Integrable probability and Macdonald processes

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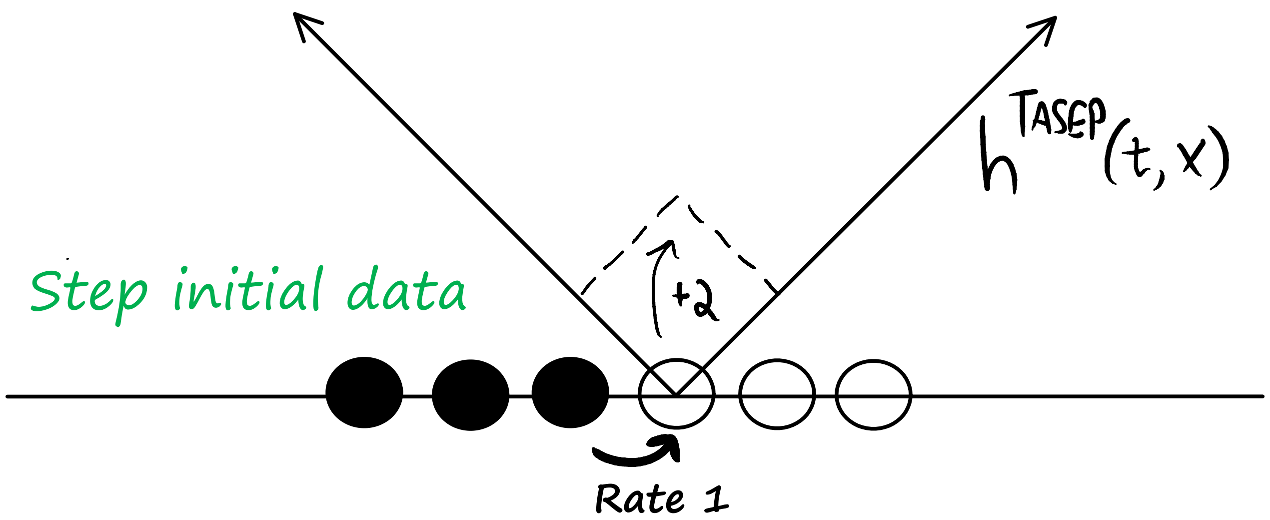
Integrable probabilistic systems have two characteristics:

1. Concise and exact formulas for expectations of rich class of interesting observables.
2. Scaling limits of systems and formulas provide access to exact descriptions of large universality classes of physical and mathematical systems.

Focus on the Kardar-Parisi-Zhang universality class where representation theory (Macdonald symmetric functions) serves as a significant source of integrable probabilistic systems

Totally asymmetric simple exclusion process (TASEP)



Object of study:

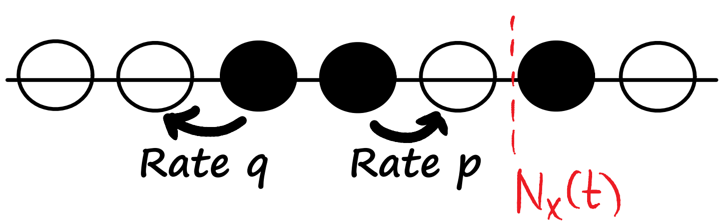
Height $h(t,x)$ above site x or equivalently,

current $N_x(t)$ of particles to pass site x.

(simulation courtesy of Patirk Ferrari)

Two (1-parameter) deformations:

ASEP [Spitzer '70]

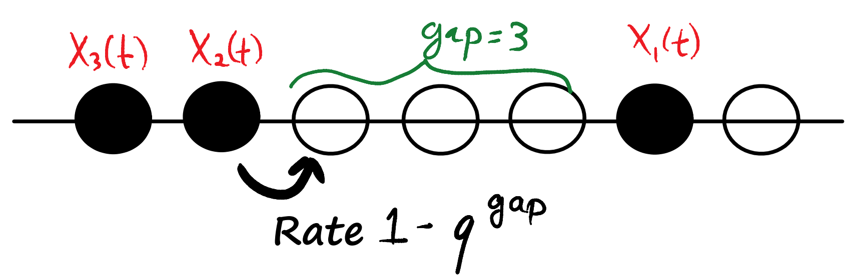


$p+q=1$

$p-q=\gamma$

$q/p=\tau$

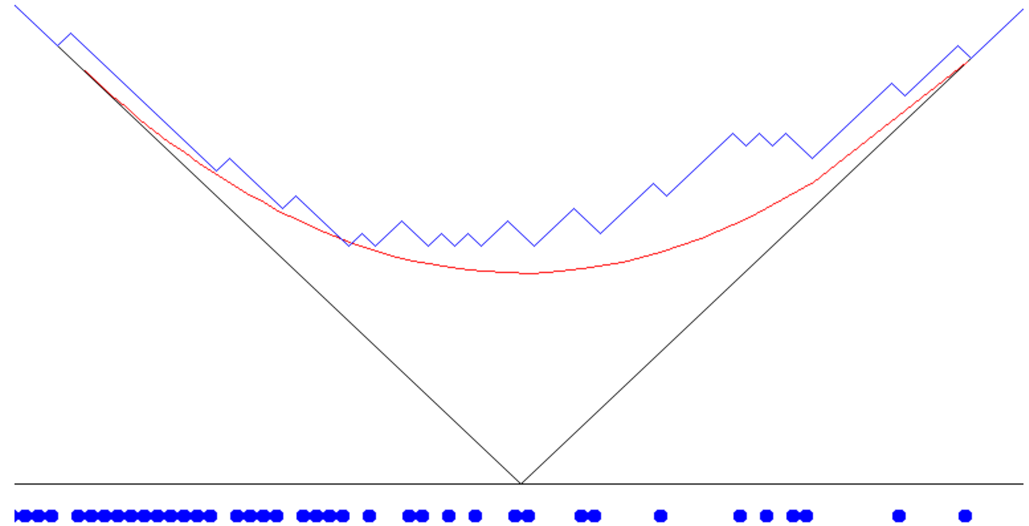
q-TASEP [Borodin-C '11]



