Stochastic quantum integrable systems

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A physicist's guide to solving the Kardar–Parisi–Zhang equation

KPZ: $\frac{\partial h}{\partial t} = \frac{1}{2}\frac{\partial^2 h}{\partial x^2} + \left(\frac{\partial h}{\partial x}\right)^2 + \dot{W}$ SHE: $\frac{\partial Z}{\partial t} = \frac{1}{2}\frac{\partial^2 Z}{\partial x^2} + \dot{W}\cdot Z$

1. Think of the Cole–Hopf transform instead: $Z = e^h$ solves the SHE
2. Look at the moments $\langle Z(t,x_1)\cdots Z(t,x_k)\rangle$. They are solutions of the quantum delta Bose gas evolution [Kardar '87], [Molchanov '87].

$$\frac{\partial}{\partial t}\langle Z(t,x_1)\cdots Z(t,x_k)\rangle = \frac{1}{2}\left(\sum_{i=1}^{k}\frac{\partial^2}{\partial x_i^2} + \sum_{i\neq j}\delta(x_i - x_j)\right)\langle Z(t,x_1)\cdots Z(t,x_k)\rangle$$

3. Use Bethe ansatz to solve it [Bethe '31], [Lieb–Liniger '63], [McGuire '64], [Yang '67], [Oxford '79] [Heckman–Opdam '97]
4. Reconstruct solution using the known moments: The replica trick. [Calabrese–Le Doussal–Rosso '10+], [Dotsenko '10+]

Possible mathematician's interpretation. Be wise – discretize!

1. Start with a good discrete system that converges to KPZ.

2. Find 'moments' that solve an integrable autonomous system of equations (i.e., find a Markov duality).

3. Use Bethe ansatz to solve, for arbitrary initial data

4. Reconstruct the solution using the known 'moments' and take the limit to KPZ/SHE. A mathematically rigorous replica trick.

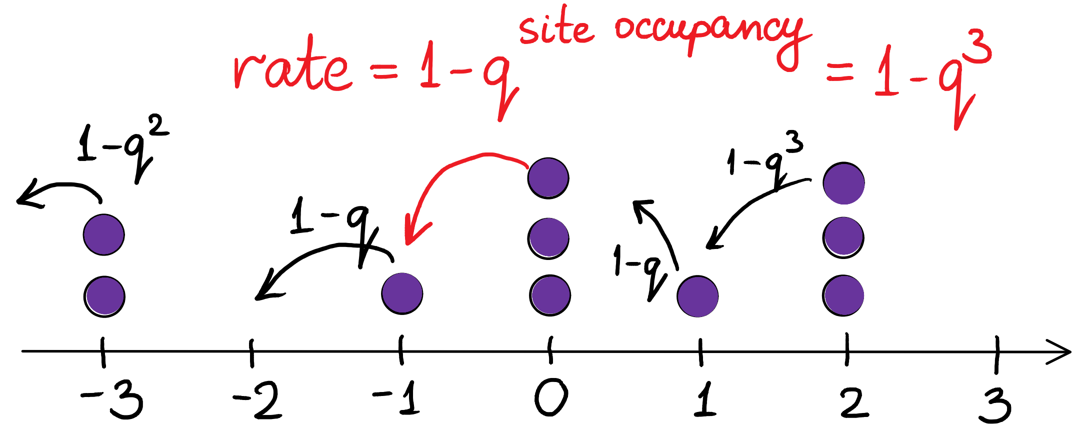
We can do 1-3 for a few systems: q-TASEP, ASEP, q-Hahn TASEP, higher-spin vertex models. So far we can do 4 only for very special initial conditions.

q-Boson process [Sasamoto-Wadati '98]

At each location top particle jumps to the left by one indep. with rates $1-q^{\#\text{ of particles at the site}}$.

The generator is $\left(\vec{n}_j^- = (\dots, n_j - 1, \dots)\right)$

$$(Hf)(\vec{n}) = \sum_{\text{clusters } i} \left(1-q^{c_i}\right)\left(f\left(\vec{n}^-_{c_1+\dots+c_i}\right) - f(\vec{n})\right)$$



particles at $\vec{n} = (2,2,2,1,0,0,0,-1,-3,-3)$

clusters $\vec{c} = (3,1,3,1,2)$

Restricted to k particles, if $u: \mathbb{Z}^k \to \mathbb{C}$ satisfies boundary conditions

$$\left(\nabla_i - q\nabla_{i+1}\right) u \Big|_{n_i = n_{i+1}} = 0 \qquad \text{for all } 1 \le i \le k-1$$

then restricted to ordered $\vec{n}$, $(Hu)(\vec{n}) = (\mathcal{L}u)(\vec{n})$ where

$$(\mathcal{L}u)(\vec{n}) = (1-q)\sum_{i=1}^{k} (\nabla_i u)(\vec{n}),$$

∇_i is $(\nabla f)(x) = f(x-1) - f(x)$ acting in n_i

q-Boson eigenfunctions

Bethe ansatz and PT-invariance yields $(z_1, \ldots, z_k \in \mathbb{C} \setminus \{1\})$

$$\Psi^{l}_{\vec{z}}(\vec{n}) = \sum_{\sigma \in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(\frac{1}{1 - z_{\sigma(j)}} \right)^{n_j}$$

$$\Psi^{r}_{\vec{z}}(\vec{n}) = \frac{1}{C_q(\vec{n})} \sum_{\sigma \in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q^{-1} z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(1 - z_{\sigma(j)} \right)^{n_j}$$

with $C_q(\vec{n}) = (-1)^k q^{-k(k-1)/2} \frac{(q;q)_{c_1}}{(1-q)^{c_1}} \frac{(q;q)_{c_2}}{(1-q)^{c_2}} \cdots$, and eigenvalues

$$H \Psi^{l}_{\vec{z}} = (q-1) \sum_{j=1}^{k} z_j \, \Psi^{l}_{\vec{z}}, \qquad H^{T} \Psi^{r}_{\vec{z}} = (q-1) \sum_{j=1}^{k} z_j \, \Psi^{r}_{\vec{z}} .$$

Direct and inverse Fourier type transforms

Let $\mathcal{W}^k = \{ f : \{ n_1 \geq \ldots \geq n_k \mid n_j \in \mathbb{Z} \} \to \mathbb{C} \text{ of compact support} \}$

$\mathcal{C}^k = \mathbb{C}\left[\left(\frac{1}{1-z_1}\right)^{\pm 1}, \ldots, \left(\frac{1}{1-z_k}\right)^{\pm 1} \right]^{S(k)}$ = symmetric Laurent poly's in $\left(\frac{1}{1-z_j}\right)$, $1 \leq j \leq k$.

