Assignment 15

Due on Monday, February 8, 2016

- In this assignment, all the manifolds are connected.
- [dC] = do Carmo, Riemannian Geometry.

1. Let $H^n = \{ (y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_n > 0 \}$ be the n-dimensional upper half space. For any $\alpha \in \mathbb{R}$, define a Riemannian metric $g_\alpha$ on $H^n$ by

   $$g_\alpha = y_n^\alpha (dy_1^2 + \cdots + dy_n^2).$$

   By [dC] Chapter 8 Section 3, $(H^n, g_{-2})$ is a complete Riemannian manifold. Prove that the Riemannian manifold $(H^n, g_\alpha)$ is complete if and only if $\alpha = -2$. (Hint: consider the geodesic $\gamma(t)$ with $\gamma(0) = (0, \ldots, 0, 1)$ and $\gamma'(0) = \partial \partial y_n$.)

2. A Riemannian manifold $M$ is said to be homogeneous if given $p, q \in M$ there exists an isometry of $M$ which takes $p$ into $q$. Prove that any homogeneous Riemannian manifold is complete.

3. Let $M$ be a Riemannian manifold with non-positive sectional curvature. Prove that, for any $p \in M$, the conjugate locus $C(p)$ is empty. (Hint: See [dC] page 119-120, Exercise 3.)

4. Show that the point $p = (0, 0, 0)$ of the paraboloid

   $$S = \{ (x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2 \}$$

   is a pole of $S$ and, nevertheless, the sectional curvature of $S$ is positive.