Integrable probability and Macdonald processes

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Integrable probabilistic systems have two characteristics:

1. Concise and exact formulas for expectations of rich class of interesting observables.
2. Scaling limits of systems and formulas provide access to exact descriptions of large universality classes of physical and mathematical systems.

Focus on the Kardar-Parisi-Zhang universality class where representation theory (Macdonald symmetric functions) serves as a significant source of integrable probabilistic systems.
Totally asymmetric simple exclusion process (TASEP)

Object of study:
- **Height** $h(t,x)$ above site $x$ or equivalently, 
- **current** $N_x(t)$ of particles to pass site $x$.

(simulation courtesy of Patirk Ferrari)

Two (1-parameter) deformations:

**ASEP [Spitzer '70]** 
- $p+q=1$
- $p-q=\bar{q}$
- $q/p=\tau$

**$q$-TASEP [Borodin-C '11]** 
- $0<q<1$
- $X_i(t), X_i(t)$
- $\text{Rate } 1 - q^{\text{gap}}$
- $\text{Rate } q$
Key properties of models in the KPZ universality class

1. Local growth
2. Smoothing mechanism
3. Slope dependent growth rate
4. Independent space–time noise

Many probabilistic/physical systems share these features
Kardar-Parisi-Zhang equation ‘86 in 1+1 dimension:

$$\partial_t h = \frac{\nu}{2} \partial_x^2 h + \lambda (\partial_x h)^2 + \sqrt{D} \xi \quad h(t,x)$$

(smoothing, slope dep rate, space-time white noise)

(standard scalings: $\nu = 1$, $\lambda = \frac{1}{2}$, $D = 1$)

Hopf-Cole solution [Bertini-Cancrini ’95, Bertini-Giacomin ’97]

Define: $h(t,x) := \log Z(t,x)$ where $Z$ solves the (well-posed) multiplicative stochastic heat equation (SHE)

$$\partial_t Z = \frac{1}{2} \partial_x^2 Z + Z \xi$$
Rescaling the KPZ equation: 

$$h_\varepsilon(t, x) := \varepsilon^b h(\varepsilon^{-2} t, \varepsilon^{-1} x)$$

$$\partial_t h_\varepsilon = \frac{1}{2} \varepsilon^{2-z} \partial_x^2 h_\varepsilon + \frac{1}{2} \varepsilon^{2-z-b} (\partial_x h_\varepsilon)^2 + \varepsilon^{b-\frac{3}{2} + \frac{1}{2}} \xi$$

**KPZ scaling:** $b = \frac{1}{2}$, $z = \frac{3}{2}$ [Forster-Nelson-Stephen '77]

"KPZ fixed point" $h_0 := \lim_{\varepsilon \to 0} h_\varepsilon$ is universal limit process

Two (weak) scalings preserve the KPZ equation

**Weak nonlinearity scaling:** $b = \frac{1}{2}$, $z = 2$, scale nonlinearity by $\varepsilon^{\frac{1}{2}}$

**Weak noise scaling:** $b = 0$, $z = 2$, scale noise by $\varepsilon^{\frac{1}{2}}$

Useful proxies for finding approximation schemes
1+1 dimensional semi-discrete and discrete SHE

ASEP

\[ \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{Rate } q \quad \text{Rate } p
\end{array} \]

q-TASEP

\[ \begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\text{Rate } 1 - q^{gap}
\end{array} \]

TASEP

\[ \begin{array}{c}
\text{Weak nonlinearity Scaling } \mathcal{N} \\
\text{Weak noise Scaling}
\end{array} \]

KPZ equation

Directed polymers

KPZ fixed point

Rate \( q \)  
Rate \( p \)  

Rate TASEP

Directed polymers

\[ R(\mathbf{x},t) \]

8.0 sec  
500 microns

18.0 sec

\[ R(\mathbf{x},t) \]

-1000 -500 0 500 1000 (um)

28.0 sec

\[ R(\mathbf{x},t) \]

1000 500 0 -500 -1000

\[ 0 \quad 12h \quad 24h \quad 36h \quad 48h \]

\[ 50h \quad 72h \quad 90h \]
Consider TASEP with step initial data
\[ h_\varepsilon (t, x) := \varepsilon^{1/2} h_{\varepsilon} (\varepsilon^{-3/2} t, \varepsilon^{-1} x) - \frac{\varepsilon^{-1} t}{2} \]

