Beilinson–Bloch conjecture for unitary Shimura varieties

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The BSD conjecture

- $E : y^2 = x^3 + Ax + B$ an elliptic curve over $\mathbb{Q}$.
- Algebraic rank: the rank of the finitely generated abelian group $E(\mathbb{Q})$

$$r_{\mathrm{alg}}(E) := \mathrm{rank}\, E(\mathbb{Q}).$$

- Analytic rank: the order of vanishing of $L(E, s)$ at the central point $s = 1$

$$r_{\mathrm{an}}(E) := \mathrm{ord}_{s=1} L(E, s).$$

Conjecture (Birch–Swinnerton-Dyer, 1960s)

(1) (Rank) $$r_{\mathrm{an}}(E) \stackrel{?}{=} r_{\mathrm{alg}}(E),$$

(2) (Leading coefficient) For $r = r_{\mathrm{an}}(E)$,

$$\frac{L^{(r)}(E,1)}{r!} \stackrel{?}{=} \frac{\Omega(E)R(E)\prod_p c_p(E) \cdot |Ш(E)|}{|E(\mathbb{Q})_{\mathrm{tor}}|^2}$$

where $R(E) = \det(\langle P_i, P_j \rangle_{\mathrm{NT}})_{r\times r}$ is the regulator for the Néron–Tate height pairing

$$\langle\ ,\ \rangle_{\mathrm{NT}} : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$$

and $Ш(E)$ is the Tate–Shafarevich group.

Remark (Tate, *The Arithmetic of Elliptic Curves*, 1974)

This remarkable conjecture relates the behavior of a function L at a point where it is not at present known to be defined to the order of a group *Ш* which is not known to be finite!

What is known about BSD?

The BSD conjecture is still widely open in general, but much progress has been made in the rank 0 or 1 case.

Theorem (Gross–Zagier, Kolyvagin, 1980s)

$$r_{\mathrm{an}}(E) = 0 \Rightarrow r_{\mathrm{alg}}(E) = 0, \quad r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) = 1,$$

Remark. When $r = r_{\mathrm{an}}(E) \in \{0, 1\}$, many cases of the formula for $L^{(r)}(E, 1)$ are known.

The proof combines two inequalities:

(1) (Gross–Zagier formula)

$$r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) \geq 1.$$

(2) (Kolyvagin's Euler system)

$$r_{\mathrm{an}}(E) \in \{0, 1\} \Rightarrow r_{\mathrm{alg}}(E) \leq r_{\mathrm{an}}(E).$$

Both steps rely on Heegner points on modular curves.

The Beilinson–Bloch conjecture

- X: smooth projective variety over a number field K.
- $\mathrm{CH}^m(X)$: the Chow group of algebraic cycles of codimension m on X.
- $\mathrm{CH}^m(X)^0 \subseteq \mathrm{CH}^m(X)$: the subgroup of geometrically cohomologically trivial cycles.
- Beilinson–Bloch height pairing

$$\langle\ ,\ \rangle_{\mathrm{BB}} : \mathrm{CH}^m(X)^0 \times \mathrm{CH}^{\dim X+1-m}(X)^0 \to \mathbb{R}.$$

- $L(H^{2m-1}(X), s)$: the motivic L-function for $H^{2m-1}(X_{\bar{K}}, \mathbb{Q}_\ell)$.

Conjecture (Beilinson–Bloch, 1980s)

(1) (Rank) $\qquad \mathrm{ord}_{s=m} L(H^{2m-1}(X), s) \overset{?}{=} \mathrm{rank}\, \mathrm{CH}^m(X)^0.$

(2) (Leading coefficient) $\qquad L^{(r)}(H^{2m-1}(X), m) \overset{?}{\sim} \det(\langle Z_i, Z_j' \rangle_{\mathrm{BB}})_{r\times r}$

Example ($X/K = E/\mathbb{Q}$ and $m = 1$)

BB recovers the BSD conjecture as

$$\mathrm{CH}^1(E)^0 \simeq E(\mathbb{Q}), \quad L(H^1(E), s) = L(E, s), \quad \langle\ ,\ \rangle_{\mathrm{BB}} = -\langle\ ,\ \rangle_{\mathrm{NT}}.$$

Remark. In general both sides are only conditionally defined!

(1) $L(H^{2m-1}(X), s)$ is not known to be analytically continued to the central point $s = m$.

(2) $\mathrm{CH}^m(X)^0$ is not known to be finitely generated.

