(1) Let $M$ be an $n$-dimensional submanifold in an $(n+1)$-dimensional Riemannian manifold $(\bar{M}, \bar{g})$. Let $g$ be the induced Riemannian metric on $M$. Suppose that $\eta \in C^\infty(M, T^*M)$ is a unit normal vector field along $M$, so that $p := H_{\eta}$ (defined on page 128 of do Carmo’s book) is a symmetric bilinear form on $M$. Let $(x_1, \ldots, x_n)$ be local coordinates on $M$, and write

$$g = \sum_{i,j} g_{ij} dx_i dx_j, \quad p = \sum_{i,j} p_{ij} dx_i dx_j,$$

$$\nabla_{\frac{\partial}{\partial x_k}} p = \sum_{i,j,k} p_{ij,k} dx_i dx_j.$$

The norm square $|p|^2$ of $p$ and the trace $\text{tr} p$ of $p$ are smooth functions on $M$ given by

$$|p|^2 = \sum_{i,j} p_{ij} p_{kl} g^{ik} g^{jl}, \quad \text{tr} p = \sum_{i,j} p_{ij} g^{ij}.$$

Let $R$ and $\bar{R}$ denote the scalar curvatures of $g$ and $\bar{g}$, respectively, and let $\text{Ric}$ denote the Ricci curvature of $\bar{g}$ (defined as in Chapter 4, Section 4 in do Carmo’s book).

(a) Use the Gauss equation to prove that

$$n(n+1) \bar{R} - 2n \text{Ric}(\eta, \eta) = n(n-1)R + |p|^2 - (\text{tr} p)^2.$$

(b) Use the Codazzi’s equation to prove that

$$n \text{Ric}(\eta, \frac{\partial}{\partial x_i}) = -\sum_{j,k} g^{jk} p_{ij,k} + \frac{\partial}{\partial x_i} \text{tr} p.$$

(2) Let $N$ be a 4-manifold equipped with a pseudo Riemannian metric $\bar{g}_{\mu\nu}$ of signature $-,+,+,+$ on $N$. Let $s_{\mu\nu}$ be a symmetric $(0,2)$ tensor on $N$ such that $\bar{g}_{\mu\nu}(t) := \bar{g}_{\mu\nu} + ts_{\mu\nu}$ is a pseudo Riemannian metric of signature $-,+,+,+$ on $N$ when $t \in (-\epsilon, \epsilon)$, where $\epsilon$ is some positive number. Let $\bar{R}$ and $\bar{R}(t)$ denote the scalar curvatures defined by $\bar{g}_{\mu\nu} = \bar{g}_{\mu\nu}(0)$ and $\bar{g}_{\mu\nu}(t)$, respectively; let $\bar{R}_{\mu\nu}, \bar{R}_{\mu\nu}(t)$ denote the Ricci curvatures defined by $\bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu}(t)$, respectively. We use Einstein’s summation convention: “$\bar{g}^{\mu\nu}s_{\mu\nu}$” means “$\sum_{\mu,\nu=0}^{3} \bar{g}^{\mu\nu}s_{\mu\nu}$”, etc.

(a) Prove that

$$\frac{d}{dt} \bigg|_{t=0} \sqrt{-\det(\bar{g}(t))} = \frac{1}{2} \bar{g}^{\mu\nu} s_{\mu\nu} \sqrt{-\det(\bar{g})}.$$ 

(b) Define a smooth function $\text{Tr}(s)$ on $N$ and a vector field $X^\mu$ on $N$ by

$$\text{Tr}(s) := \bar{g}^{\mu\nu} s_{\mu\nu}, \quad X^\mu = (s^{\mu\nu} - \bar{g}^{\mu\nu} \text{Tr}(s))_{,\nu}.$$

where $_{,\nu}$ denotes the covariant derivative with respect to $\frac{\partial}{\partial x^\nu}$. Prove that

$$\frac{d}{dt} \bigg|_{t=0} \bar{R}(t) = -\bar{R}^{\mu\nu} s_{\mu\nu} + \text{div} X.$$ 

(c) Use (a) and (b) to prove that

$$\frac{d}{dt} \bigg|_{t=0} \left( \bar{R}(t) \sqrt{-\det(\bar{g}(t))} \right) = \left( -\bar{R}^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} \bar{R} s_{\mu\nu} + \text{div} X \right) \sqrt{-\det(\bar{g})}.$$