Direct tranform: $\mathcal{F} : \mathcal{W}^k \to \mathcal{C}^k$

$$\mathcal{F} : f \longmapsto \sum_{n_1 \geq \ldots \geq n_k} f(\vec{n}) \cdot \Psi^r_{\vec{z}}(\vec{n})$$

Inverse transform: $\mathcal{J} : \mathcal{C}^k \to \mathcal{W}^k$

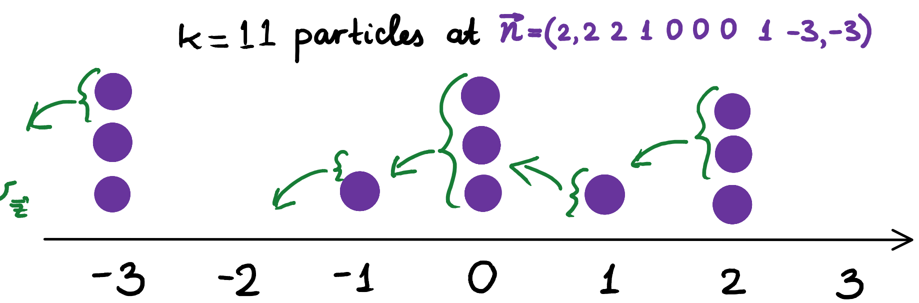
$$\mathcal{J} : G \longmapsto (q-1)^k q^{-\frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k k!} \oint_{|w_j| = R > 1} \cdots \oint \det\left[\frac{1}{q w_i - w_j} \right]_{i,j=1}^{k} \prod_{j=1}^{k} \frac{w_j}{1 - w_j} \Psi^\ell_{\vec{w}}(\vec{n}) G(\vec{w}) \, d\vec{w}$$

Theorem [Borodin-C-Petrov-Sasamoto '13] On spaces $\mathcal{W}^k$ and $\mathcal{C}^k$, operators $\mathcal{F}$ and $\mathcal{J}$ are mutual inverses of each other.

Back to the q-Boson particle system

Corollary The (unique) solution of the q-Boson evolution equation

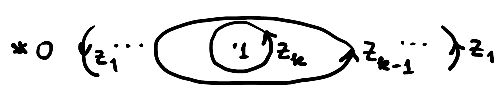
$\partial_t f(t,\vec{n}) = H f(t,\vec{n})$ with $f(0,\vec{n}) = f_0(\vec{n})$ is



eigenvalue of H corr. to $\Psi_{\vec{z}}$

$$f(t,\vec{n}) = \mathcal{J}\left(e^{\overbrace{(q-1)\sum_{j=1}^{k} z_j}\, t}\, \mathcal{F} f_0\right)$$

$$= \frac{1}{(2\pi i)^k} \oint \cdots \oint \prod_{a<b} \frac{z_a - z_b}{z_a - q z_b} \prod_{j=1}^{k} \left(\frac{1}{1-z_j}\right)^{n_j+1} e^{\overbrace{(q-1)\sum_{j=1}^{k} z_j}\, t}\, \mathcal{F} f_0(\vec{z})\, d\vec{z}$$

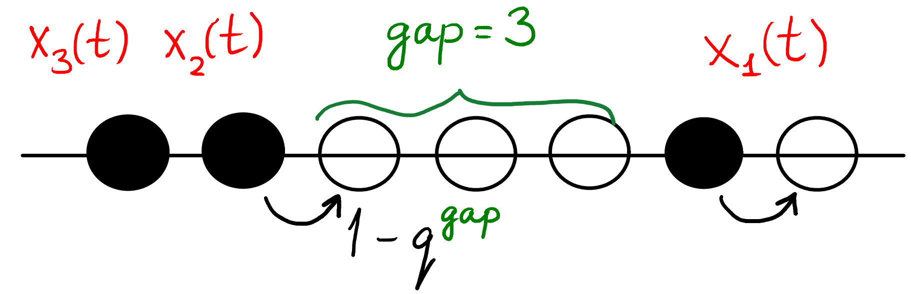


The computation of $\mathcal{F} f_0$ can still be difficult. It is, however, automatic if $f_0 = \mathcal{J} G \Rightarrow \mathcal{F} f_0 = \mathcal{F}\mathcal{J} G = G$.

Eg: For initial data $f_0(\vec{n}) = \mathbb{1}_{\{n_i \geq 1,\, 1 \leq i \leq k\}}$, $G(\vec{z}) = q^{k(k-1)/2} \prod_{j=1}^{k} \frac{z_j - 1}{z_j}$.

q-TASEP [Borodin-C '11]

Particles jump right by one according to exponential clocks of rate $1-q^{gap}$*.*



Proposition [Borodin-C-Sasamoto '12] For q-TASEP with finitely many particles on the right, $f(t,\vec{n}) = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(t)+n_j}\right]$ is the unique solution of

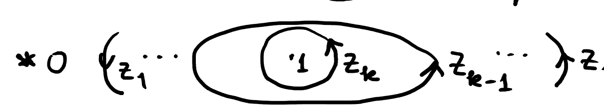
$$\frac{d}{dt} f(t,\vec{n}) = (Hf)(t,\vec{n}), \qquad f(0,\vec{n}) = \mathbb{E}\left[\prod_{j=1}^{k} q^{x_{n_j}(0)+n_j}\right].$$

Theorem [B-C '11], [B-C-Sasamoto '12], [B-C-Gorin-Shakirov '13]

For the q-TASEP with step initial data $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\left[q^{(x_{n_1}(t)+n_1)+\dots+(x_{n_k}(t)+n_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)t z_j}}{(1-z_j)^{n_j}} \frac{dz_j}{z_j}$$

$(n_1 \geq n_2 \geq \dots \geq n_k)$



Starting point for KPZ asymptotics.

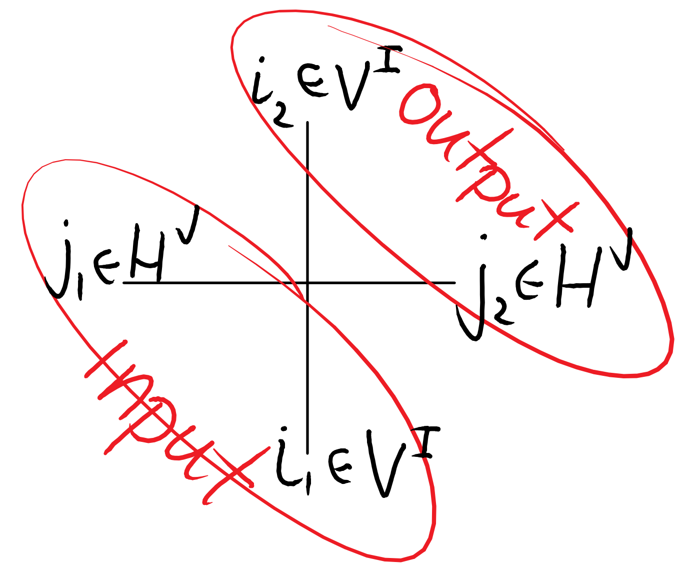
Defining the L-matrix

<u>Vector spaces</u>: $$V^I = \begin{cases} \mathrm{span}(0,1,\dots,I) & , I \in \mathbb{Z}_{\geq 0} \\ \mathrm{span}(0,1,\dots\ ) & , \text{else} \end{cases}$$

(H^J likewise)

<u>L-matrix</u>: Indexed by complex parameters q, α, I, J such that

$$L : V^I \otimes H^J \to V^I \otimes H^J .$$

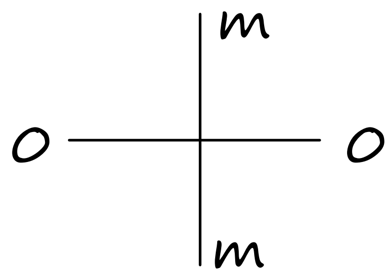


<u>L-matrix elements</u>: $L(i_1, j_1 ; i_2, j_2)$ indexed by $i_1, i_2 \in V^I$ and $j_1, j_2 \in H^J$.