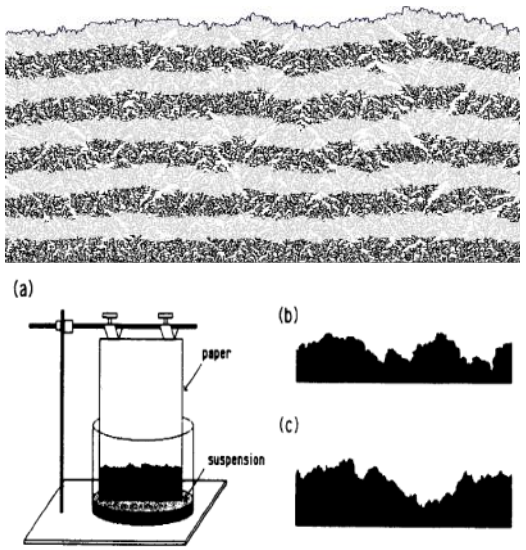
$0<q<1$

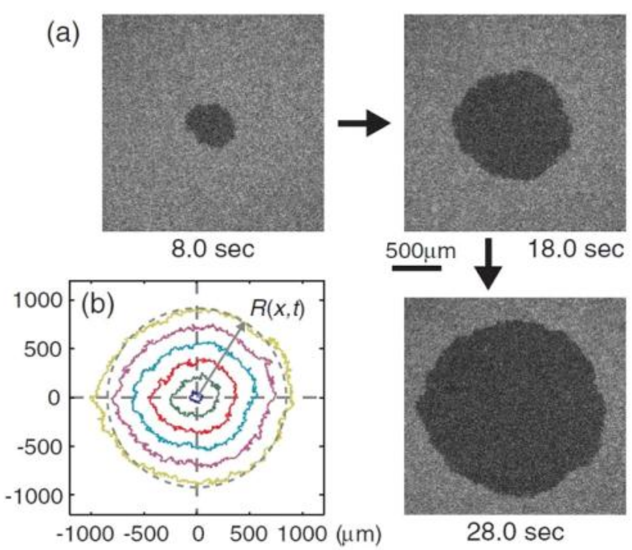
Key properties of models in the KPZ universality class

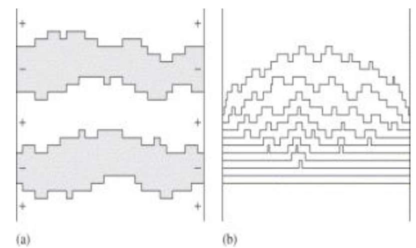
1. Local growth
2. Smoothing mechanism
3. Slope dependent growth rate
4. Independent space-time noise

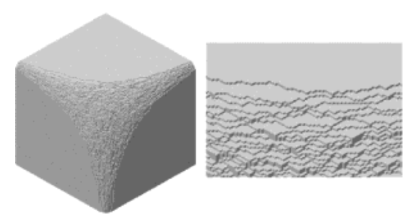


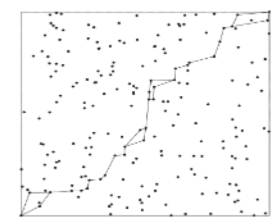
Many probabilistic/physical systems share these features

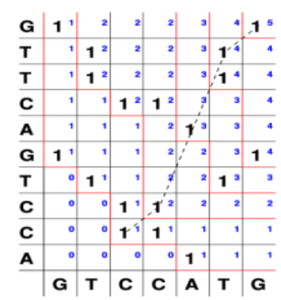


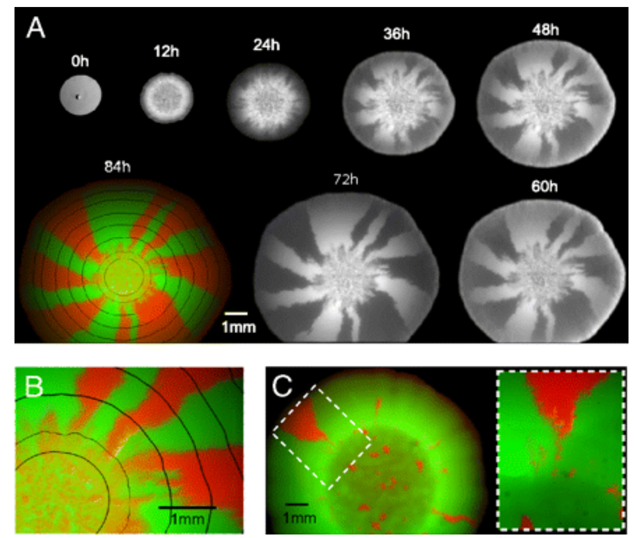






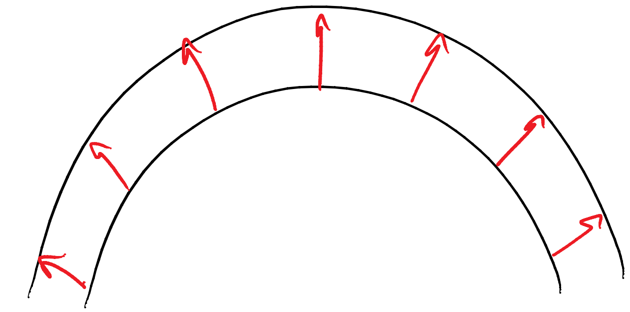






Kardar-Parisi-Zhang equation '86 in 1+1 dimension:

$$\partial_t h = \frac{\nu}{2}\partial_x^2 h + \lambda(\partial_x h)^2 + \sqrt{D}\,\xi \qquad h(t,x)$$



smoothing   slope dep rate   space-time white noise

(standard scalings: $\nu = 1$, $\lambda = 1/2$, $D = 1$)

Hopf-Cole solution [Bertini-Cancrini '95, Bertini-Giacomin '97]

Define: $h(t,x) := \log Z(t,x)$ where Z solves the (well-posed) multiplicative stochastic heat equation (SHE)

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z + Z\xi$$

Rescaling the KPZ equation: $h_\varepsilon(t,x) := \varepsilon^b h(\varepsilon^{-z}t, \varepsilon^{-1}x)$

$$\partial_t h_\varepsilon = \frac{1}{2}\varepsilon^{2-z}\partial_x^2 h_\varepsilon + \frac{1}{2}\varepsilon^{2-z-b}(\partial_x h_\varepsilon)^2 + \varepsilon^{b-z/2+1/2}\xi$$

KPZ scaling: $b = 1/2$, $z = 3/2$ [Forster-Nelson-Stephen '77]

"KPZ fixed point" $h_0 := \lim_{\varepsilon \searrow 0} h_\varepsilon$ is universal limit process

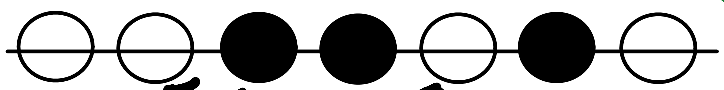
Two (weak) scalings preserve the KPZ equation

Weak nonlinearity scaling: $b = 1/2$, $z = 2$, scale nonlinearity by $\varepsilon^{1/2}$