Testable BB conjecture: X = Shimura varieties

- Langlands–Kottwitz/Langlands–Rapoport: express the motivic L-functions of Shimura varieties $X = \mathrm{Sh}_G$, as a product of automorphic L-functions $L(s, \pi)$ on G,
$$L(H^{2m-1}(X), s+m) = \prod_{\pi} L(s+1/2, \pi).$$
- Assume from now (the most interesting case):
  (1) $2m - 1 = \dim X$ (middle cohomology).
  (2) π is tempered cuspidal.
- Analytic properties of $L(s, \pi)$ can be established.
- $\mathrm{CH}^m(X)^0$ is not known to be finitely generated, but we can test if it is nontrivial.

Unconditional prediction of BB conjecture, in the same spirit of Gross–Zagier:

Conjecture (Beilinson–Bloch for Shimura varieties)

$$\mathrm{ord}_{s=1/2} L(s, \pi) = 1 \overset{?}{\Longrightarrow} \mathrm{rank}\, \mathrm{CH}^m(X)^0_\pi \geq 1.$$

Remark. Conjecture was only known for:

(1) X = modular curves (Gross–Zagier)
(2) X = Shimura curves (S. Zhang, Kudla–Rapoport–Yang, Yuan–Zhang–Zhang, Liu).
(3) $X = \mathrm{U}(1,1) \times \mathrm{U}(2,1)$ Shimura threefolds and π = endoscopic (Xue).

Theorem A (L.-Liu, impressionist version)

Conjecture holds for $\mathrm{U}(2m-1, 1)$-Shimura varieties and π satisfying local assumptions.

The Hermitian symmetric space for $\mathrm{U}(n-1,1)$

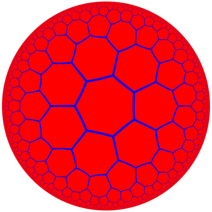
- Hermitian symmetric space for $\mathrm{U}(n-1,1)$,

$$\mathbb{D}_{n-1} := \{z \in \mathbb{C}^{n-1} : |z| < 1\} \cong \frac{\mathrm{U}(n-1,1)}{\mathrm{U}(n-1)\times \mathrm{U}(1)}.$$

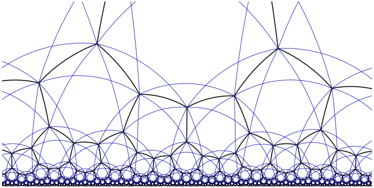
- We have an action

$$\mathrm{U}(n-1,1) \curvearrowright \mathbb{D}_{n-1}.$$

- Notice $\mathbb{D}_1$ is isomorphic to the upper half plane $\mathbb{H}$.



$$\longleftrightarrow$$



Unitary Shimura varieties X

- E/F: CM extension of a totally real number field.
- $\mathbb{V}$: totally definite incoherent $\mathbb{A}_E/\mathbb{A}_F$-hermitian space of rank n.
- Incoherent: $\mathbb{V}$ is not the base change of a global E/F-hermitian space, or equivalently $\prod_v \varepsilon(\mathbb{V}_v) = -1$, where $\mathbb{V}_v := \mathbb{V} \otimes_{\mathbb{A}_F} F_v$.
- Any place $w|\infty$ of F gives a nearby coherent E/F-hermitian space V such that
$$V_v \cong \mathbb{V}_v, v \neq w, \quad \text{but } V_w \text{ has signature } (n-1, 1).$$
- $G = \mathrm{U}(\mathbb{V})$.
- $K \subseteq G(\mathbb{A}_F^\infty) \cong \mathrm{U}(V)(\mathbb{A}_F^\infty)$: open compact subgroup.
- X: unitary Shimura variety of dimension $n-1$ over E such that for any place $w|\infty$ inducing $\iota_w : E \hookrightarrow \mathbb{C}$,
$$X(\mathbb{C}) = \mathrm{U}(V)(F)\backslash[\mathbb{D}_{n-1} \times \mathrm{U}(V)(\mathbb{A}_F^\infty)/K].$$
- X is a Shimura variety of abelian type.
- Its étale cohomology and L-function are computed in the forthcoming work of Kisin–Shin–Zhu, under the help of the endoscopic classification for unitary groups (Mok, Kaletha–Minguez–Shin–White).

Automorphic representations π

- $W = E^{2m}$: the standard E/F-skew-hermitian space with matrix $\begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$.
- $\mathrm{U}(W)$: quasi-split unitary group of rank $n = 2m$.
- π: cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$.