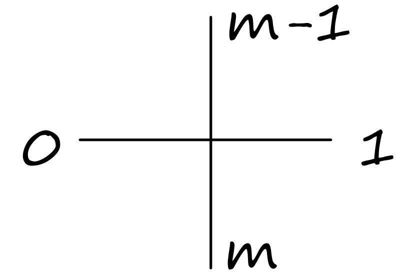
For most of the talk, we will set $J=1, \nu = q^{-I}$ and write $L_\alpha^{(J)}$.

L-matrix elements

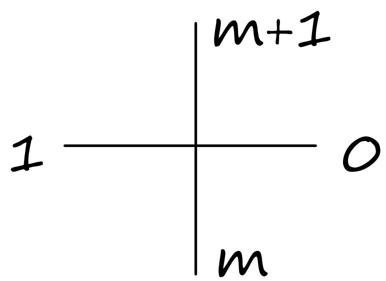
Definition: For $J=1$ and $m \geq 0$, the non-zero entries of $L^{(1)}_{\alpha}$ are:



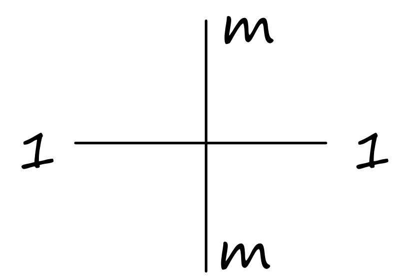
$$L^{(1)}_{\alpha}(m,0;m,0) = \frac{1+\alpha q^{m}}{1+\alpha}$$



$$L^{(1)}_{\alpha}(m,0;m-1,0) = \frac{\alpha(1-q^{m})}{1+\alpha}$$



$$L^{(1)}_{\alpha}(m,1;m+1,0) = \frac{1-\nu q^{m}}{1+\alpha}$$



$$L^{(1)}_{\alpha}(m,1;m,1) = \frac{\alpha+\nu q^{m}}{1+\alpha}$$

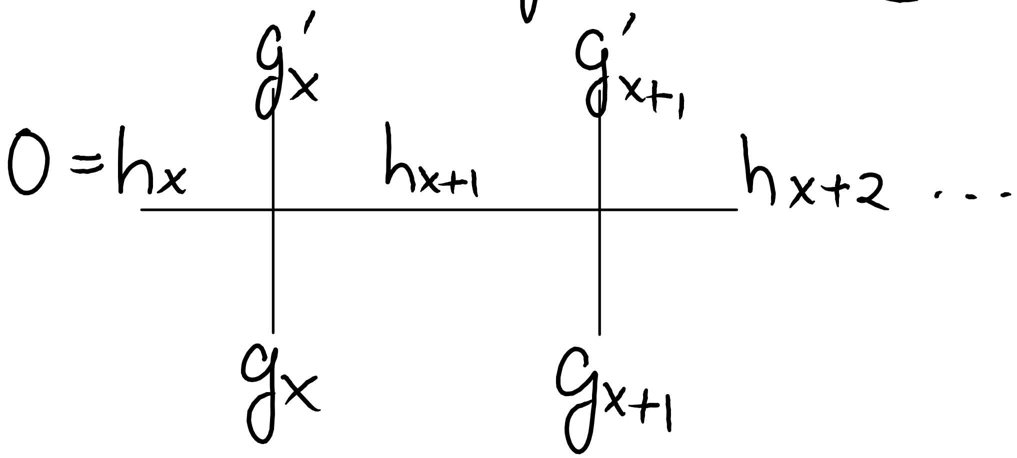
Particle conservation: sum of inputs i_1+j_1 equals sum of outputs i_2+j_2.

Stochasticity: Given i_1, j_1, sum over i_2, j_2 equals 1; positive entries if:

1) $q, \nu \in (0,1)$, $\alpha > 0$,
2) $q \in (1,\infty)$, $\nu = q^{-I}$ for $I \in \mathbb{Z}_{\geq 1}$, $\alpha \in (-\nu, 0)$.

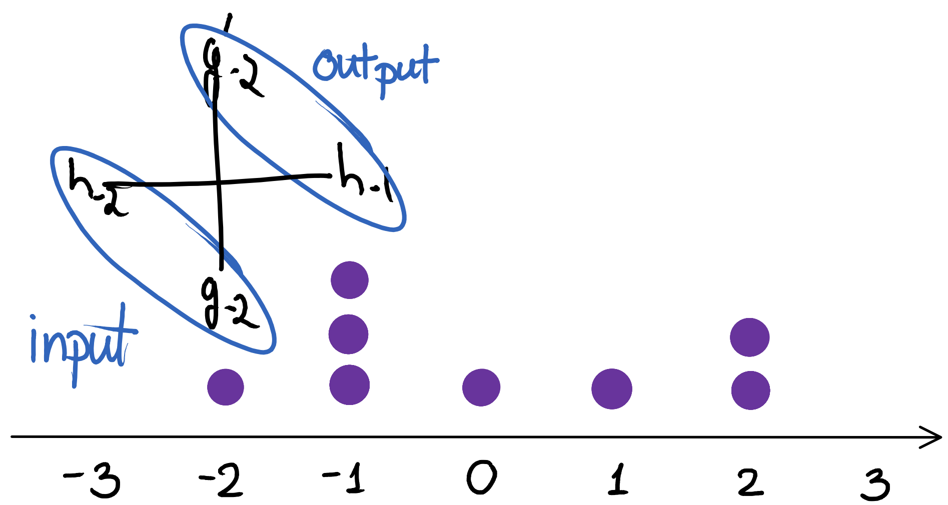
Zero range process (stochastic transfer matrix)

<u>ZRP</u>: State $\vec{g} = (g_i)_{i \in \mathbb{Z}}$, $g_i \in \mathbb{Z}_{\geq 0}$, $\exists x$ s.t. $g_x > 0$, $g_y = 0$ $\forall y < x$.



Seq'n (left->right) update via $L^{(1)}_\alpha$ Markov chain so given g_x, h_x choose g'_x, h_{x+1} with probability $L^{(1)}_\alpha(g_x, h_x; g'_x, h_{x+1})$.

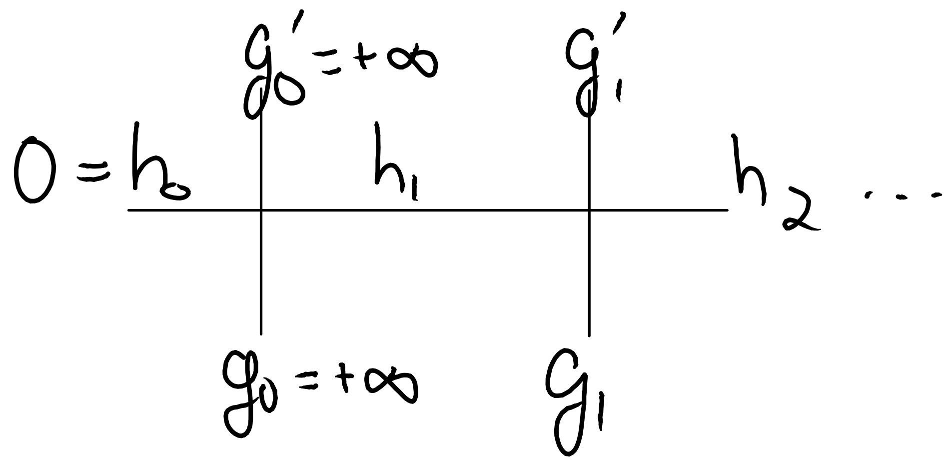
(dynamics conserve sum of g's, and h's =0 or 1)



Call $B^{\alpha, q\alpha}(\vec{g}, \vec{g}')$ transition probability / matrix and define its space reversal $\widetilde{B}^{\alpha, q\alpha}(\vec{y}, \vec{y}')$ with state variables $\vec{y}$.

