Classification Problems in Higher Dimensions

Recall: We defined surfaces as spaces that locally look like 2-space. We completely classified closed surfaces.

The higher-dimensional analog of a surface is called an $n$-manifold.

**Def:** A $n$-manifold is a topological space that locally looks like $\mathbb{R}^n$ annually

i.e. locally the space is homomorphic to $\mathbb{R}^n$.

**Note:** Surfaces are 2-mflds

**Question:** Is there a classification of $n$-mflds? i.e. can we classify $n$-mflds up to homeomorphism?

Overview of Classification of $n$-mflds up to homeo

$n=1$: Any connected 1-mfld is homeomorphic to the circle or $\mathbb{R}$

$n=2$: We have seen that every compact 2-mfld (surface) is homeomorphic to the sphere, connected sum of tori, or connected sum of $\mathbb{RP}^2$.

This has been known since early 20th century.

$n=3$: In 1970s, Thurston conjectured that any compact connected 3-mflds can be split into pieces that fall into 8 groups. This is known as Thurston's geometrization conjecture. It was proved in 2003 by Perelman.

$n \geq 4$: The classification problem in dimensions $\geq 4$ is impossible.

This doesn't mean very difficult, it means that we can prove that the problem can't be solved.
Given the lack of classification in $d \geq 4$, are we left hopeless? Not completely!

**Poincaré Conjecture**

We are left asking slightly simpler questions. In 1904, Poincaré asked "can we recognize spheres". More precisely, one way of studying a space is by looking at maps of spheres into your space. These notions are formalized in objects called homology groups.

Poincaré conjectured that "If an $n$-manifold looks like $S^n$ from this point of view, then it is actually $S^n$!"

Formally, if an $n$-manifold has the same homology groups as an $S^n$, it is actually $S^n$!

Poincaré's Conjecture has an even more intuitive explanation in dimension 3. To state it we need to talk about simple connectivity.

**Simple Connectivity**

Observe the following property of $S^2$:

If I have a closed curve on $S^2$ (i.e. a curve that starts and ends at the same point), then I can continuously deform this curve to a single point. (An example of this is shown to the right.)
Def: A space in which every closed curve can be continuously deformed to a point is called simply connected.

Ex:
1) The 2-disk is simply connected
   (convince yourself of this)

2) The annulus is not simply connected
   For example, the curve drawn to the right cannot be deformed to a point because it goes around a hole.

3) The cylinder is not simply connected

4) 3-space with a ball removed is simply connected
   We can move any curve away from the hole

• These examples should suggest that simple connectivity somehow measures if there is a "big" hole in our space.
5) \( T^2 \) is not simply connected.

For example, the cone drawn cannot be deformed to point

6) Similarly \( T^2 \ldots \# T^2 \) is not simply connected.

**Exercise:** Think about why \( \mathbb{RP}^2 \) is not simply connected.

**Hint:** Think of \( \mathbb{RP}^2 \) as \( S^2 \) with antipodal point identified.

When we think of \( \mathbb{RP}^2 \) like this, what is a closed curve?

Based on our classification and the examples above, we see that the only closed simply connected 2-mfld is \( S^2 \)

**Poincaré Conjecture in 3-dim:** If \( M \) is a closed, connected, simply connected 3-mfld,
then \( M \) is homeomorphic to \( S^3 \)

Poincaré conjecture eluded mathematicians for decades, though during this time the subject of topology grew greatly.

In 60s Smale shocked mathematicians by proving the Poincaré Conjecture in dimensions \( \geq 5 \).

In 80s Freedman proved the Poincaré Conjecture in dim 4.

In 2003 Perelman proved Geometrization Conjecture, which in turn implied the Poincaré Conjecture in dimension 3.
Examples of higher dimensional spaces

Higher dimension spheres
Let's recall the different descriptions of $S^2$ we have.

$S^2 = \text{set of points } x^2 + y^2 + z^2 = 1 \text{ in 3-space}$

$\cong 2$-ball with points on boundary all identified

($= \text{points}$

$x^2 + y^2 \leq 1$)

$\cong 2$-space with a point at $\infty$ added

(i.e. $S^2$ minus a point is $\mathbb{R}^2$)

We can describe higher dimensional spaces in analogous way.

$S^n = \text{set of points } x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1 \text{ in } (n+1)$-space

$\cong n$-ball with boundary identified

($= \text{points}$

$x_1^2 + \ldots + x_n^2 \leq 1$)

$\cong n$-space with a point at $\infty$ added

(i.e. $S^n$ minus a point is $\mathbb{R}^n$)
Product spaces

We can make new higher dimensional manifolds by taking product of lower dimensional manifolds.

Def: If $X$ and $Y$ are spaces, their product $X \times Y = \{(x, y) | x \in X \text{ and } y \in Y\}$.

Informally, you might think of this as "an $X$'s worth of $Y$'s" or "a $Y$'s worth of $X$'s".

Or you could like that there are cross-sections of $X \times Y$ that look like $X$ and cross-sections that look like $Y$.

Examples (in lower dimensions):

1) $X = \mathbb{R}$
   $Y = \mathbb{R}$

   $X \times Y = \text{box}$

2) $X = S^1$
   $Y = \mathbb{R}$

   $X \times Y = \text{cylinder}$
3) $X = S'$, $Y = S'$

We can think of $S'$ as the interval $[0, 1]$ with 0

Thus, $X \times Y = \{(x, y) \mid x \in [0, 1], y \in [0, 1], \theta = 1\} = \mathbb{T}^2$

box with edges identified as pictured

$S' \times S' = \mathbb{T}^2$

We can use products to form new manifolds of higher dimension

Examples:

1) $S' \times S' \times S'$ is a 3-manifold.

It can be represented as a filled in cube with opposite sides identified.

2) $S^2 \times [0, 1] = \text{thick sphere}$
3) $S^1 \times S^1 \times [0, 1] = \text{thick torus}$

- We can also form connect sums of manifolds.
  Instead of removing disks and gluing along their boundary (a circle),
  for an $n$-manifold we remove $n$-balls and glue along their
  boundary ($S^{n-1}$).

Exercise 5: Think about examples 1-3 above.
These are 3-manifolds. What are some examples of 4-manifolds.