Assumptions.

(1) E/F is split at all 2-adic places and $F \neq \mathbb{Q}$. Assume that $E/\mathbb{Q}$ is Galois or contains an imaginary quadratic field (for simplicity).

(2) For $v|\infty$, π_v is the holomorphic discrete series with Harish-Chandra parameter $\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{-n+3}{2}, \frac{-n+1}{2}\}$.

(3) For $v \nmid \infty$, π_v is tempered.

(4) For $v \nmid \infty$ ramified in E, π_v is spherical with respect to the stabilizer of $O_{E_v}^{2m}$.

(5) For $v \nmid \infty$ inert in E, π_v is unramified or almost unramified. If π_v is almost unramified, then v is unramified over $\mathbb{Q}$.

Remark. π_v is almost unramified: π_v has a nonzero Iwahori-fixed vector and its Satake parameter contains $\{q_v, q_v^{-1}\}$ and $2m-2$ complex numbers of norm 1. Equivalently, the theta lift of π_v to the non-quasi-slit unitary group of same rank is spherical with respect to the stabilizer of an almost self-dual lattice.

Main result A: BB conjecture

Let $S_\pi = \{v \text{ inert} : \pi_v \text{ almost unramified}\}$. Then under Assumptions, the global root number for the (complete) standard L-function $L(s,\pi)$ equals

$$\varepsilon(\pi) = (-1)^{|S_\pi|} \cdot (-1)^{m[F:\mathbb{Q}]}$$

by epsilon dichotomy (Harris–Kudla–Sweet, Gan–Ichino). When $\mathrm{ord}_{s=1/2} L(s,\pi) = 1$:

- $\varepsilon(\pi) = -1$,
- $\mathbb{V} = \mathbb{V}_\pi$: totally definite incoherent space of rank $n = 2m$ such that for $v \nmid \infty$,
$$\varepsilon(\mathbb{V}_v) = -1 \text{ exactly for } v \in S_\pi.$$
- Associated unitary Shimura variety X of dimension $n - 1 = 2m - 1$ over E.
- $\mathrm{CH}^m(X)^0_\pi$ the localization of $\mathrm{CH}^m(X)^0_{\mathbb{C}}$ at the maximal ideal $\mathfrak{m}_\pi$ of the Hecke algebra associated to π.

Theorem A (L.–Liu, 2020, 2021)

Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions. Then the implication

$$\mathrm{ord}_{s=1/2} L(s,\pi) = 1 \Longrightarrow \mathrm{rank}\, \mathrm{CH}^m(X)^0_\pi \geq 1$$

holds when the level $K \subseteq G(\mathbb{A}_F^\infty)$ is sufficiently small.

Example: Symmetric power L-function of elliptic curves

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Example

Let A/F be a modular elliptic curve without CM such that

(1) A has bad reduction only at places v split in E.

(2) $\mathrm{Sym}^{2m-1} A_E$ is automorphic (Newton–Thorne, Clozel–Thorne, ...)

Then there exists π satisfying Assumptions such that

$$L(s+1/2,\pi) = L(\mathrm{Sym}^{2m-1} A_E, s+m).$$

As $S_\pi = \varnothing$ and $\varepsilon(\pi) = (-1)^{m[F:\mathbb{Q}]}$, Theorem A applies to π when $m[F:\mathbb{Q}]$ is odd.

Nontrivial cycles constructed via the method of arithmetic theta lifting (Kudla, Liu).
Next: a baby example of Heegner points.

The Gross–Zagier formula

- Modular curve

$$X_0(N) = \Gamma_0(N)\backslash\mathbb{H} \cup \{\text{cusps}\} = \{(E_1 \xrightarrow[N\text{-isogeny}]{\text{cyclic}} E_2)\}$$

- For certain imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-d})$, get a Heegner divisor

$$Z(d) := \{(E_1 \to E_2) \text{ with endomorphisms by } O_K\} \in \mathrm{CH}^1(X_0(N)).$$

- The theory of complex multiplication: $Z(d)$ is defined over K.
- $E/\mathbb{Q}$ elliptic curve of conductor N has a modular parametrization

$$\varphi_E : X_0(N) \to E.$$

- Get a Heegner point

$$P_K \in \varphi_E(Z(d) - \deg Z(d) \cdot \infty) \in E(K).$$

Theorem (Gross–Zagier, 1980s)

Up to simpler nonzero factors,

$$L'(E_K, 1) \sim \langle P_K, P_K \rangle_{\mathrm{NT}}.$$

Remark

(1) Choosing K suitably gives the implications $r_{\mathrm{an}}(E) = 1 \Rightarrow r_{\mathrm{alg}}(E) \geq 1$.

