(1) Let $G_1$ and $G_2$ be Lie groups, and let $e_1 \in G_1$ and $e_2 \in G_2$ be the identity elements. Suppose that $f : G_1 \to G_2$ is a group homomorphism and a smooth map. Prove that $df_{e_1} : T_{e_1}G \to T_{e_2}H$ is a Lie algebra homomorphism.

(2) Let $H$ be a closed Lie subgroup of a Lie group $G$, and let $G/H = \{aH \mid a \in G\}$ be the set of left cosets of $H$ in $G$. (In other words, let $H$ act on $G$ by right multiplication and let $G/H$ be the quotient.) $G$ acts on $G/H$ on the left by $G \times G/H \to G/H$, $(a,bH) \mapsto abH$. Let $g$ be a right invariant Riemannian metric on $G$. By the theorems stated in class, there is a unique Riemannian metric $\hat{g}$ on $G/H$ such that $\pi : (G,g) \to (G/H,\hat{g})$ is a Riemannian submersion. Prove that if $g$ is left invariant then $G$ acts isometrically on $(G/H,\hat{g})$.

(3) Let $g_n$ be the bi-invariant metric on $SO(n)$ defined in Problem 3 of Assignment 6. We have seen in class that there is a diffeomorphism $f : S^n \to SO(n+1)/SO(n)$. Let $\tilde{g}$ be the unique Riemannian metric on $SO(n+1)/SO(n)$ such that 

$\pi : (SO(n+1),g_{n+1}) \to (SO(n+1)/SO(n),\hat{g})$

is a Riemannian submersion. Prove that $f^*\tilde{g} = \lambda g_n$ for some $\lambda > 0$, and find $\lambda$. (Hint: What is the horizontal space $H_{T_{n+1}} \subset T_{T_{n+1}}SO(n+1)?$)