Weak noise scaling: $b = 0$, $z = 2$, scale noise by $\varepsilon^{1/2}$

Useful proxies for finding approximation schemes

ASEP



Rate q Rate p

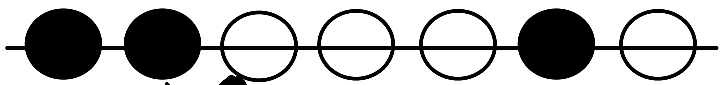
weak nonlinearity scaling

weak noise scaling

1+1 dimensional semi-discrete and discrete SHE

KPZ equation

q-TASEP

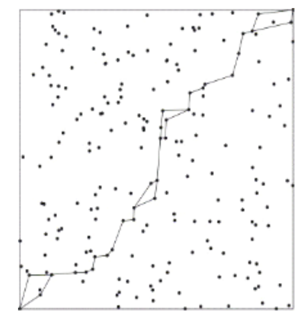


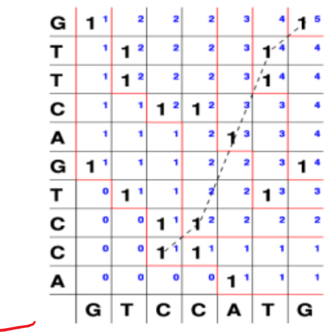
Rate $1-q^{gap}$

KPZ scaling

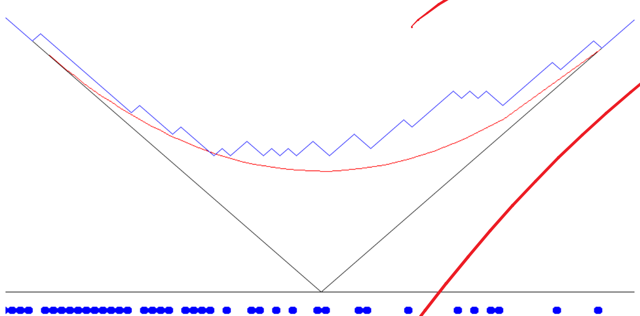
KPZ fixed point

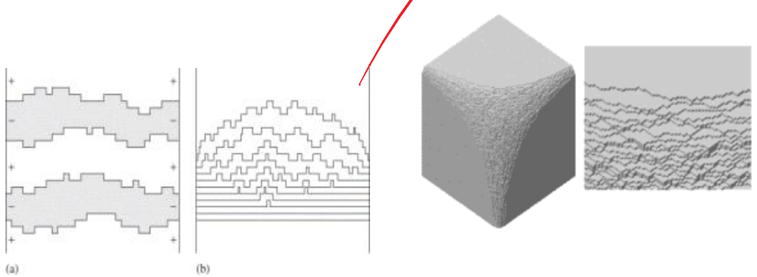
Directed polymers

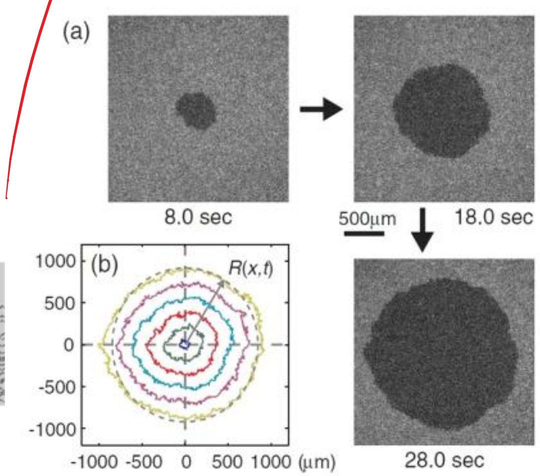


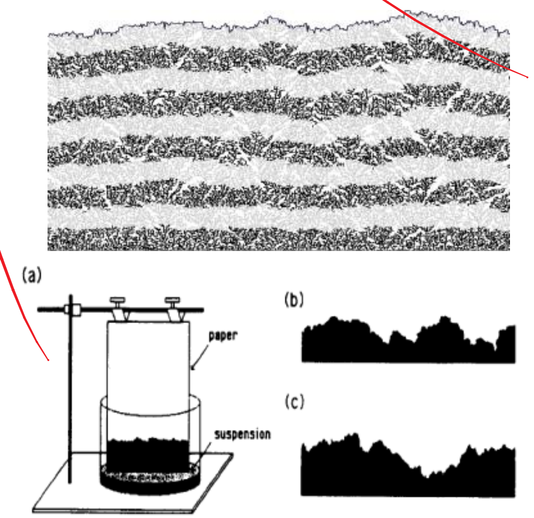


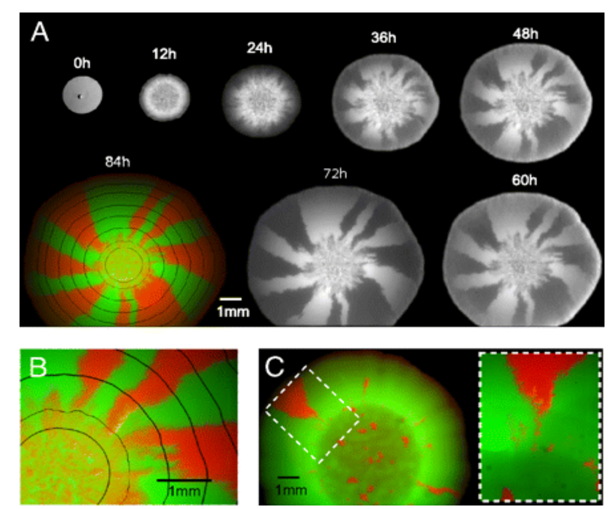
TASEP





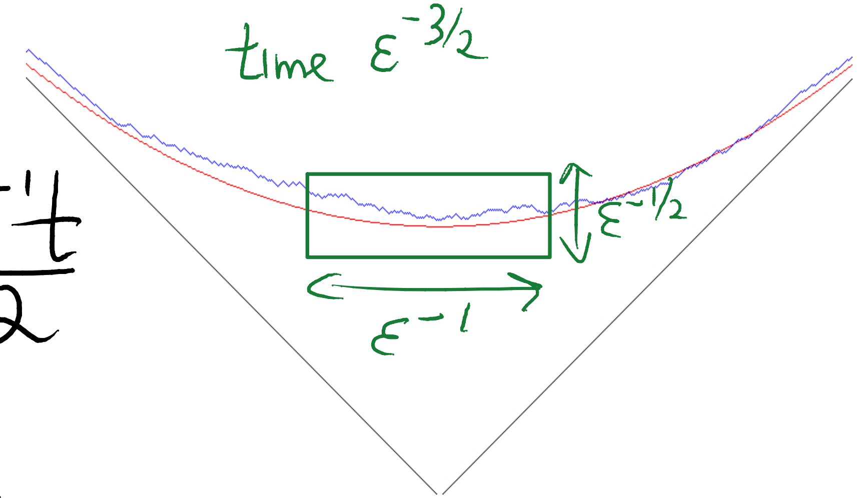






Consider TASEP with step initial data

$$h_\varepsilon^{\mathrm{TASEP}}(t,x) := \varepsilon^{1/2} h^{\mathrm{TASEP}}(\varepsilon^{-3/2}t, \varepsilon^{-1}x) - \frac{\varepsilon^{-1}t}{2}$$



Theorem (Johansson '99): For TASEP with step initial data,

$$\mathbb{P}\left(h_\varepsilon^{\mathrm{TASEP}}(1,0) \geq -s\right) \xrightarrow[\varepsilon \searrow 0]{} F_{GUE}(2^{1/3}s)$$

largest eigenvalue statistic for $N \times N$ Hermitian random matrix as $N \nearrow \infty$

See also [Baik-Deift-Johansson '99, Prahofer-Spohn '02]

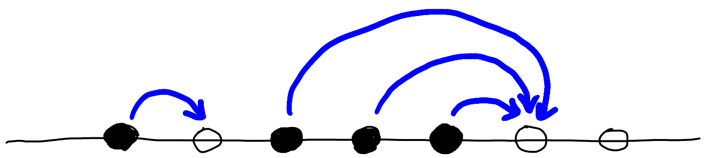
Source of integrability: determinantal structure of Schur measure and process [Okounkov-Reshetikhin '03, Borodin-Ferrari '08]

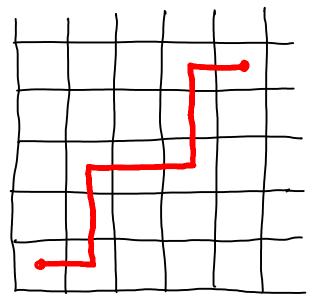
$h_\varepsilon^{\mathrm{TASEP}}(\cdot,\cdot)$ tight [Cator-C-Pimentel-Quastel '12]->KPZ fixed point exists

TASEP is one of a few growth models in the KPZ class that can be analyzed via the techniques of determinantal point processes (or free fermions, nonintersecting paths, Schur processes).