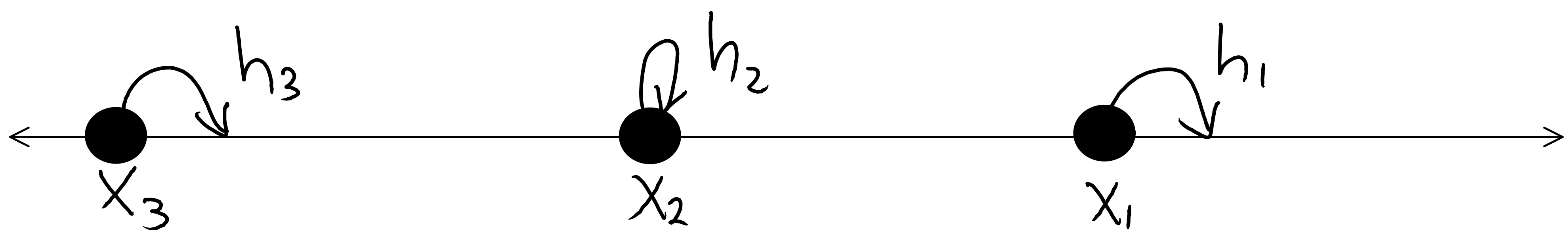
Asymmetric exclusion process

AEP: State $\vec{x} = (x_1 > x_2 > \cdots)$, $x_i \in \mathbb{Z}$, $x_i \equiv +\infty$, $i \le 0$ (need V^I inf. dim)



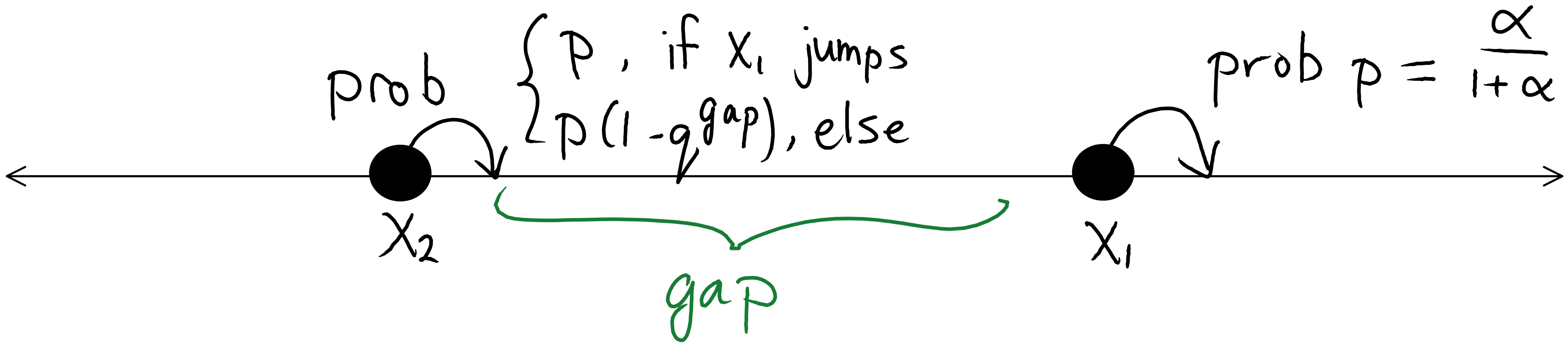
ZRP on gaps: Let $g_i = x_i - x_{i+1} - 1$ and update $\vec{g} \to \vec{g}'$ via ZRP. Set $x_i' = x_i + h_i$.

Call $T^{\alpha, q\alpha}(\vec{x}, \vec{x}')$ transition probability / matrix for the AEP.

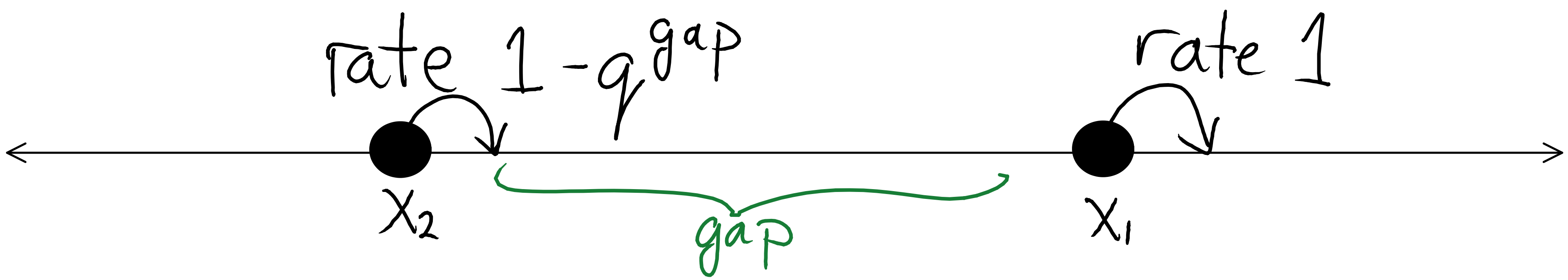


Bernoulli q-TASEP

Take $\nu = 0$, $q \in (0,1)$, $\alpha > 0$ then the AEP becomes



Taking p->0, jumps become seldom and speeding up by 1/p we recover the continuous time q-TASEP [Borodin-C '11]

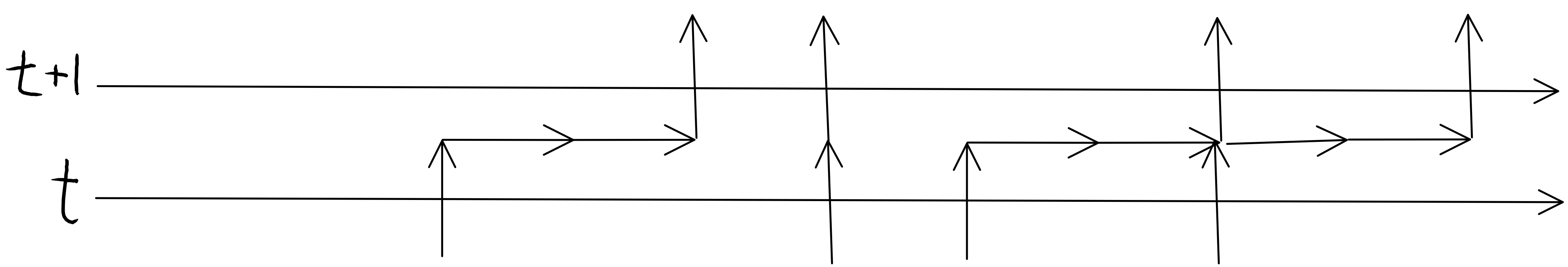


Stochastic six vertex model

Take $\nu = q^{-1}$ $(I=1)$, $q \in (1,\infty)$, $\alpha \in (-\nu, 0)$. The six non-zero weights depend on q, α and can be reparameterized via $b_1, b_2 \in (0,1)$ as

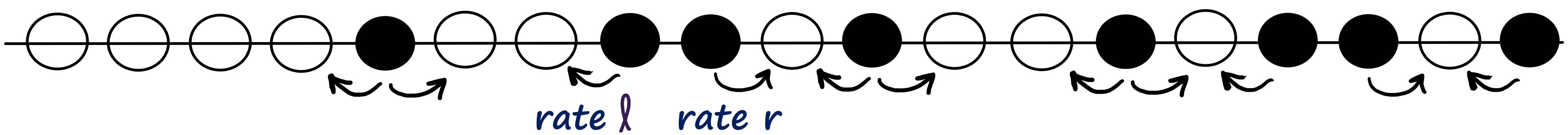
weight 1 1 b_1 $1-b_1$ b_2 $1-b_2$

ZRP obeys exclusion rule [Gwa-Spohn '92], [Borodin-C-Gorin '14].



