Assignment 19

Due on Monday, March 21, 2016

(1) Let \( f : \mathbb{R}^n - \{p_0\} \rightarrow \mathbb{R}^n - \{p_0\} \) be the inversion

\[
f(p) = \frac{p - p_0}{|p - p_0|^2} + p_0.
\]

(a) Let \( P \) be a hyperplane in \( \mathbb{R}^n \). Show that if \( p_0 \in P \) then \( f(P - p_0) = P - \{p_0\} \). Show that if \( p_0 \notin P \) then the closure of \( f(P) \) is an \((n - 1)\)-sphere passing through \( p_0 \).

(b) Let \( S \) be an \((n - 1)\)-sphere in \( \mathbb{R}^n \). Show that if \( p_0 \in S \) then \( f(S - \{p_0\}) \) is a hyperplane. Show that \( f(S) \) is an \((n - 1)\)-sphere if \( p_0 \notin S \).

(2) Let \( S = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\} \) be the graph of a \( C^\infty \) function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). (Note that this is a special case of Assignment 14 (1), and this includes Assignment 15 (4) as a special case.) Let

\[
\begin{align*}
    f_x &= \frac{\partial f}{\partial x}, & f_y &= \frac{\partial f}{\partial y}, & f_{xx} &= \frac{\partial^2 f}{\partial^2 x}, & f_{xy} &= \frac{\partial^2 f}{\partial y \partial x}, & f_{yy} &= \frac{\partial^2 f}{\partial^2 y}.
\end{align*}
\]

Prove the following formulae.

(a) The second fundamental form of \( S \) in \( \mathbb{R}^3 \) with respect to the upward unit normal is given by

\[
h = \frac{f_{xx} dx^2 + 2f_{xy} dxdy + f_{yy} dy^2}{\sqrt{1 + f_x^2 + f_y^2}}.
\]

(b) The sectional curvature of \( S \) is given by

\[
K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}.
\]

(In particular, if \( f(x, y) = x^2 + y^2 \) then \( K = \frac{4}{(1 + 4x^2 + 4y^2)} \).)

(3) do Carmo Chapter 8 Exercise 6 (page 182-184). To solve part b), you may use Exercise 5, which is equivalent to Assignment 16 (1). Please read page 178 of do Carmo before you do part d).