Theorem (Johansson '99): For TASEP with step initial data,
\[ P(h_{\varepsilon} (1, 0) \geq -S) \xrightarrow{\varepsilon \to 0} F_{\text{GUE}} \left( \frac{2}{3} S \right) \]

See also [Baik-Deift-Johansson '99, Prahofer-Spohn '02]

Source of integrability: determinantal structure of Schur measure and process [Okounkov-Reshetikhin '03, Borodin-Ferrari '08]

\[ h_{\varepsilon} (\cdot, \cdot) \text{ tight} \quad [\text{Cator-C-Pimentel-Quastel '12}] \rightarrow \text{KPZ fixed point exists} \]
TASEP is one of a few growth models in the KPZ class that can be analyzed via the techniques of **determinantal point processes** (or free fermions, nonintersecting paths, Schur processes).

Other examples include

- Discrete time TASEPs with sequential/parallel update
- PushASEP or long range TASEP
- Directed last passage percolation in 2d with geometric/Bernoulli/exponential weights
- Polynuclear growth processes
Recent advances on the KPZ front:

1. Strong **experimental evidence** that real life systems follow the KPZ class universal laws

2. Direct **well-posedness** of the KPZ equation and some weak universality of the equation

3. **Non-determinantal** models whose large time behaviour has been analyzed
Non-determinantal models whose large time behaviour has been analyzed:

- **ASEP** [Tracy-Widom, 2009], [Borodin-C-Sasamoto, 2012]
- **KPZ equation / stochastic heat equation (SHE)**
  - [Amir-C-Quastel, 2010], [Sasamoto-Spohn, 2010], [Dotsenko, 2010+],
  - [Calabrese-Le Doussal-Rosso, 2010+], [Borodin-C-Ferrari, 2012]
- **q-TASEP** [Borodin-C, 2011+]
- **Semi-discrete stochastic heat equation**
- **Fully discrete log-Gamma polymer (stochastic heat equation)**
  - [C-O'Connell-Seppalainen-Zygouras, 2011] [Borodin-C-Remenik, 2012]
Theorem (Borodin-C '11): q-TASEP step initial data

\[ E \left[ \frac{1}{(q \cdot x_n(t))_\infty} \right] = \text{det} \left( I + K_{q-TASEP}^q \right) \]

where

\[ K_{q-TASEP}^q(w, w') = \frac{1}{2\pi i} \int \frac{\pi}{\sin(-\pi s)} (-s)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'} \, ds \]

\[ g(w) = \frac{e^{-tw}}{(w \cdot q)_{\infty}} \] \quad (a \cdot q)_{\infty} = (1-a)(1-q^a)(1-q^2a) \ldots

q-Laplace transform [Hahn '49] identifies \( X_n(t) \) distribution

Good for asymptotics [Borodin-C-Ferrari '12]
Theorem (Borodin-C-Sasamoto '11): ASEP step initial data

\[ E \left[ \frac{1}{(\xi \zeta N_x(t) : \zeta)_\infty} \right] = \det \left( I + K_8^{ASEP} \right) \]

where \( K_8^{ASEP} \) same as \( K_8^{q-TASEP} \) upto \( q \to \zeta \), \( g(\omega) = e^{\frac{H \zeta}{\zeta + \omega}} \left( \frac{\zeta}{\zeta + \omega} \right)^x \).

Both \( q \)-TASEP and ASEP have second type of Fredholm determinant formula (harder for asymptotics).

For ASEP, second formula matches [Tracy-Widom '09].
Discrete time q-TASEPs

q-TASEP

log-Gamma discrete polymer

semi-discrete stochastic heat eqn.

