((s_k^1), (s_k^2)\) such that

\[
\lim_{k \to \infty} v_{nk}(s + s_k^1, t) = v(s, t) \quad \text{and} \quad \lim_{k \to \infty} v_{nk}(s + s_k^2, t) = w(s, t)
\]

Again, \( v \) and \( w \) do not depend on \( t \) by the same proof. Thus, by \([1.7]\), \((dF)_v\) and \((dF)_w\) are surjective. By the gluing theorem (9.2.3), there exists an embedding

\[
\hat{\psi} : [\rho_0, \infty) \to L(x, y)
\]

such that

\[
\lim_{\rho \to \infty} \hat{\psi}(\rho) = (\hat{v}, \hat{w}) \in L(x, z) \times L(z, y).
\]

Next, \( v \) and \( w \) are also Morse trajectories of the Morse-Smale vector field \(-JX_H\). The gluing theorem in Morse theory (3.2.6) says that the broken trajectory \((\hat{u}, \hat{v})\) is a point on a manifold with boundary of dimension 1 formed by Morse trajectories joining \( x \) and \( y \). But there are also Floer trajectories, so we get a map

\[
\phi : [\rho_0, \infty)M(x, y, H)
\]

such that

\[
\hat{\phi} : [\rho_0, \infty) \to L(x, y)
\]

is an embedding. Additionally, when \( \rho \to \infty \), \( \hat{\phi} \to (\hat{v}, \hat{w}) \) (in the sense of convergence of broken trajectories floes in Morse theory). In particular, this means that

\[
\lim_{\rho \to \infty} \phi(\rho)(s^+_\rho) = v(s^+)
\]

21
where \((s_\rho^+)\) and \(s^+\) are points on the trajectories \(\phi(\rho)\) and \(v\) respectively outside a neighborhood of \(x\). This implies that in the \(C^{\infty}_{\text{loc}}\) topology,

\[
\lim_{\rho \to \infty} \phi(\rho)(s + s_\rho^+) = v(s + s^+).
\]

For convenience we write

\[
\phi(\rho)(s + s_\rho^+) = \Phi_s(\phi(\rho))(s + s_\rho^+) = v(s + s^+) = \Phi_s(v(s^+))
\]

where \(\Phi_s\) is the flow of \(-JX_H\).

The same reasoning also applies to \(w\). We also have that

\[
\lim_{\rho \to \infty} \hat{\phi}(\rho) = (\hat{v}, \hat{w})
\]

(in the sense of broken trajectories in Floer theory).

Thus, we have 3 different convergences to \((\hat{v}, \hat{w})\).

(1) \((v_{n_k})\) for \(k\) tending to \(\infty\).

(2) \(\hat{\psi}(\rho)\) for \(\rho \to \infty\). Gluing in Floer theory.

(3) \(\phi(\rho)\) for \(\rho \to \infty\). Gluing in Morse theory.

We now apply the uniqueness of gluing (9.2.6, second part). This implies that for large \(k\), \((v_{n_k})\) is in the image of \(\hat{\psi}\). Similarly, for large \(\rho\), \(\phi(\rho) \in \text{Im} \hat{\psi}\).
(say for $\rho \geq \rho_0$). Moreover, $\hat{\phi}([\rho, \infty))$ is an interval in $\mathcal{L}(x, y)$ (a manifold of dimension 1). Finally

$$\lim_{\rho \to \infty} \hat{\phi}(\rho) = \lim_{\rho \to \infty} \hat{\psi}(\rho) = (\hat{v}, \hat{w}) \notin \mathcal{L}(x, y),$$

so $\hat{\psi}(\rho) \in \text{Im} \hat{\phi}$ for large $\rho$. Thus, $\hat{v}_{n_k} \in \text{Im} \hat{\phi}$ for large $k$. This contradicts the fact that $v_{n_k}$ depending on $t$. This completes the proof of Case (b), Case II, and hence $1.9$. □

Therefore, we have shown that the Floer solutions connecting critical points of index difference $\leq 2$ are Morse trajectories and $(d\mathcal{F})_v$ is surjective along these trajectories. Therefore, we get the equalities of the desired complexes.