Let $\Gamma = \{ H_p : p \in S^{2n+1} \}$ be the connection on the principal $U(1)$-bundle $\pi : S^{2n+1} \to \mathbb{P}_n(\mathbb{C})$ defined in Assignment 21 (2), and let $\hat{g}$ be the Riemannian metric on $\mathbb{P}_n(\mathbb{C})$ defined in Assignment 21 (2). We now specialize to the case $n = 1$. Let $\mathbb{C}$ be oriented by the volume form $dx \wedge dy$, where $z = x + \sqrt{-1}y$ is the complex coordinate on $\mathbb{C}$. We choose an orientation on $\mathbb{P}_1(\mathbb{C})$ such that \( f : \mathbb{C} \to \mathbb{P}_1(\mathbb{C}), \quad z \mapsto \left( \frac{1}{\sqrt{1+|z|^2}}, \frac{z}{\sqrt{1+|z|^2}} \right) \) is orientation preserving. Let $\nu \in \Omega^2(\mathbb{P}_1(\mathbb{C}))$ be the volume form determined by this orientation and the Riemannian metric $\hat{g}$. Let $c$ be a real constant. Find $c$.

Let $E$ be a $C^\infty$ complex vector bundle over a $C^\infty$ manifold $M$, and let $\nabla$ be a connection on $E$. The induced connection on $\text{End}(E) = E^* \otimes E$ satisfies

$$\nabla_X(\phi(s)) = (\nabla_X \phi)(s) + \phi(\nabla_X s)$$

for all $\phi \in C^\infty(M, \text{End}(E))$, $X \in \mathfrak{X}(M)$, $s \in C^\infty(M, E)$. The exterior covariant derivative $d_{\nabla} : \Omega^r(M, \text{End}(E)) \to \Omega^{r+1}(M, \text{End}(E))$ satisfies

$$d_{\nabla}(\alpha \otimes \phi) = d\alpha \otimes \phi + (-1)^r \alpha \wedge \nabla \phi$$

for all $\alpha \in \Omega^k(M)$, $\phi \in C^\infty(M, \text{End}(E))$. The trace map $\text{End}(E) \to \mathbb{C}$, where $\mathbb{C} = M \times \mathbb{C}$ is the product (trivial) complex line bundle over $M$, induces $\text{tr} : \Omega^k(M, \text{End}(E)) \to \Omega^k(M, \mathbb{C})$. Prove that if $\phi \in \Omega^k(M, \text{End}(E))$ then

$$d(\text{tr}\phi) = \text{tr}(d_{\nabla}\phi).$$

Let $(E, h)$ be a hermitian vector bundle over a $C^\infty$ manifold $M$, and let $\nabla$ be a unitary connection. Let $F \in \Omega^2(M, \text{End}(E))$ be the curvature 2-form of $\nabla$. Use (2) and the Bianchi identity to show that

$$\text{ch}_k(E, \nabla) := \text{tr}\left( \frac{\sqrt{-1}}{2\pi} F \right)^k$$

is a closed 2k form for any positive integer $k \leq \dim M/2$.

Let $(E, h)$ be a hermitian vector bundle over a $C^\infty$ manifold $M$, and let $\nabla$ be a unitary connection. We also use $\nabla$ to denote the induced connection on the dual vector bundle $E^*$, and on $\Lambda^k E$. The determinant line bundle of $E$ is defined to be $\det E := \Lambda^r E$, where $r = \text{rank}_E E$. Prove the following identities of Chern forms.

(a) $c_k(E^*, \nabla) = (-1)^k c_k(E, \nabla)$.

(b) $c_1(E, \nabla) = c_1(\det E, \nabla)$. 
