Vector fields

First we will consider vector fields on \( \mathbb{R}^n \) (my pictures will be \( n=2 \)). A vector field is an assignment to each point in \( \mathbb{R}^n \) a vector coming out of it (this is topology, so the vectors should vary continuously).

Example

You have probably already seen this in your calculus class in the context of slope fields (i.e. at each point \((x,y)\), you give a vector of slope \( \frac{dy}{dx}(x,y) \)). These slope fields are involved in solving differential equations. We will come back to the relationship of vector fields and differential equations.

Now vector fields on manifolds.

A vector field on a manifold assigns to each point of the manifold a vector that is tangent to the manifold.

Example

Examples:

1) 2) Velocity of fluid flow
3) Magnetic field
4) Wind
Basic Question: On a given surface, does there exist a vector field that is everywhere non-zero?

Answer: It depends on the topology of the surface.

Let's first consider the torus $T^2$.

Exercise: Can you find a nonvanishing vector field on $T^2$? Draw pictures.

One way to think about this is "can you comb a torus?"

Think of a torus with hairs at every point.

This isn't a vector field b/c the hairs aren't tangent yet.

To make it a vector field you need to "comb" the hairy torus.

What about $S^2$?

Exercise: Can you find a nonvanishing vector field on $S^2$?

Draw example (previous example 1 is an example with 2 zeros).
The (Hairy Ball Theorem): You cannot comb a hairy sphere.

We there is no nonvanishing vector field on $S^2$.

We will present two proofs of this. The first will be direct/straightforward, but will only apply to $S^2$. The second will be far more general.

**Pf 1:** Suppose we have a nonvanishing vector field on $S^2$. Think of this as giving a differential equation. i.e. at each point, I want to move in the direction of the vector field at that point (I want a curve through that point with derivative equal to the vector field)

You can do this for tiny time. This gives a map $f: S^2 \rightarrow S^2$. It is homotopic to the identity.

Because the vector field is nonvanishing, this map has no fixed points.

We will not homotope $f$ to the antipodal map, $f(x) \neq x$ so there is a unique great circle through $x$, $f(x)$, $-x$.

Our homotopy is to move from $f(x)$ to $-x$.

Thus, $\text{Id} \sim f \sim$ antipodal map.
We now prove that this cannot be the case we do this by proving that \( \text{deg}(\text{id}) \neq \text{deg}(\text{antipodal}) \)

Recall: \( \text{deg}(f) \) is defined to be the number of points in \( f^{-1}(y) \)
for a regular value \( y \) counted with signs depending on whether orientation is preserved \((+1)\) or reversed \((-1)\).

Clearly \( \text{deg}(\text{id}) = 1 \).

Now note that antipodal map \(((\text{rotation by } \pi) \circ \text{reflection})\)
Each value of the antipodal map has 2 preimages. We want to show orientation is reversed.
This is b/c reflection reverses orientation and rotation does not.
Therefore, \( \text{deg}(\text{antipodal}) = -1 \)

This is a contradiction.

Consequences of the Hairy Ball theorem

1) At some point on Earth, there is no wind.
So we have seen that $T^2$ has a nonvanishing vector field, $S^2$ does not and every picture we drew had at least 2 "problem points"

**Exercise**: What about genus 2 surface? 

What about genus 3?

We notice that the number of problem points is at least $K(1,1)$. This is not a coincidence. To make things more precise, we need some more background.

**Index of a vector field**

Suppose we have a vector field $V$ and an isolated zero of this vector field $x$. We want to define the index of $V$ at $x$, $\text{ind}_V(x)$. This is done as follows:

Look at a small circle around $x$. At each point $y$ on this circle, consider $\frac{V(y)}{||V(y)||}$. This is a unit vector, so we can consider its end point, which is a point on the unit circle.

Thus, this gives us a map $S^1 \to S^1$.

The index $\text{ind}_V(x)$ is the degree of this map.

Informally, we think of the index as "how many times the vector field wraps around the zero."
Exercise: Come up with index -2, 0, and 2 zeros at a vector field.

Theorem (Poincaré-Hopf Theorem): Let \( V \) be a vector field with isolated zeros on \( M \)

\[
\sum_{x \text{ zero of } V} i_V(x) = \chi(M)
\]

In particular, \( \sum_i i_V(x) \) is a topological invariant (independent of \( V \)).

Corollary: Hairy ball theorem

Because \( \chi(S^2) = 2 \), so this implies any vector field must have a zero.

Some goes for genus 2, 3,... surfaces.
Some for \( \mathbb{RP}^2 \), \( \mathbb{RP}^2 \# \cdots \# \mathbb{R}^2 \) any number other than 2.
We already constructed a nonvanishing vector field on $T^2$

Exercise: Construct a nonvanishing vector field on Klein bottle.

This answers our question for all closed surfaces.
This theorem also applies to nonclosed surfaces as long as the number of zeros is finite.

**Thm (Hopf):** If $\chi(M)$, $M$ connected
then there exists a nonvanishing vector field on $M$.

Exercise: Construct a nonvanishing vector field on the manifolds you know Euler characteristic $0$ (e.g. circle, annulus, Möbius band, any 3-manifold you know)

**Idea of Proof of Poincaré-Hopf**

**Step 1:** Prove that $\sum_x i_{\nu}(x)$ is independent of the vector field $\nu$.

**Step 2:** Construct a vector field $\nu$ with $\sum_x i_{\nu}(x) = \chi(M)$.
Do this by triangulating $\nu$ and on each $n$-triangle have a zero with index $n$. 