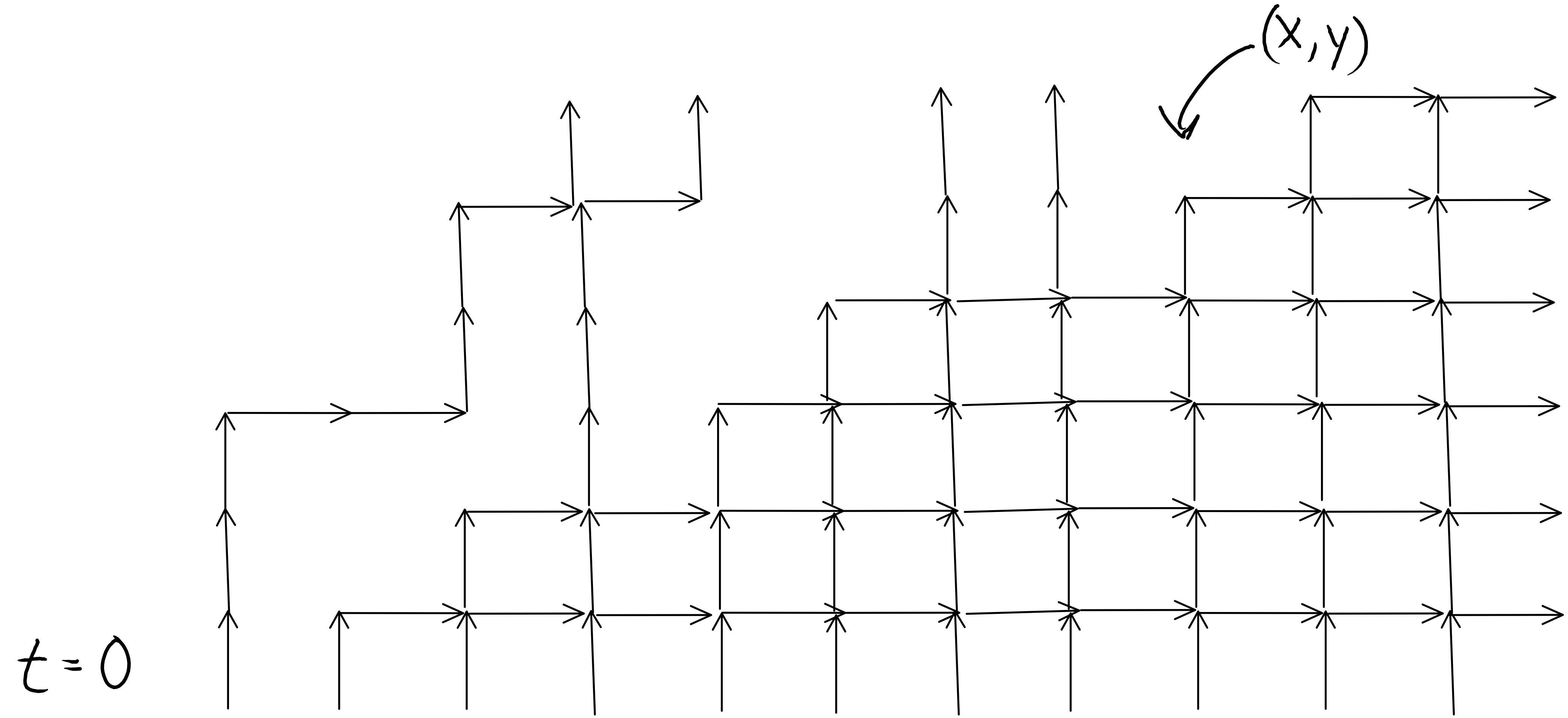
ASEP limits

The ratio $\frac{b_2}{b_1}=\nu$. Fixing this and taking $b_1, b_2 \searrow 0$ $(\alpha \searrow -\nu)$ Particles almost always follow a trajectory. Subtracting this diagonal motion and speeding up time by $1/b$ we arrive at ASEP with left jump rate ℓ and right jump rate r having ratio $\frac{r}{\ell}=\nu$.



Thus we have **united q-TASEP and ASEP as processes**.

Half domain wall boundary conditions (step initial data)



Start stochastic six-vertex with $g_i(0) = 1_{i \geq 0}$ and define a height function: $H(x,y) = \#$ lines left of (x,y).

Asymptotics

Theorem [Borodin-C-Gorin '14]: For $0 < b_2 < b_1 < 1$, $\kappa := \frac{1-b_1}{1-b_2}$ we have

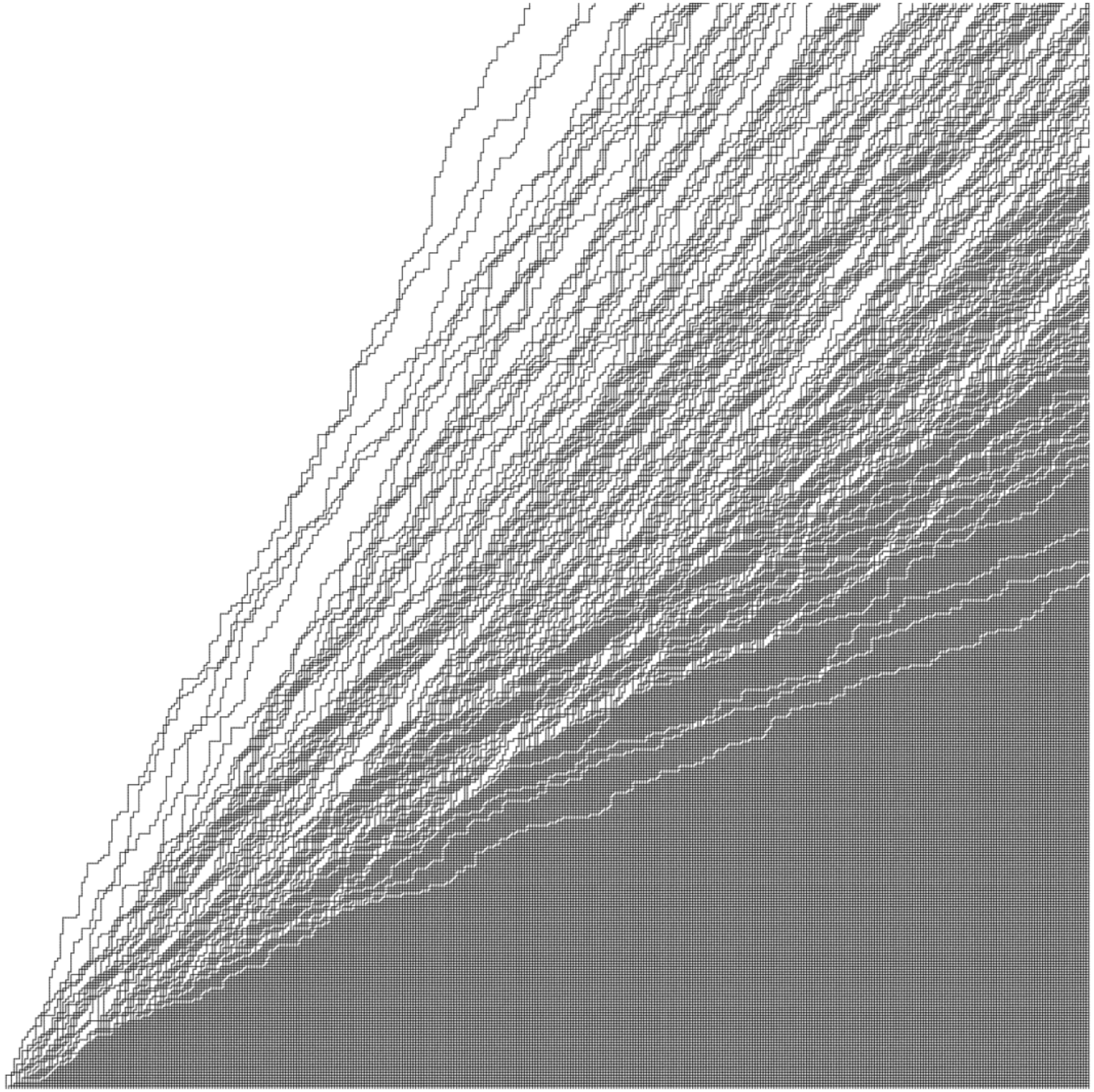
Law of large numbers:

