(1) Let $X$ be a left invariant vector field on a Lie group $G$, and let $\gamma : I \to G$ be an integral curve of $X$. Prove that $\gamma$ is a geodesic with respect to any bi-invariant metric on $G$. (Hint: see do Carmo page 80, Exercise 3)

(2) Let $F : M \to N$ be a smooth map between smooth manifolds, and define $F_* : \mathcal{X}(M) \to C^\infty(M, F^*TN)$ by
\[
(F_*X)(p) = dF_p(X(p)) \in T_{F(p)}N = (F^*TN)_p
\]
where $X \in \mathcal{X}(M)$ and $p \in M$. Let $h$ be a Riemannian metric on $N$, let $\nabla$ be an affine connection on $(N, h)$, and let $D = F^*\nabla$ be the pull back connection on $F^*TN$. Prove the following statements:

(a) Suppose that $\nabla$ is symmetric. Then for all $X, Y \in \mathcal{X}(N)$,
\[
D_X(F_*Y) - D_Y(F_*X) = F_*([X, Y]).
\]

(b) Suppose that $\nabla$ is compatible with the Riemannian metric $h$. Then for all $X \in \mathcal{X}(M)$ and $V, W \in C^\infty(M, F^*TN)$,
\[
X\langle V, W \rangle = \langle D_XV, W \rangle + \langle V, D_XW \rangle.
\]

(3) (geodesic frame) Let $(M, g)$ be a Riemannian manifold of dimension $n$ and let $p \in M$. Show that there exists an open neighborhood $U \subset M$ of $p$ and $n$ vector fields $E_1, \ldots, E_n \in \mathcal{X}(U)$ such that (i) for all $q \in U$, $\{E_1(q), \ldots, E_n(q)\}$ is an orthonormal basis of $T_qM$, and (ii) $(\nabla_{E_i}E_j)(p) = 0$.

(4) (normal coordinates) Let $(M, g)$ be a Riemannian manifold of dimension $n$ and let $p \in M$. Show that there exist local coordinates $x_1, \ldots, x_n$ on an open neighborhood $U \subset M$ of $p$ such that $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$ for $i, j, k \in \{1, \ldots, n\}$. (Hint: see do Carmo page 86, Problem 14.)