Recall: Last time we define $\deg f$ for a map $f : M \to N$ closed between manifolds of the same dimension. (This case we will be most interested in will be maps $S^1 \to S^1$.) We said that when $\dim M = \dim N$, $f^{-1}(\eta)$ is a 0-dim closed manifold for "most $\eta$ in $N$" (those $\eta$ were called regular values), i.e. $f^{-1}(\eta)$ is finitely many points.

We discussed that $\# f^{-1}(\eta)$ is a locally constant function on $N$.

We defined $\deg f = \# f^{-1}(\eta) \mod 2$.

Properties of $\deg f$

1) $\deg f$ is independent of the $\eta$ we chose.

2) If $f \sim g$, then $\deg f = \deg g$.

Remark: We only defined the degree of a map modulo 2. The degree can actually be defined to be an integer. We won't discuss the necessary modifications, but the difference is that we need to keep track of orientations (signs).

For example: Maps $S^1 \to S^1$.

In addition to counting points, we give each point a + or - based on if the graph at the point has positive or negative slope.
There is a generalization of this to higher dimensions.

We may use \( \text{deg} f \) (not \( \text{mod} \, 2 \)) and all theorems stating about \( \text{deg} g \) is also true for \( \text{deg} f \).

### Applications

1) Brouwer Fixed Point Theorem

(See last week's notes.)

2) An application of looking at \# pre-images

**Fundamental Theorem of Algebra**: Every nonconstant polynomial in the complex numbers, \( P(z) \), has a zero.

(This implies that \( P(z) \) factors as \( P(z) = (z - \lambda_1) \cdots (z - \lambda_n) \).

**Proof**: \( P: \mathbb{C} \to \mathbb{C} \)

Remember we saw that \( S^2 \times \mathbb{R}^3 \cong \mathbb{R}^2 \).

Thus, we can consider \( P: S^2 \times \mathbb{R}^3 \to S^2 \times \mathbb{R}^3 \).

We would like to view \( P \) as a map \( P: S^2 \to S^2 \) (because we have been dealing with closed manifolds).

If you like of the missing point as "\( \infty \)" and define \( P(\infty) = \infty \).
Then \( P: S^2 \rightarrow S^2 \) is continuous (Convince yourself this is true).

Now observe that \( P \) has only finitely many zeros (b/c it is a polynomial). It also has only finitely many critical points (places where \( P'(z) = 0 \)) because \( P' \) is also a polynomial. Thus, the set of critical values of \( P \) is only finitely many points. So the regular values of \( P \) are the sphere with finitely many points removed, in particular, the set of regular values is connected.

We noted that \( \#P^{-1}(y) \) is a locally constant function of \( y \). (Constant when you stay away from critical values.) Because the set of regular values is connected, this implies that \( \#P^{-1}(y) \) is actually constant. Since \( \#P^{-1}(y) \) can't be zero everywhere, we conclude it is zero nowhere. Thus, \( P \) is one to one and hence \( P \) has a zero. \( \square \)

**Exercise:** How does this proof fail for polynomials

\[ P: \mathbb{R} \rightarrow \mathbb{R} \]
3) Borsuk-Ulam Theorem: If \( f: S^2 \rightarrow \mathbb{R}^2 \)
then for some \( x \in S^2 \), \( f(x) = f(-x) \).

We will see this implies many other interesting theorems
and is interesting on its own.

For example let \( f \) be the function \( f: S^2 \rightarrow \mathbb{R}^2 \)
\( f(x) = (\text{temperature at } x, \text{ barometric pressure at } x) \).

The Borsuk-Ulam theorem implies the “Theorem of meteorology”:

Some antipodal points \( x, -x \) have the same temperature and
barometric pressure.

Proof of Borsuk-Ulam: We need the following lemma.

Lemma: If \( g: S^1 \rightarrow S^1 \) is an odd map, i.e. \( g(-x) = -g(x) \)
then \( \text{deg}(g) \) is odd.

Pf.: Remember, \( \text{deg}(g) \) is “the number of times \( g \) wraps around
Consider \( g \) on half of the circle
(b/c \( g \) is odd, \( g \) is determined
by its image on a half)
Since \( g \) is odd, \( g \) wraps around the circle a half.
Thus, we double the half circle to a full circle, it wraps around an odd number of times.

Back to the proof Baranov-Kuon

Suppose for contradiction that \( f(x) \neq f(-x) \) for every \( x \).

Define a new function \( h: S^2 \rightarrow S^1 \)

\[ h(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|} \]

Note that \( h(-x) = -h(x) \). (*)

Now let \( g: S^1 \rightarrow S^1 \) be the restriction of \( h \) to the equator. (*) implies \( g \) is odd and hence by the lemma, has odd degree.

On the other hand, \( g \) extends to \( D^2 \rightarrow D^1 \) (the upper half sphere) and is thus homotopic to a constant map and hence has degree \( 0 \) (as discussed in the proof of Brouwer).

Contradiction.
Applications using Borsuk-Ulam

4) Ham Sandwich theorem: For 3 bounded closed subset of $\mathbb{R}^3$ (2 pieces of bread and a slice of ham), you can find a plane dividing the area of a 3 pieces in half (you can cut a ham sandwich in half).

5) If $S^2$ is covered by 3 sets, then one must contain a pair of antipodal points.

6) $\mathbb{R}^2$ is not homeomorphic to $S^2$

Exercise: Derive 4 - 6 using the Borsuk-Ulam theorem