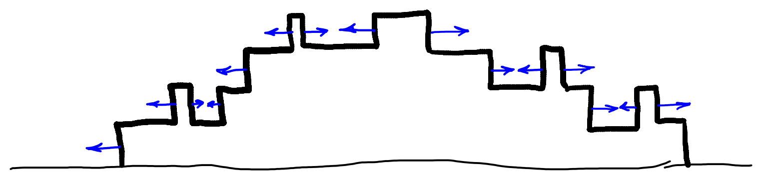
Other examples include

- Discrete time TASEPs with sequential/parallel update
- PushASEP or long range TASEP





- Directed last passage percolation in 2d with geometric/Bernoulli/exponential weights
- Polynuclear growth processes



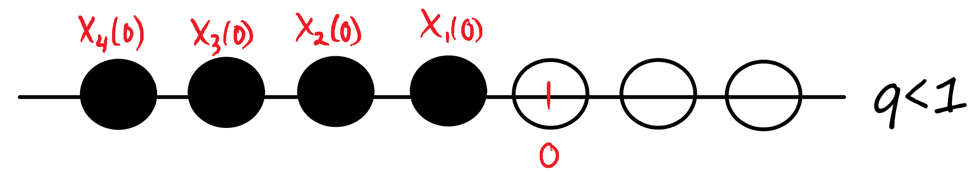
Recent advances on the KPZ front:

1. Strong experimental evidence that real life systems follow the KPZ class universal laws
2. Direct well-posedness of the KPZ equation and some weak universality of the equation
3. Non-determinantal models whose large time behaviour has been analyzed

Non-determinantal models whose large time behaviour has been analyzed:

- ASEP [Tracy-Widom, 2009], [Borodin-C-Sasamoto, 2012]
- KPZ equation / stochastic heat equation (SHE)
  [Amir-C-Quastel, 2010], [Sasamoto-Spohn, 2010], [Dotsenko, 2010+], [Calabrese-Le Doussal-Rosso, 2010+], [Borodin-C-Ferrari, 2012]
- q-TASEP [Borodin-C, 2011+]
- Semi-discrete stochastic heat equation
  [O'Connell, 2010], [Borodin-C, 2011, Borodin-C-Ferrari, 2012]
- Fully discrete log-Gamma polymer (stochastic heat equation)
  [C-O'Connell-Seppalainen-Zygouras, 2011] [Borodin-C-Remenik, 2012]

Theorem (Borodin-C '11): q-TASEP step initial data



$$\mathbb{E}\left[\frac{1}{(\zeta q^{x_n(t)};q)_\infty}\right] = \det\left(I + K_\zeta^{q\text{-TASEP}}\right)$$

where

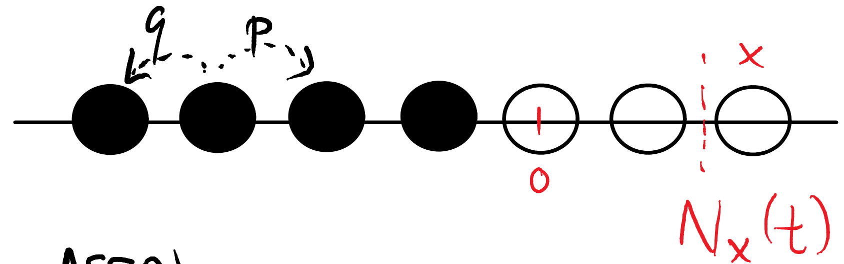
$$K_\zeta^{q\text{-TASEP}}(w,w') = \frac{1}{2\pi i}\int \frac{\pi}{\sin(-\pi s)}(-\zeta)^s \frac{g(w)}{g(q^s w)} \frac{1}{q^s w - w'}\,ds$$

$$g(w) = \frac{e^{-tw}}{(w;q)_\infty^n}, \quad (a;q)_\infty = (1-a)(1-qa)(1-q^2a)\cdots$$

q-Laplace transform [Hahn '49] *identifies* $x_n(t)$ *distribution*

Good for asymptotics [Borodin-C-Ferrari '12]

Theorem (Borodin-C-Sasamoto '11): ASEP step initial data



$$\mathbb{E}\left[\frac{1}{(\zeta\tau^{N_x(t)};\tau)_\infty}\right] = \det\left(I + K_\zeta^{ASEP}\right)$$

$p-q=\gamma>0$

$q/p=\tau<1$

where K_ζ^{ASEP} same as $K_\zeta^{q\text{-TASEP}}$ upto $q\to\tau,\ g(w)=e^{\gamma t\frac{\tau}{\tau+w}}\left(\frac{\tau}{\tau+w}\right)^x.$

Both q-TASEP and ASEP have second type of Fredholm determinant formula (harder for asymptotics).

For ASEP, second formula matches [Tracy-Widom '09].

Discrete time q-TASEPs

q-TASEP

log-Gamma discrete polymer

ASEP

semi-discrete stochastic heat eqn.