$$\lim_{L\to\infty} \frac{H(Lx,Ly)}{L} = \mathcal{H}(x,y) := \begin{cases} 0 & , \frac{x}{y} < \kappa \\ \left(\sqrt{y(1-b_1)} - \sqrt{x(1-b_2)}\right)^2 & , \kappa < \frac{x}{y} < \frac{1}{\kappa} \\ x-y & , \frac{1}{\kappa} < \frac{x}{y} \end{cases}$$

Central limit theorem: For $\kappa < \frac{x}{y} < \frac{1}{\kappa}$,

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{\mathcal{H}(x,y)L - H(Lx,Ly)}{\sigma_{x,y} L^{1/3}} \le s \right) = F_{GUE}(s)$$

Asymptotics (simulations by Leonid Petrov)



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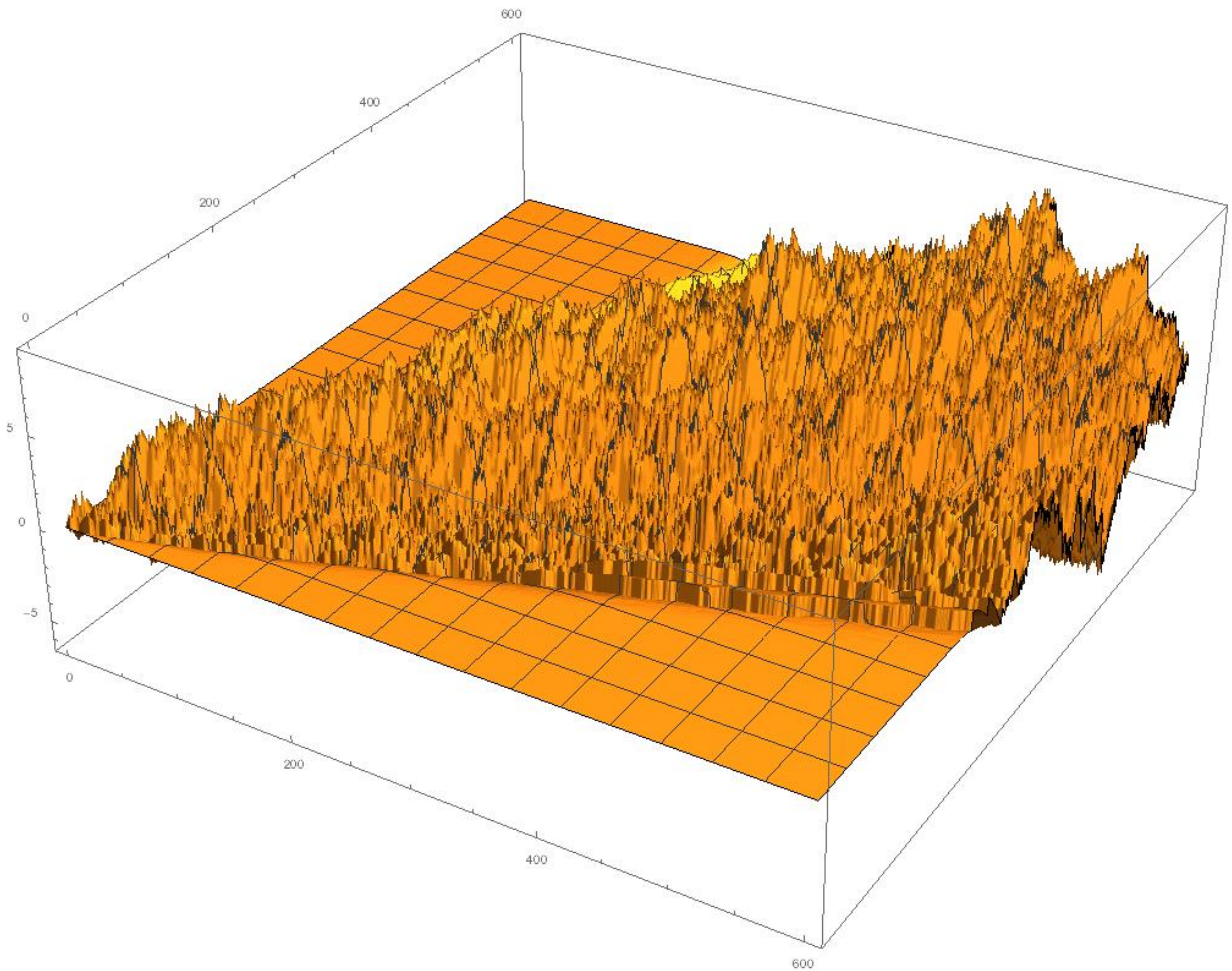


Bethe ansatz diagonalization

Consider the space-reverse ZRP with k particles ($\sum y_i = k$) and label stated by $(n_1 \geq n_2 \geq \cdots \geq n_k) = \vec{n}$. Define the left eigenfunction:

$$\Psi^{\ell}_{\vec{z}}(\vec{n}) = \sum_{\sigma \in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(\frac{1 - \nu z_{\sigma(j)}}{1 - z_{\sigma(j)}} \right)^{n_j}$$

indexed by $z_1, \ldots, z_k \in \mathbb{C} \setminus \{1, \nu^{-1}\}$ and depending on q, ν only.

Theorem [Borodin '14]: For $z_i : \left| \frac{1-z_i}{1-\nu z_i} \cdot \frac{\alpha+\nu}{\alpha+1} \right| < 1, \quad i = 1, \ldots, k$

$$\left(\widetilde{B}^{\alpha, q\alpha} \Psi^{\ell}_{\vec{z}} \right)(\vec{n}) = \prod_{i=1}^{k} \frac{1 + q\alpha z_i}{1 + \alpha z_i} \Psi^{\ell}_{\vec{z}}(\vec{n}).$$

Plancherel theory given in [Borodin-C-Petrov-Sasamoto '14] can be used to solve Kolmogorov forward and backward equation.