(2) BSD formula for E_K reduces to a precise relation between P_K and $\text{Ш}(E_K)$.

Generating series of Heegner points

Take $P_d = \mathrm{tr}_{K/\mathbb{Q}} P_K \in E(\mathbb{Q})$. It may depend on the choice of d, even when $E(\mathbb{Q}) \cong \mathbb{Z}$.

Example ($E = 37a1 = X_0^+(37) : y^2 + y = x^3 - x$)

- $E(\mathbb{Q}) \cong \mathbb{Z}$ with a generator $P = (0, 0)$.
- E corresponds to the modular form $f \in S_2(37)$,

$$f = \sum_{n \geq 1} a_n q^n = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} - 6q^{12} - 2q^{13} + \cdots$$

- Table of Heegner points P_d:

d	3	4	7	11	12	16	27	$\cdots$	67	$\cdots$
P_d	$(0, -1)$	$(0, -1)$	$(0, 0)$	$(0, -1)$	$(0, 0)$	$(1, 0)$	$(-1, -1)$	$\cdots$	$(6, -15)$	$\cdots$
c_d	-1	-1	1	-1	1	2	3	$\cdots$	-6	$\cdots$

where $P_d = c_d \cdot P$.

Miracle. The coefficients c_d appear as the Fourier coefficients of $\phi \in S^+_{3/2}(4 \cdot 37)$,

$$\phi = \sum_{d \geq 1} c_d q^d = -q^3 - q^4 + q^7 - q^{11} + q^{12} + 2q^{16} + 3q^{27} + \cdots - 6q^{67} + \cdots,$$

which maps to f under the Shimura–Waldspurger–Kohnen correspondence

$$\theta : S^+_{3/2}(4N) \to S_2(N), \quad \theta(\phi) = f.$$

Arithmetic theta lifting

- The generating series of Heegner points

$$\sum_{d\geq 1} P_d \cdot q^d = \sum_{d\geq 1} c_d P \cdot q^d = \phi \cdot P \in S^+_{3/2}(4\cdot 37)\otimes E(\mathbb{Q})_{\mathbb{C}}$$

  is a modular form valued in $E(\mathbb{Q})_{\mathbb{C}}$.
- More generally, we may define a generating series of Heegner divisors on $X_0(N)$,

$$Z := \sum_d Z(d) q^d \in M_{3/2}(4N)\otimes \mathrm{CH}^1(X_0(N))_{\mathbb{C}},$$

  which may be viewed as an arithmetic theta series.
- Use Z as the kernel to define arithmetic theta lifting

$$\Theta(\phi) := (Z,\phi)_{\mathrm{Pet}} \in \mathrm{CH}^1(X_0(N))^0_{f,\mathbb{C}} = E(\mathbb{Q})_{\mathbb{C}}.$$

- Now $\Theta(\phi)$ does not depend on any particular choice of d or K.

Theorem (Gross–Kohnen–Zagier, 1980s)

Up to simpler nonzero factors,

$$L'(E,1) \sim \langle \Theta(\phi), \Theta(\phi)\rangle_{\mathrm{NT}}.$$

Special cycles on X

- For any $y \in V$ with $(y, y) > 0$. Its orthogonal complement $V_y \subseteq V$ has rank $n - 1$. The embedding $\mathrm{U}(V_y) \hookrightarrow \mathrm{U}(V)$ defines a Shimura subvariety of codimension 1

$$\mathrm{Sh}_{\mathrm{U}(V_y)} \to X = \mathrm{Sh}_{\mathrm{U}(V)}\,.$$

- For any $x \in V(\mathbb{A}_F^\infty)$ with $(x, x) \in F_{>0}$, there exists $y \in V$ and $g \in \mathrm{U}(V)(\mathbb{A}_F^\infty)$ such that $y = gx$. Define the the special divisor

$$Z(x) \to X$$

  to be g-translate of $\mathrm{Sh}_{\mathrm{U}(V_y)}$.

