A proof that Euler missed

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§1. Basel problem (1644).

$$\sum_{n\geq 1} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = ?$$

Jacob Bernoulli (1689) "It's less than 2. Help us out!"

Johann Bernoulli (1721) "It's about $\frac{8}{5}$."

difficulty: series converges very slowly!

$$= 1.6\ 4\ 4\ 9\ 3\ 4\ 0 \ \cdots$$

$n \sim 10^2 \quad 10^3 \quad 10^6$

Thm (Euler, 1735)

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Pf. Consider the function $\sin(x)$.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

Euler's factorization formula:

$$\sin(x) = x \cdot \prod_{n\geq 1} \left(1 - \frac{x^2}{n^2\pi^2}\right),$$

($\sin(x)$ has zero $x = 0, \pm\pi, \pm 2\pi, \cdots \pm n\pi, \cdots$).

Compare the coefficients before x^3

$$-\frac{1}{3!} = -\sum_{n\geq 1} \frac{1}{n^2\pi^2}$$

$$\Rightarrow \sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \square.$$

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Rem. 1) Euler's proof is not rigorous (e.g. $e^x \sin(x)$, $\sin(x)$ have same set of zeros). Rigorous pf can be established using complex analysis (Weierstrass factorization theorem) or Fourier analysis.

2). specialize to $x = \frac{\pi}{2}$.

$$1 = \frac{\pi}{2} \cdot \prod_{n \geq 1} \left(1 - \frac{1}{4n^2}\right).$$

$$\Rightarrow \quad \frac{\pi}{2} = \prod_{n \geq 1} \frac{4n^2}{4n^2 - 1} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}$$

(Wallis formula for π, 1656).

3). Compare x^5- coefficient, get.

$$\frac{1}{5!} = \sum_{n > m \geq 1} \frac{1}{n^2 \cdot m^2 \pi^4}.$$

$$\Rightarrow \quad \sum_{n > m \geq 1} \frac{1}{n^2 \cdot m^2} = \frac{\pi^4}{120}.$$

$$\Rightarrow \quad \sum_{n \geq 1} \frac{1}{n^4} = \left(\sum_{n \geq 1} \frac{1}{n^2}\right)^2 - 2 \sum_{n > m \geq 1} \frac{1}{n^2 m^2}$$

$$= \left(\frac{\pi^2}{6}\right)^2 - \frac{\pi^4}{60}$$

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

§2. Higher powers.

Def. Riemann zeta function

$$\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s} \quad (\text{converges when } s > 1)$$

Euler:
$$\zeta(2) = \frac{\pi^2}{6}.$$
$$\zeta(4) = \frac{\pi^4}{90}.$$
$$\vdots$$
$$\zeta(2k) \in \pi^{2k}\cdot \mathbb{Q}.$$

In particular: $\zeta(2k) \notin \mathbb{Q}$. $\forall k \geq 1$.

Q. What about $\zeta(2k+1)$?

$$\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3} = 1 + \frac{1}{8} + \frac{1}{27} + \cdots = 1.2020569\ldots$$

Expect: $\zeta(2k+1) \notin \mathbb{Q}$. $\zeta(2k+1) \notin \pi^{2k+1}\cdot\mathbb{Q}$

Nothing was known about $\zeta(2k+1)$ until:

Thm (Apéry, 1979). $\zeta(3) \notin \mathbb{Q}$.

Rem. 1) We don't know similiar irrationality for $\zeta(2k+1)$, $k \geq 2$.

2). Rivoal (2002) proved that infinitely many $\zeta(2k+1)$ are irrational.

3). Zudilin (2003) proved that at least one of
$$\zeta(5),\ \zeta(7),\ \zeta(9),\ \zeta(11)$$
is irrational.

§3. Irrationality proofs.

Key idea: A rational number cannot be approximated by other rational numbers "too well".

Let $x = \frac{a}{b} \neq y = \frac{p}{q}$. $a, b, p, q \in \mathbb{Z}$.

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Then $|x - y| = \left|\frac{a}{b} - \frac{p}{q}\right| = \left|\frac{aq - bp}{bq}\right| \geq \frac{1}{|bq|}$.

In particular, fix $x \in \mathbb{Q}$, then for $C > 1$.

there exists only finitely many $y = \frac{p}{q} \in \mathbb{Q}$ s.t.