KPZ equation / stochastic heat equation

universal limits (Tracy-Widom distributions, Airy processes)
q-TASEP: \[ q \cdot TASEP: \quad \partial_t q^n(t) + q^n(t) \partial^\uparrow q^n(t) + q^n(t) \partial^\downarrow q^n(t) + q^n(t) \partial^\downarrow \bar f(n) = f(n-1) - f(n) \quad \text{martingale} \]

Semi-discrete SHE: \[ \partial_t Z(t,n) = \nabla Z(t,n) + \beta Z(t,n) dB_n(t) \]

Continuum SHE: \[ \partial_t Z(t,x) = \frac{1}{2} \partial^2_x Z(t,x) + \xi(t,x) Z(t,x) \quad \text{space-time white noise} \]

ASEP: \[ \partial_t \bar Z^N_x(t) = \Delta^\uparrow \bar Z^N_x(t) + \bar Z^N_x(t) dB_x(t) \]
Let \( h(t, x) := \log \xi(t, x) \) with \( \xi(0, x) = \delta_{x=0} \).
Formally, \( h \) solves the KPZ equation
\[
\partial_t h = \frac{1}{2} \partial_x^2 h + \frac{1}{2} (\partial_x h)^2 + \xi
\]

**Theorem (Amir-C-Quastel '10):** Let \( \mathcal{F}_t(s) := \mathcal{P}( h(t, x) + \frac{x^2}{2t} + \frac{t}{2y} \leq s^\frac{3}{2} ) \) then
\[
\mathcal{F}_t(s) = \int \frac{d\mu}{\mu} e^{-\mu} \text{det}(I - K_{t, \mu})_t^2(\tau \geq \frac{s}{3})
\]

**Corollary:** KPZ equation is in KPZ universality class.
\[
\mathcal{P}( \xi^{1/2} h(\xi^{-3/2} t, \xi^{-1/4} x) + \frac{x^2}{2t} + \xi^{-1/4} \frac{t}{2y} \leq s ) \xrightarrow{\epsilon \to 0} \mathcal{F}_{\text{GUE}}(s)
\]

Formula discovered independently and in parallel in non-rigorous work of [Sasamoto-Spohn '10].
Scaled one point marginal distribution for KPZ equation
Macdonald processes $q, t \in [0, 1)$

Ruijsenaars-Macdonald system
Representations of Double Affine Hecke Algebras

q-Whittaker processes $t=0$

$q$-TASEP, 2d dynamics
$q$-deformed quantum Toda lattice
Representations of $\hat{gl}_N$, $U_q(\hat{gl}_N)$

discussed so far

General $\beta$ RMT $t=q^{\beta/2} \rightarrow 1$

Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$
Calogero-Sutherland, Jack polynomials

Hall-Littlewood processes $q=0$

Random matrices over finite fields
Spherical functions for p-adic groups

Whittaker processes $t=0$

$q$-TASEP, 2d dynamics
$q$-deformed quantum Toda lattice
Representations of $\hat{gl}_N$, $U_q(\hat{gl}_N)$

Directed polymers and their hierarchies
Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$

Schur processes $q=t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE
Characters of symmetric, unitary groups

Kingman partition structures

Cycles of random permutations $q=0$
Poisson-Dirichlet distributions $t=1$
Macdonald processes: a source of (many parameter) integrable probabilistic systems. Specializations and degenerations include q-TASEP, continuum/semi-discrete/discrete SHE, KPZ equation

ASEP does not fit. But it does share certain parallel formulas

Structure leads to

suggests new

Formulas
(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand–Tsetlin patterns)

\[ P(\lambda^{(k)}) = \frac{P_{\lambda^{(k)}}(a_1, \ldots, a_k) \prod_{\lambda_1} Q_{\lambda^{(k)}}(b_1, \ldots, b_M)}{\prod(a_1, \ldots, a_k; b_1, \ldots, b_M)} \]

\[ \lambda_j \in \mathbb{Z}_{\geq 0} \]

Macdonald polynomials

Normalization constant

two groups of parameters
Macdonald polynomials \( P_\lambda(x_1, \ldots, x_N) \in \mathbb{Q}(q,t)[x_1, \ldots, x_N]^{S(N)} \) with partitions \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0) \) form a basis in symmetric polynomials in \( N \) variables over \( \mathbb{Q}(q,t) \). They diagonalize

\[
(D_1 f)(x_1, \ldots, x_N) = \sum_{i=1}^{\lambda_1} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} f(q^{\lambda_1+1} x_i, \ldots, q^{\lambda_N} x_N)
\]

with (generically) pairwise different eigenvalues

\[
D_1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \ldots + q^{\lambda_N}) P_\lambda.
\]

They have many remarkable properties that include orthogonality (dual basis \( Q_\lambda \)), simple reproducing kernel (Cauchy type identity), Pieri and branching rules, index/variable duality, explicit generators of the algebra of (Macdonald) operators commuting with \( D_1 \), etc.
We are able to do two basic things:

• Construct relatively explicit Markov operators that map Macdonald processes to Macdonald processes;

• Evaluate averages of a broad class of observables.

