(1) Let $(\overline{M}, \overline{g})$ be a Riemannian manifold of dimension $n + 1$. Let $f : M \to \mathbb{R}$ be a smooth function, and let $a \in \mathbb{R}$ be a regular value of $f$ such that $M = f^{-1}(a)$ is nonempty. Then $M$ is an $n$-dimensional submanifold of $\overline{M}$. Let $i : M \to \overline{M}$ be the inclusion map, and let $g = i^* \overline{g}$, so that $i : (M, g) \to (\overline{M}, \overline{g})$ is an isometric embedding. Let $\text{Hess} f = \nabla df$ be the Hessian of $f$, where $\nabla$ is the Levi-Civita connection on $(\overline{M}, \overline{g})$. Prove the following statements.
(a) For any $p \in M$ and $x, y \in T_p M$, we have
\[
\text{grad} f(p) \in (T_p M)^\perp, \quad H_{\text{grad} f(p)}(x, y) = -\text{Hess} f(p)(x, y).
\]
(b) Let $H$ be the mean curvature of $M$ in $\overline{M}$ with respect to the unit normal $\text{grad} f |_{\text{grad} f}$. Then
\[
H = -\frac{1}{n} \text{div} \left( \frac{\text{grad} f}{|\text{grad} f|} \right)
\]
where the gradient and the divergence are defined by $\overline{g}$.

(2) Let $(M, g)$ be a Riemannian manifold of dimension $n$, and let $f \in C^\infty(M)$. Let $M \times \mathbb{R}$ be equipped with the product metric $\tilde{g}$. In local coordinates $(x_1, \ldots, x_n)$ on $\overline{M}$,
\[
g = g_{ij} dx_i dx_j, \quad \tilde{g} = g_{ij} dx_i dx_j + dt^2, \quad df = f_i dx_i, \quad \text{Hess}(f) = f_{ij} dx_i dx_j
\]
where the Hessian is defined by the metric $g$. The map $\phi_f : M \to M \times \mathbb{R}$ defined by $x \mapsto (x, f(x))$ is a smooth embedding, so the image $\Gamma_f = \phi_f(M)$ is an $n$-dimensional submanifold of $M \times \mathbb{R}$.
(a) Let $\tilde{g} = \phi_f^* \tilde{g}$, and write $\tilde{g} = \tilde{g}_{ij} dx_i dx_j$. Show that
\[
\tilde{g}_{ij} = g_{ij} + f_i f_j, \quad \tilde{g}^{ij} = g^{ij} - \frac{f^i f^j}{1 + |df|^2}
\]
where $f^i = g^{ij} f_j$ and $|df|^2 = g^{ij} f_i f_j$.
(b) Let $\tilde{H}$ be the mean curvature of $\Gamma_f \subset M \times \mathbb{R}$ with respect to the upward unit normal. Show that
\[
n \phi_f^* \tilde{H} = \frac{\tilde{g}^{ij} f_{ij}}{\sqrt{1 + |df|^2}}
\]

(3) Recall that the cartesian coordinates $(x, y, z)$ and the spherical coordinates $(r, \phi, \theta)$ on $\mathbb{R}^3$ are related by
\[
x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi.
\]
Consider a Riemannian metric on $\mathbb{R}^3$ of the form
\[
g = u(r)^2 dr^2 + r^2 (d\phi^2 + \sin^2 \phi d\theta^2)
\]
where $u = u(r)$ is a positive, smooth function on $\mathbb{R}^3$ which depends only on $r$. Note that $u = 1$ corresponds to the Euclidean metric. Given any $\rho > 0$, let $S_\rho$ be the sphere defined by $r = \rho$.

(a) Find the scalar curvature of $(\mathbb{R}^3, g)$.
(b) Find the second fundamental form and the mean curvature of $S_\rho$ with respect to the inner unit normal.
(c) Find the scalar curvature of $S_\rho$. 