KPZ equation / stochastic heat equation

universal limits (Tracy-Widom distributions, Airy processes) q-TASEP: $$\partial_t q^{x_n(t)+n} = (1-q)\nabla q^{x_n(t)+n} + q^{x_n(t)+n}\, dM_n(t)$$

$(\nabla f)(n) = f(n-1)-f(n)$ martingale

[Borodin-C '11,
Borodin-C-Sasamoto '12]

$$\Big\Downarrow \quad q \nearrow 1,\ t \nearrow \infty,\ n \in \mathbb{N} \text{ fixed}$$

Semi-discrete SHE: $$\partial_t Z(t,n) = \nabla Z(t,n) + \beta Z(t,n)\, dB_n(t)$$

inv. temp ind. BMs

[Alberts-Khanin-Quastel '12],
[Moreno-Remenik-Quastel '12]

$$\Big\Downarrow \quad t \nearrow \infty,\ n \nearrow \infty,\ \beta \searrow 0$$

weak noise

Continuum SHE: $$\partial_t Z(t,x) = \tfrac{1}{2}\partial_x^2 Z(t,x) + \xi(t,x) Z(t,x)$$

space-time white noise

[Bertini-Giacomin '97],
[Gartner '88]

$$\Big\Uparrow \quad t \nearrow \infty,\ x \nearrow \infty,\ \tau \nearrow 1$$

weak nonlinearity

ASEP: $$\partial_t \tau^{N_x(t)} = \Delta^{p,q} \tau^{N_x(t)} + \tau^{N_x(t)}\, dM_x(t)$$

Let $h(t,x) := \log Z(t,X)$ with $Z(0,X) = \delta_{X=0}$

Formally h solves the KPZ equation $\partial_t h = \frac{1}{2}\partial_x^2 h + \frac{1}{2}(\partial_x h)^2 + \xi$

Theorem (Amir-C-Quastel '10): Let $F_t(s) := \mathbb{P}\left(h(t,x) + \frac{x^2}{2t} + \frac{t}{24} \le 2^{-1/3} s \right)$ then

$$F_t(s) = \int \frac{d\mu}{\mu} e^{-\mu} \det\left(I - K_{t,\mu}\right)_{L^2(t^{-1/3}s,\infty)}$$

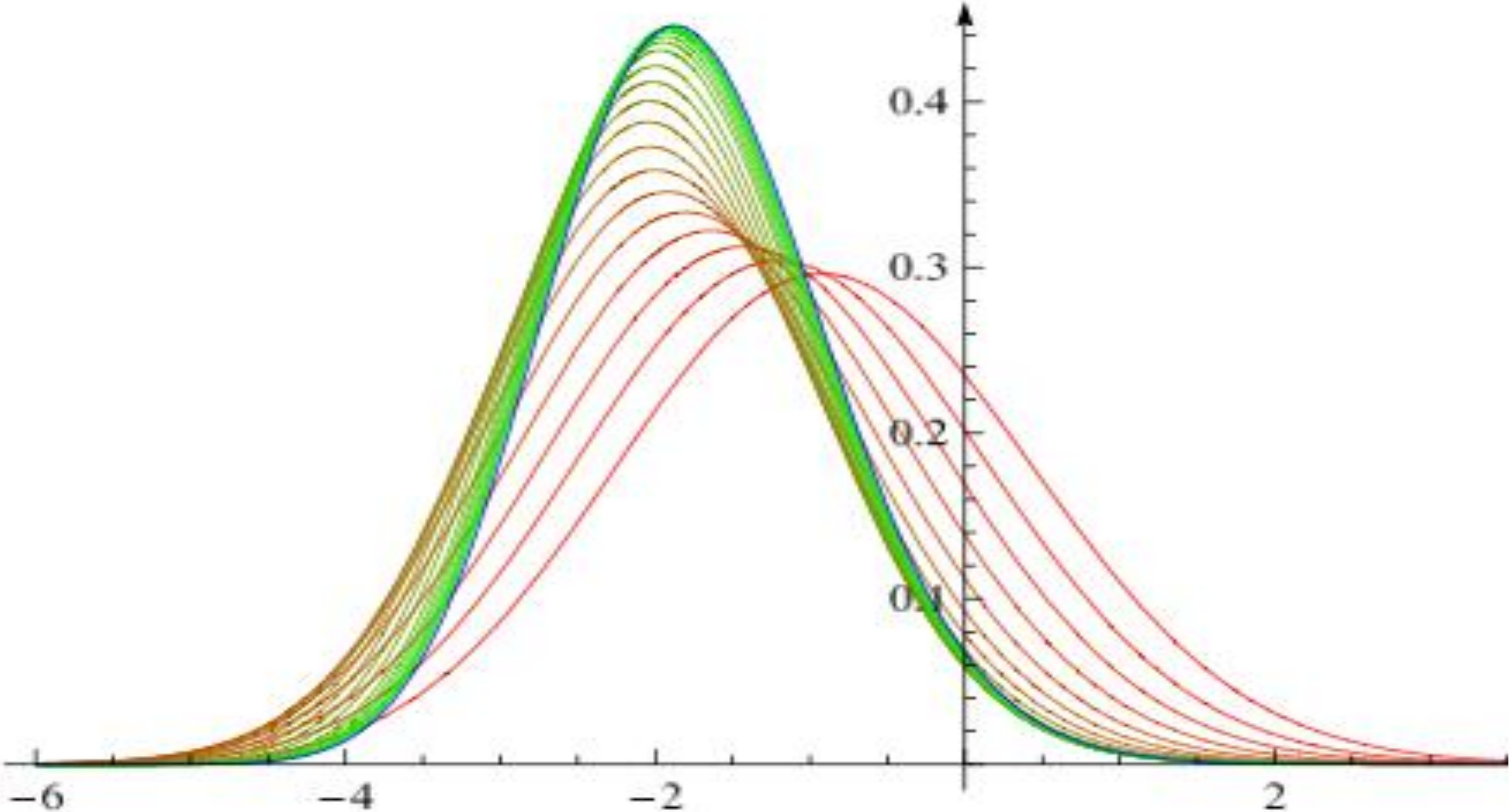
$K_{t,\mu}(x,y) = \int \frac{\mu}{\mu - e^{-t^{1/3}r}} Ai(x+r) Ai(y+r)\, dr$