The construction is based on commutativity of Markov operators

\[
\begin{align*}
\mathbb{P}(\lambda \to \mu) &= \frac{P'_\mu(x_1, \ldots, x_{n-1})}{P'_\lambda(x_1, \ldots, x_n)} \cdot P_{\lambda/\mu}(x_n), \\
\mathbb{P}(\lambda \to \nu) &= \frac{P'_\nu(x_1, \ldots, x_m)}{P'_\lambda(x_1, \ldots, x_m)} \cdot \frac{P_{\lambda/\nu}(u)}{\Pi(x_i, u)}
\end{align*}
\]

an idea from [Diaconis-Fill '90], and Schur process dynamics from [Borodin-Ferrari '08].
Evaluation of averages is based on the following observation. Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

$$\mathcal{D} \mathcal{P}_\lambda = d_\lambda \mathcal{P}_\lambda .$$

Applying it to the Cauchy type identity

$$\sum_\lambda \mathcal{P}_\lambda (a) \mathcal{Q}_\lambda (b) = \prod (a;b)$$

we obtain

$$\mathbb{E}[d_\lambda] = \frac{\mathcal{D}^{(a)} \prod (a;b)}{\prod (a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Contrast with the lack of explicit formulas for the Macdonald polynomials.
Here is an example of a Markov process preserving the class of the $q$-Whittaker processes (Macdonald processes with $t=0$).

Each coordinate of the triangular array jumps by 1 to the right independently of the others with

\[
\text{rate } (\lambda_k^{(m)}) = \frac{(1 - q \lambda_k^{(m-1)} - \lambda_k^{(m)}) (1 - q \lambda_k^{(m)} - \lambda_{k+1}^{(m)} + 1)}{(1 - q \lambda_k^{(m)} - \lambda_k^{(m-1)})}.
\]

The set of coordinates $\{\lambda_m^{(m)} - m\}_{m>1}$ forms $q$-TASEP.
Taking the observables corresponding to powers of the first Macdonald operator yields

\[
\mathbb{E}\left[ (q \lambda_N^{(n)}(z_1)^k \right] = \frac{(-1)^k}{(2\pi i)^k} \oint \cdots \oint \prod_{A < B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^k \frac{e^{(q-1)\tau z_j}}{(1 - z_j)^N} \frac{dz_j}{z_j}
\]

\[
\mathbb{E}\left[ \sum_{k=0}^{\infty} \frac{(q \lambda_N^{(w)})^k}{(1-q)^k \cdots (1-q^k)} \right] = \mathbb{E}\left[ \frac{1}{(5q \lambda_N^{(w)}; q)_\infty} \right] = \text{det} (I + K)_{L^2(N \times \mathbb{C})}
\]

with \( q \)-Laplace transform of \( q \lambda_N^{(w)} \)

\[
K(n_1, w_1 ; n_2, w_2) = \frac{f(w_1) \cdots f(q^{n_1-1} w_1)}{q^{n_1} w_1 - w_2}, \quad f(w) = \frac{e^{(q-1)\tau w}}{(1-w)^N}
\]

A rigorous version of the physics "replica trick"
\[ q\text{-TASEP} \quad q^x_n(t) + n \quad \longrightarrow \quad \text{Semi-discrete SHE} \quad \overline{Z}(t, n) \quad \longrightarrow \quad \text{Continuum SHE} \quad Z(t, x) \]

[Molchanov '86] [Kardar '87] observe

\[ \overline{Z}(t; x_1, \ldots, x_k) := \mathbb{E}\left[ \prod_{i=1}^{k} Z(t, x_i) \right] \]

satisfies

\[ \partial_t \overline{Z} = \frac{1}{a} \left( \sum_{i=1}^{k} \partial^2_{x_i} + c \sum_{i \neq j} \delta_{x_i - x_j} \right) \overline{Z} \quad \text{"delta Bose gas"} \]

Bethe ansatz [\( c < 0 \) Lieb-Liniger '63, \( c > 0 \) McGuire '64] gives eigenbasis