Direct and inverse Fourier type transforms

Let

$$\mathcal{W}^k = \{ f : \{ n_1 \geq \ldots \geq n_k \mid n_j \in \mathbb{Z} \} \to \mathbb{C} \text{ of compact support} \}$$

$$\mathcal{C}^k = \mathbb{C}\left[\left(\frac{1-\nu z_1}{1-z_1}\right)^{\pm 1}, \ldots, \left(\frac{1-\nu z_k}{1-z_k}\right)^{\pm 1} \right]^{S(k)} = \text{symmetric Laurent poly's in } \left(\frac{1-\nu z_j}{1-z_j}\right),\ 1 \leq j \leq k.$$

Direct tranform: $\mathcal{F} : \mathcal{W}^k \to \mathcal{C}^k$

$$\mathcal{F} : f \longmapsto \sum_{n_1 \geq \ldots \geq n_k} f(\vec{n}) \cdot \Psi^r_{\vec{z}}(\vec{n}) =: \langle f, \Psi^r_{\vec{z}} \rangle_{\mathcal{W}}$$

Inverse transform: $\mathcal{J} : \mathcal{C}^k \to \mathcal{W}^k$

$$\mathcal{J} : G \longmapsto (q-1)^k q^{-\frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k k!} \underset{\substack{|w_j| = R \in (1, \nu^{-1}) \\ j=1,\ldots,k}}{\oint \cdots \oint} \det \left[\frac{1}{q w_i - w_j} \right]_{i,j=1}^{k} \prod_{j=1}^{k} \frac{w_j}{(1-w_j)(1-\nu w_j)} \Psi^{\ell}_{\vec{w}}(\vec{n}) G(\vec{w}) \, d\vec{w}$$

$$=: \langle \Psi^{\ell}(\vec{n}), G \rangle_{\mathcal{C}}$$

Plancherel isomorphism theorem

Theorem [Borodin-C-Petrov-Sasamoto '14] On spaces $\mathcal{W}^k$ and $\mathcal{C}^k$, operators $\mathcal{F}$ and $\mathcal{J}$ are mutual inverses of each other.

Isometry:

$$\langle f, g\rangle_{\mathcal{W}} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{\mathcal{C}} \quad \text{for } f,g \in \mathcal{W}^k$$

$$\langle F, G\rangle_{\mathcal{C}} = \langle \mathcal{J}F, \mathcal{J}G\rangle_{\mathcal{W}} \quad \text{for } F,G \in \mathcal{C}^k$$

Biorthogonality:

$$\langle \Psi^{\ell}_{\bullet}(\vec{m}), \Psi^{r}_{\bullet}(\vec{n})\rangle_{\mathcal{C}} = \delta_{\vec{m},\vec{n}}$$

in a certain weak sense

$$\langle \Psi^{\ell}_{\vec{z}}(\cdot), \Psi^{r}_{\vec{w}}(\cdot)\rangle_{\mathcal{W}} = \prod_{a\neq b} \frac{z_a - q z_b}{z_a - z_b} \prod_{j=1}^{k} (1-z_j)(1-\nu z_j) \det\left[\delta(z_i - w_j)\right]_{i,j=1}^{k}$$

Proof of $\mathcal{J}\mathcal{F} = \mathrm{Id}$ uses residue calculus in nested contour version of $\mathcal{J}$, while proof of $\mathcal{F}\mathcal{J} = \mathrm{Id}$ uses existence of simultaneously diagonalized family of matrices

ASEP/XXZ degeneration of the Plancherel theorem

Specializing $\nu = q^{-1}$ (require some care since now $\nu > 1$) yields ASEP eigenfunctions:

- $\mathcal{J}\mathcal{F} = \mathrm{Id}$ becomes equivalent to the time zero version of [Tracy-Widom '08] k-particle ASEP transition probability result

$$P_{\vec{y}}(t;\vec{x}) = \sum_{\sigma \in S(k)} \oint \cdots \oint A_\sigma(\xi) \prod_{j=1}^{k} \xi_{\sigma(j)}^{x_j - y_{\sigma(j)} - 1} e^{(p\xi_j^{-1} + q\xi_j - 1)t} d\xi_j \,, \qquad A_\sigma(\xi) = \prod_{\substack{\{\alpha,\beta\} \\ \text{inversions} \\ \text{of } \sigma}} - \frac{p + q\xi_\alpha\xi_\beta - \xi_\alpha}{p + q\xi_\beta\xi_\alpha - \xi_\beta}$$

- $\mathcal{F}\mathcal{J}G = G$ applied to a certain G yields TW 'magical-identity'.

XXZ in k-magnon sector is similarity transform of ASEP, so we recover results of [Babbitt-Gutkin '90] (the proof of which seems to be lost in the literature).

Further limit to XXX in k-magnon sector [Babbitt-Thomas '77]

AEP – ZRP Markov duality

Define a duality functional $H(\vec{x},\vec{y}) := \prod_{i\in\mathbb{Z}} q^{(x_i+i)y_i}$ $\left(= 0 \text{ if } y_i > 0 \text{ for any } i \le 0\right)$

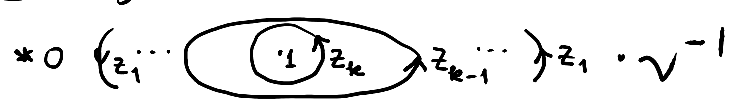
Theorem [C-Petrov '15]: $T^{\alpha,q\alpha} H = H\left(\tilde{B}^{\alpha,q\alpha}\right)^T$

Corollary: $\mathbb{E}\left[H(\vec{x}(t),\vec{y})\right] = \left(\tilde{B}^{\alpha,q\alpha}\right)^t \mathbb{E}\left[H(\vec{x}(0),\vec{y})\right]$

Corollary: For the AEP with step initial data $\{x_n(0)=-n\}_{n\ge 1}$

$$\mathbb{E}\left[q^{(x_{n_1}(t)+n_1)+\dots+(x_{n_k}(t)+n_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{A<B} \frac{z_A-z_B}{z_A-qz_B} \prod_{j=1}^{k} \left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j} \left(\frac{1+q\alpha z_j}{1+\alpha z_j}\right)^t \frac{dz_j}{z_j}$$

$(n_1 \ge n_2 \ge \dots \ge n_k)$



This is the starting point for distributional formulas and asymptotics.

J>1 via fusion

Define the higher horizontal spin ZRP transition operator as

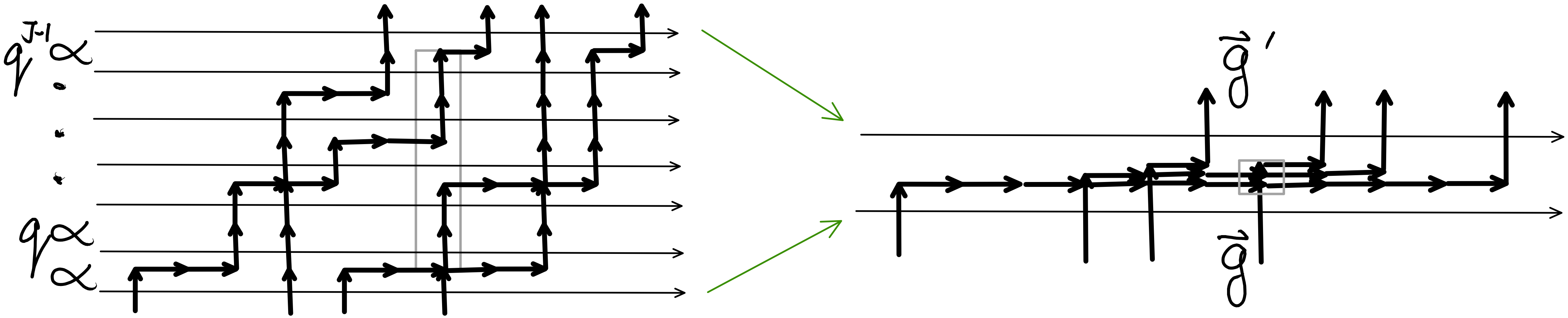
$$B^{\alpha, q^J\alpha} := B^{\alpha, q\alpha} B^{q\alpha, q^2\alpha} \cdots B^{q^{J-1}\alpha, q^J\alpha}.$$

Clearly this is still stochastic (if each B was) and it is diagonalized via the same eigenfunctions with eigenvalue $\prod_{j=1}^{k} \frac{1+q^J\alpha z_j}{1+\alpha z_j}$.