- For any $\mathbf{x} = (x_1, \ldots, x_m) \in V(\mathbb{A}_F^\infty)^m$ with $T(\mathbf{x}) = ((x_i, x_j)) \in \mathrm{Herm}_m(F)_{>0}$, define the special cycle (of codimension m)

$$Z(\mathbf{x}) = Z(x_1) \cap \cdots \cap Z(x_m) \to X.$$

- More generally, for a Schwartz function $\varphi \in \mathcal{S}(V(\mathbb{A}_F^\infty)^m)^K$ and $T \in \mathrm{Herm}_m(F)_{>0}$, define the weighted special cycle

$$Z_\varphi(T) = \sum_{\substack{\mathbf{x} \in K \backslash V(\mathbb{A}_F^\infty)^m \\ T(\mathbf{x}) = T}} \varphi(\mathbf{x}) Z(\mathbf{x}) \in \mathrm{CH}^m(X)_{\mathbb{C}}.$$

- With extra care, we can also define $Z_\varphi(T) \in \mathrm{CH}^m(X)_{\mathbb{C}}$ for any $T \in \mathrm{Herm}_m(F)_{\geq 0}$.

Arithmetic theta lifting

Define Kudla's generating series of special cycles

$$Z_\varphi(\tau) = \sum_{T \in \mathrm{Herm}_m(E)_{\geq 0}} Z_\varphi(T) q^T.$$

Conjecture (Kudla's modularity, 1990s)

The formal generating series $Z_\varphi(\tau)$ converges absolutely and defines a modular form on $\mathrm{U}(W)$ valued in $\mathrm{CH}^m(X)_\mathbb{C}$.

Remark

(1) The analogous modularity in Betti cohomology is known (Kudla–Millson, 1980s).

(2) Conjecture is known for $m = 1$. For general m, the modularity follows from the absolute convergence (Liu, 2011).

(3) The analogous conjecture for orthogonal Shimura varieties over $\mathbb{Q}$ is known (Bruinier–Raum, 2014).

(4) Conjecture is known when E/F is a norm-Euclidean imaginary quadratic field (Xia, 2021).

Assuming Kudla's modularity conjecture, for $\phi \in \pi$, define arithmetic theta lifting

$$\Theta_\varphi(\phi) = (Z_\varphi(\tau), \phi)_{\mathrm{Pet}} \in \mathrm{CH}^m(X)^0_\pi.$$

Main result B: Arithmetic inner product formula

Theorem B (L.–Liu, 2020, 2021)

Let π be a cuspidal automorphic representation of $\mathrm{U}(W)(\mathbb{A}_F)$ satisfying Assumptions. Assume $\varepsilon(\pi) = -1$. Assume Kudla's modularity. Then for any $\phi \in \pi$ and $\varphi \in \mathcal{S}(\mathbb{V}(\mathbb{A}_F^\infty)^m)$, up to simpler factors depending on ϕ and φ,

$$L'(1/2, \pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}}.$$

Remark. The simpler factors can be further made explicit. For example, if

- π: unramified or almost unramified at all finite places,
- $\phi \in \pi$: holomorphic newform such that $(\phi, \overline{\phi})_\pi = 1$,
- φ: characteristic function of self-dual or almost self-dual lattices at all finite places.

Then $$\frac{L'(1/2,\pi)}{\prod_{i=1}^{2m} L(i, \eta_{E/F}^i)} C_m^{[F:\mathbb{Q}]} \prod_{v \in S_\pi} \frac{q_v^{m-1}(q_v+1)}{(q_v^{2m-1}+1)(q_v^{2m}-1)} = (-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}},$$

where $C_m = 2^{m(m-1)} \pi^{m^2} \frac{\Gamma(1)\cdots\Gamma(m)}{\Gamma(m+1)\cdots\Gamma(2m)}$.

Remark

- Riemann hypothesis predicts $L'(1/2, \pi) \geq 0$.
- Beilinson's Hodge index conjecture predicts $(-1)^m \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}} \geq 0$.

Compatible with our formula!