"Replica trick" [Dotsenko '10, Calabrese-Le Doussal-Rosso '10]

\[ \mathbb{E}[e^{s \overline{Z}}] = \sum_{k=0}^{\infty} \frac{\mathbb{E}[Z^k]}{k!} s^k \]

Divergent series! Risky to draw conclusions (originally obtained incorrect answer)
Theorem (Borodin-Corwin, '11): For \( x_1 < x_2 < \ldots < x_K \), the integral

\[
U(t; x_1, \ldots, x_K) := \int \cdots \int \prod_{A < B} \frac{Z_A - Z_B}{Z_A - Z_B - c} \prod_{j=1}^k e^{\frac{1}{2} z_j^2 + x_j z_j} \frac{dz_j}{2\pi i}
\]

solves the delta Bose gas for all \( c \in \mathbb{R} \) and for \( U(0; x_1, \ldots, x_K) = \prod_{i=1}^K \delta_{x_i = 0} \)

(Here the \( z_j \)-integration is over \( \mathcal{C}_{\alpha_j + i \mathbb{R}} \) with \( \alpha_1 > \alpha_2 + c > \alpha_3 + 2c > \ldots \))

- **Clear symmetry between attractive (c>0) and repulsive (c<0) cases**
- **Bethe eigenstates are very different in attractive/repulsive cases**
- **Formula can be found in [Heckman-Opdam '97] Plancherel theorem for delta Bose gas; ideas trace back to [Harish Chandra, Helgason]**
For semi-discrete SHE, \( \mathbb{E} \left[ \prod_{i=1}^{\kappa} Z(t, n_i) \right] \) satisfies [Borodin-C '11]

\[
\partial_t V(t; n_1, \ldots, n_\kappa) = \left( \sum_{i=1}^{\kappa} \nabla_i + \sum_{i < j} 1 \right) V(t; n_1, \ldots, n_\kappa)
\]

\( \nabla_i \) acts as \((\nabla_i f)(n) = f(n-\delta_i) - f(n)\)
in the \(n_i\) coordinate

For q-TASEP, \( \mathbb{E} \left[ \prod_{i=1}^{\kappa} q^{X_{n_i}(t)+n_i} \right] \) satisfies [Borodin-C-Sasamoto '12]

\[
\partial_t V(t; n_1, \ldots, n_\kappa) = (1 - q) \left( \sum_{i=1}^{\kappa} \nabla_i + (1 - q^{-1}) \sum_{i < j} 1 \right) V(t; n_1, \ldots, n_\kappa)
\]

In all cases, the "nested contour integral ansatz" solves Bose gas
ASEP is not solved by Macdonald process. However,

• Self-duality of ASEP \([\text{Schutz '97}]\) \(\rightarrow\) moments satisfy Bose gas

• Nested contour integral ansatz applies \([\text{Borodin-C-Sasamoto '12}]\)

• Leads to two Fredholm determinants (one new and one TWs)

• TW compute ASEP \(k\)-particle Green’s function via Bethe ansatz

Formulas suggest search for new structure:

• For \(q\)-TASEP: Nested contour integral formulas and Bose gas are consequences of structural properties of the Macdonald polynomials

• For ASEP: No structure to predict existence of nested contour integral formulas (duality is from \(U_q(sl_2)\) symmetry)
To summarize:

- ASEP and q-TASEP are important systems in the KPZ universality class, which can be scaled to the KPZ equation.
- Macdonald processes are a source of integrable probabilistic models.
- They generalize Schur processes but are not determinantal.
- Integrability comes from structural properties of Macdonald polynomials (lead to nice Markov dynamics and concise formulas for averages).
- Turning averages into asymptotics remains challenging.
- Rigorous replica trick developed for q-TASEP and ASEP.
- Nested contour integral ansatz formulas for ASEP moments suggest search for new structure parallel to Macdonald processes.
Lecture 1: Overview and intro to symmetric functions.

Lecture 2: Schur processes

Lecture 3: Macdonald processes I

Lecture 4: Macdonald processes II

Lecture 5: Duality and Bose gas methods

Lecture 6: Analysis of ASEP, conjectures and open problems

Exercise handout and office hours (Wed., Thur. 3-5pm 3.040)

Lectures times Wed. 10-12 and Thur. 9-11.

Website: [http://math.mit.edu/~icorwin/Lipschitz.html](http://math.mit.edu/~icorwin/Lipschitz.html)