Question: Can this be realized via a sequential update Markov chain using some $L_\alpha^{(J)} : V^I \otimes H^J \to V^I \otimes H^J$?

Answer: Yes, due to [Kirillov-Reshetikhin '87] fusion procedure. This simplifies on the line and we provide a probabilistic proof.

J>1 via fusion



- The horizontal Markov chain updating column by column preserves 'q-exchangable' measures.
- Along with Markov function theory [Pitman-Rogers '80], this implies that the projection is Markov in its own filtration.
- It also provides a recursion for the higher J L-matrices.

Explicit formula for higher spin L-matrix

Based on [Mangazeev '14] we solve the recursion explicitly ($\beta := \alpha q^{J}$)

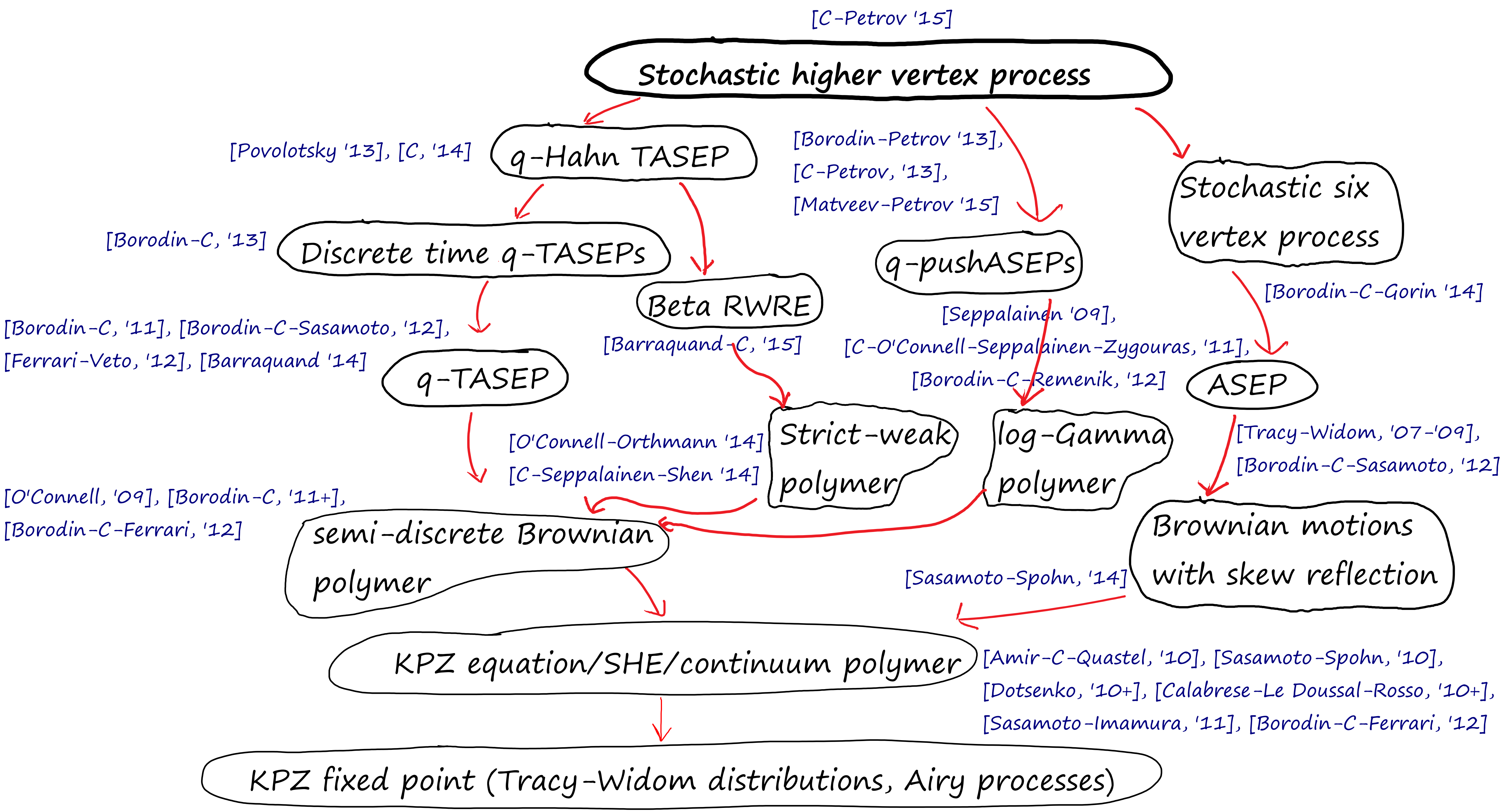
$$L_{\alpha}^{(J)}(i_1,j_1;i_2,j_2) = \mathbb{1}_{i_1+j_1=i_2+j_2}\, q^{\frac{2j_1-j_1^2}{4} - \frac{2j_2-j_2^2}{4} + \frac{i_1^2+i_2^2}{2} + \frac{i_2(j_2-1)+i_1 j_1}{2}}$$

terminating basic hypergeometric series

$$\frac{\nu^{j_1-j_2}\,\alpha^{j_2-j_1+i_2}\,(-\alpha\nu^{-1};q)_{j_2-i_1}}{(q;q)_{i_2}\,(-\alpha;q)_{i_2+j_2}\,(\beta\alpha^{-1}q^{1-J};q)_{j_1-j_2}}\ {}_4\bar{\phi}_3\left(\begin{matrix} q^{-i_2};\ q^{-i_1},\ -\beta,\ -q\nu\alpha^{-1} \\ \nu,\ q^{1+j_2-i_1},\ \beta\alpha^{-1}q^{1-i_2-j_2}\end{matrix}\,\middle|\, q,q\right).$$

Various degenerations (and analytic continuations) are possible and many remain to be investigated.

Degenerations to known integrable stochastic systems in KPZ class



Summary

- Found **stochastic L-matrix** and constructed **ZRP/AEP** from it.
- **Diagonalized** via complete Bethe ansatz basis (one the line).
- **Markov duality** enabled computation of **moment formulas**.
- Provided explicit formula for 4-parameter family of processes encompassing **all known integrable KPZ class models**.
- This method generalizes / rigorizes the polymer replica trick and removes much of the ad hoc nature.
- Many directions: asymptotics, new degenerations, other initial data, product matrix ansatz, higher rank groups, boundary conditions, connections to Macdonald-like processes...