Summary

BSD conjecture	BB conjecture
Modular curves $X_0(N)$	Unitary Shimura varieties X
Heegner points $Z(d)$	Special cycles $Z_\varphi(T)$
$Z = \sum_d Z(d)q^d \in \mathrm{CH}^1(X_0(N))_\mathbb{C}$	$Z_\varphi = \sum_T Z_\varphi(T)q^T \in \mathrm{CH}^m(X)_\mathbb{C}$
$\Theta(\phi) \in E(\mathbb{Q})_\mathbb{C}$	$\Theta_\varphi(\phi) \in \mathrm{CH}^m(X)^0_\pi$
Gross–Zagier formula $L'(E,1) \sim \langle \Theta(\phi), \Theta(\phi) \rangle_{\mathrm{NT}}$	Arithmetic inner product formula $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}}$

Proof strategy: doubling method

- Doubling method (Piatetski-Shapiro–Rallis, Yamana)

$$L(s+1/2,\pi) \sim (\phi \otimes \overline{\phi}, \mathrm{Eis}(s,g))_{\mathrm{U}(W)^2},$$

  where $\mathrm{Eis}(s,g)$ is a Siegel Eisenstein series on $\mathrm{U}(W \oplus W)$.
- By definition $\Theta_\varphi(\phi) = (Z_\varphi, \phi)_{\mathrm{Pet}}$ gives

$$\langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}} = (\phi \otimes \overline{\phi}, \langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB}})_{\mathrm{U}(W)^2}.$$

- To prove $L'(1/2,\pi) \sim \langle \Theta_\varphi(\phi), \Theta_\varphi(\phi) \rangle_{\mathrm{BB}}$, it suffices to compare

$$\mathrm{Eis}'(0,g) \stackrel{?}{=} \langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB}}.$$

  This can be viewed as an arithmetic Siegel–Weil formula.
- The Beilinson–Bloch height pairing is a sum of local indexes

$$\langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB}} = \sum_v \langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB},v}.$$

- The nonsingular Fourier coefficient decomposes as

$$\mathrm{Eis}'_T(0,g) = \sum_v \mathrm{Eis}'_{T,v}(0,g)$$

Proof strategy: comparison

- Nonsingular terms: it suffices to compare

$$\mathrm{Eis}'_{T,v}(0,g) \stackrel{?}{=} \langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB},T,v}.$$

- Gross–Zagier ($m = 1$): compute both sides explicitly.
- Explicit computation infeasible for general m.
- $v \nmid \infty$
  (1) relate $\langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB},T,v}$ to arithmetic intersection numbers.
  (2) recent proof of Kudla–Rapoport conjecture (L.–Zhang).
- $v \mid \infty$:
  (1) archimedean arithmetic Siegel–Weil formula (Liu, Garcia–Sankaran).
  (2) avoidance of holomorphic projections.

To finish:

- Kill singular terms on both sides: Prove the existence of special $\varphi \in \mathcal{S}(V(\mathbb{A}_F^\infty)^m)$ with regular support at two split places with nonvanishing local zeta integrals.
- Theorem B for special φ: comparison of nonsingular terms.
- Theorem B for arbitrary φ: multiplicity one of doubling method (tempered case).
- Theorem A: same computation without Kudla's modularity (proof by contradiction).

Remarks on Assumptions

- When $v \nmid \infty$, the local index $\langle\ ,\ \rangle_{\mathrm{BB},v}$ is defined as a ℓ-adic linking number. It is defined on a certain subspace $\mathrm{CH}^m(X)^{\langle \ell \rangle} \subseteq \mathrm{CH}^m(X)^0$ (conjecturally equal) and its independence on ℓ is not known in general.
- Find a Hecke operator $t \notin \mathfrak{m}_\pi$ such that $t^*Z \in \mathrm{CH}^m(X)^{\langle \ell \rangle}$, so BB height is defined.
- Find another Hecke operator $s \notin \mathfrak{m}_\pi$, so BB height of s^*t^*Z can be computed in terms of the arithmetic intersection number of a nice extension $\mathcal{Z}$ on $\mathcal{X}$. Here $\mathcal{X}$ is a regular integral model of a related unitary Shimura variety of PEL type. This step requires to prove certain vanishing of $\mathfrak{m}_\pi$-localized ℓ-adic cohomology of $\mathcal{X}$.
- Kudla–Rapoport conjecture: arithmetic intersection number equals $\mathrm{Eis}'_{T,v}(0, g)$.
- The ℓ-independence of $\langle Z_\varphi, Z_\varphi \rangle_{\mathrm{BB},T,v}$ then follows.
- Construction of Hecke operators and the proof of Kudla–Rapoport conjecture requires Assumptions.
- $F \neq \mathbb{Q}$ is needed to prove vanishing of $\mathfrak{m}_\pi$-localized cohomology of integral models with Drinfeld level structures at split places (with input from Mantovan, Caraiani–Scholze).