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^C}$$

Irrationality criterion: If for some $C > 1$. there exists infinitely many $y = \frac{p}{q} \in \mathbb{Q}$ such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^C}$$

then $x \notin \mathbb{Q}$.

To prove Apéry's thm, need a sequence of rat'l numbers $\frac{p_n}{q_n}$ good approximation of $\zeta(3)$:

$$\left|\zeta(3) - \frac{p_n}{q_n}\right| < \frac{1}{q_n^C} \quad \text{for some } C>1.$$

§4. Apéry's numbers

<u>Def</u>. Consider the recursion:

$$n^3 u_n = (34n^3 - 51n^2 + 27n - 5)\, u_{n-1} - (n-1)^3 u_{n-2}.$$

define $\{b_n\} = \{u_0 = 1,\ u_1 = 5,\ u_2,\ u_3 \ldots \}$.

$\{a_n\} = \{u_0 = 0,\ u_1 = 6,\ u_2,\ u_3 \ldots \}$.

Ex.

$$b_2 = \frac{(34\cdot 2^3 - 51\cdot 2^2 + 27\cdot 2 - 5)\cdot 5 - 1}{2^3}$$

$$= \frac{584}{8} = 73.$$

$$b_3 = \frac{(34\cdot 3^3 - 51\cdot 3^2 + 27\cdot 3 - 5)\cdot 73 - 2^3\cdot 5}{3^3}$$

$$= \frac{39015}{27} = 1445.$$

$$\{b_n\} = \{1,\ 5,\ 73,\ 1445,\ 33001,\ \cdots\}.$$

$$\{a_n\} = \left\{0,\ 6,\ \frac{351}{4},\ \frac{62531}{36},\ \frac{11424695}{288},\ \cdots\right\}.$$

Miracles:

1). $b_n \in \mathbb{Z}$.

2). $a_n \in \frac{1}{d_n^3}\mathbb{Z}$, where $d_n = \operatorname{lcm}(1, 2, \cdots n)$.

3). $\frac{a_n}{b_n} \to \zeta(3)$.

Ex. $n=4$.

$$\frac{a_4}{b_4} = \frac{11424695}{288}\cdot\frac{1}{33001}$$

$$= 1.2020569031578\cdots$$

$$\zeta(3) = 1.2020569031595\cdots$$

Fun exercise: prove Miracles

(Hint:

$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \cdot c_{n,k}.$$

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \cdot c_{n,k}.$$

$$c_{n,k} = \sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \Bigg).$$

Key of pf : integrality properties from a general recursion like so is very rare.

(found similar recursion to prove $\zeta(2) \notin \mathbb{Q}$).

§ 5. Explanations for the integrality.

① arithmetic of modular forms.

A modular form $f : \mathcal{H} = \{ z = x+iy : y>0 \} \to \mathbb{C}$

such that 1) f is holomorphic

2) f has a lot symmetries under the action of modular gp $SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$.

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

Beukers, 1987 :

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n) \quad \text{where } q = e^{2\pi i z}.$$

(Dedekind eta function) is modular form $(k = \frac{1}{2})$

$$t(z) = \left(\frac{\eta(z)\,\eta(6z)}{\eta(2z)\,\eta(3z)} \right)^{12} \quad \text{is modular form } (k=0).$$

$$= q - 12q^2 + 66q^3 - 220q^4 + \cdots$$

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$$f(z) = \frac{\eta(2z)^7\,\eta(3z)^7}{\eta(z)^5\,\eta(6z)^5} \quad \text{----} \quad (k=2)$$

$$= 1 + 5q + 13q^2 + 23q^3 + \cdots$$

Miracle $\quad f(z) = 1 + 5\,t(z) + 73\,t(z)^2 + 1445\,t(z)^3 + \cdots$

$$= \sum_{n\geq 1} b_n\, t(z)^n$$

$\Rightarrow \quad b_n \in \mathbb{Z}$ automatically !

② arithmetic of K3 surfaces:

Beukers-Peters. 1984 :

$$X_t : \quad 1 - z + xyz - t\,xyz(1-x)(1-y)(1-z) = 0.$$

(concretely: $\quad a_n - b_n\zeta(3)$

$$= \int_0^1\int_0^1\int_0^1 \frac{\left(xyz(1-x)(1-y)(1-z)\right)^n}{(1-z+xyz)^{n+1}}\,dx\,dy\,dz).$$

$\Rightarrow$ integrality statements.