Corollary: KPZ equation is in KPZ universality class.

$$\mathbb{P}\left(\varepsilon^{1/2} h(\varepsilon^{-3/2} t, \varepsilon^{-1} X) + \frac{X^2}{2t} + \varepsilon^{-1} \frac{t}{24} \le s \right) \xrightarrow[\varepsilon \searrow 0]{} F_{GUE}(s)$$

Formula discovered independently and in parallel in non-rigorous work of [Sasamoto-Spohn '10].

Scaled one point marginal distribution for KPZ equation



Macdonald processes $q, t \in [0, 1)$

Ruijsenaars–Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker processes $t = 0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$

General β RMT $t = q^{\beta/2} \to 1$

Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Calogero–Sutherland, Jack polynomials

Spherical functions for Riem. Symm. Sp.

Hall–Littlewood processes $q = 0$

Random matrices over finite fields

Spherical functions for p-adic groups

discussed so far

Whittaker processes $t = 0$, $q \to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n, \mathbb{R})$

Kingman partition structures

Cycles of random permutations $q = 0$, $t = 1$

Poisson–Dirichlet distributions

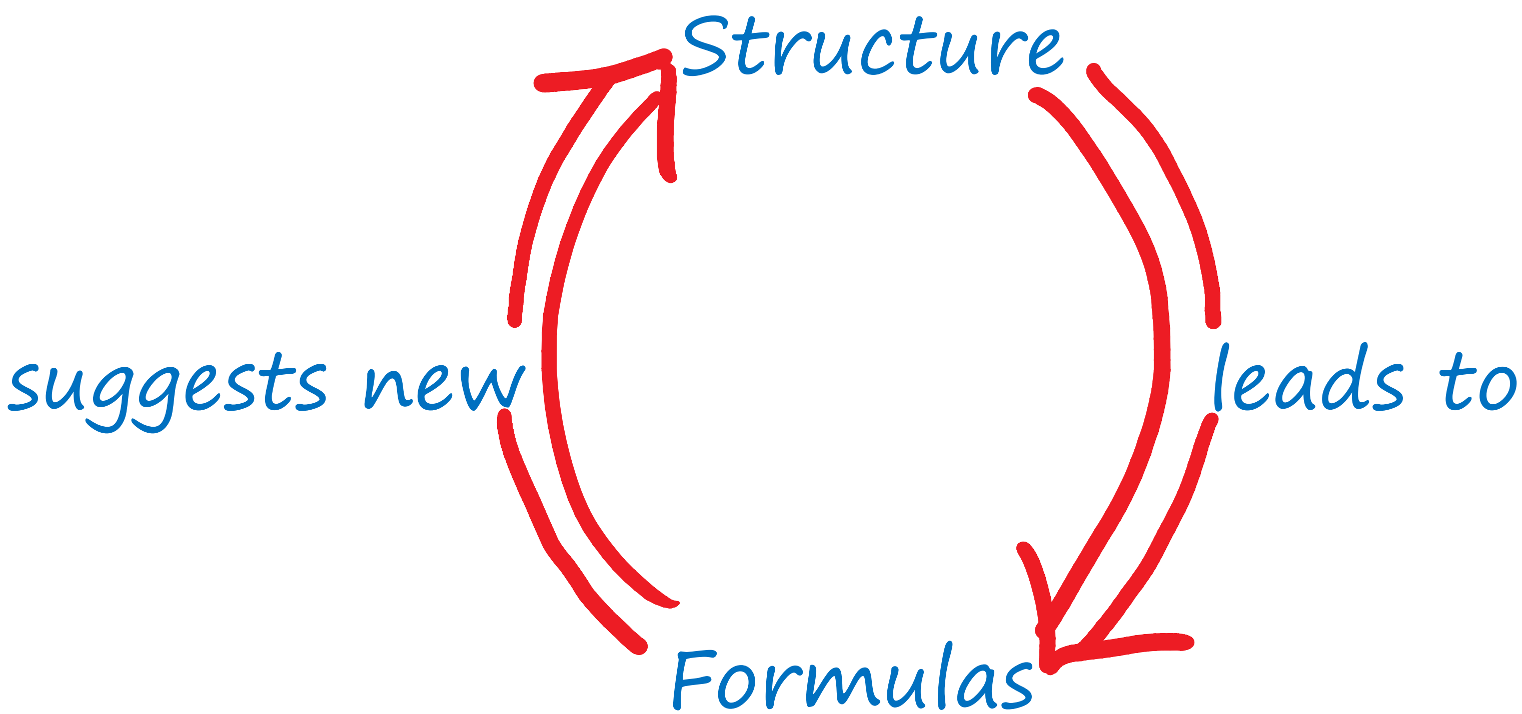
Schur processes $q = t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups

Macdonald processes: a source of (many parameter) integrable probabilistic systems. Specializations and degenerations include q-TASEP, continuum/semi-discrete/discrete SHE, KPZ equation

ASEP does not fit. But it does share certain parallel formulas



(Ascending) Macdonald processes are probability measures on interlacing triangular arrays (Gelfand-Tsetlin patterns)

$$\begin{array}{ccccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & \lambda^{(N)}_1 \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & & \cdots & & \lambda^{(N-1)}_1 & \\
& & \ddots & & \cdots & & \cdot^{\cdot^{\cdot}} & & \\
& & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & \\
& & & & \lambda^{(1)}_1 & & & &
\end{array}$$

$$\lambda^{(m)}_j \in \mathbb{Z}_{\geq 0}$$

Macdonald polynomials

$$\mathbb{P}\left(\lambda^{(k)}\right) = \frac{P_{\lambda^{(k)}}(a_1, \dots, a_k)\, Q_{\lambda^{(k)}}(b_1, \dots, b_M)}{\Pi(\underbrace{a_1, \dots, a_k}; \underbrace{b_1, \dots, b_M})}$$

normalization constant

two groups of parameters

Macdonald polynomials $P_\lambda(x_1,\dots,x_N)\in\mathbb{Q}(q,t)[x_1,\dots,x_N]^{S(N)}$ with partitions $\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$(\mathcal{D}_1 f)(x_1,\dots,x_N)=\sum_{i=1}^{n}\prod_{j\neq i}\frac{t x_i-x_j}{x_i-x_j}\, f(x_1,\dots,qx_i,\dots,x_N)$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda=(q^{\lambda_1}t^{N-1}+q^{\lambda_2}t^{N-2}+\dots+q^{\lambda_N})P_\lambda .$$

They have many remarkable properties that include orthogonality (dual basis Q_λ), simple reproducing kernel (Cauchy type identity), Pieri and branching rules, index/variable duality, explicit generators of the algebra of (Macdonald) operators commuting with $\mathcal{D}_1$, etc.

We are able to do two basic things:

- Construct relatively explicit Markov operators that map Macdonald processes to Macdonald processes;
- Evaluate averages of a broad class of observables.

The construction is based on commutativity of Markov operators

$$\mathbb{P}(\lambda\to\mu) = \frac{P_\mu(x_1,\dots,x_{n-1})}{P_\lambda(x_1,\dots,x_n)} P_{\lambda/\mu}(x_n), \qquad \mathbb{P}(\lambda\to\nu) = \frac{P_\nu(x_1,\dots,x_m)}{P_\lambda(x_1,\dots,x_m)} \frac{P_{\nu/\lambda}(u)}{\Pi(x;u)}$$

skew Macdonald polynomials

additional parameter

an idea from [Diaconis-Fill '90], and Schur process dynamics from [Borodin-Ferrari '08].

Evaluation of averages is based on the following observation. Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

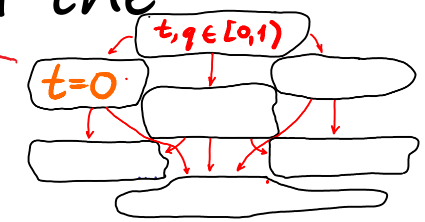
$$\mathcal{D} P_\lambda = d_\lambda P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$ we obtain

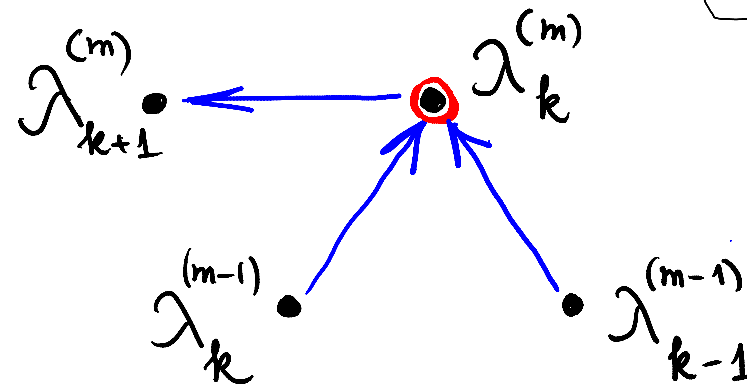
$$\mathbb{E}[d_\lambda] = \frac{\mathcal{D}^{(a)} \Pi(a;b)}{\Pi(a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Contrast with the lack of explicit formulas for the Macdonald polynomials.

Here is an example of a Markov process preserving the class of the q-Whittaker processes (Macdonald processes with $t=0$).

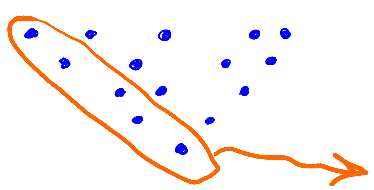


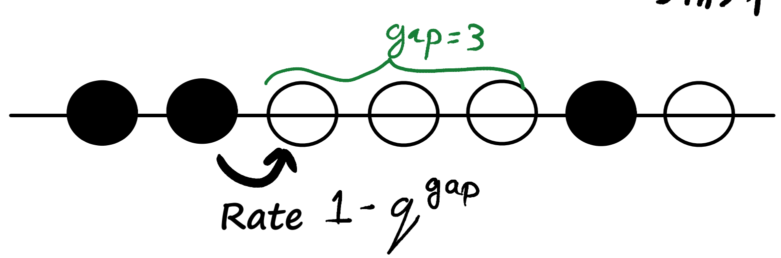
Each coordinate of the triangular array jumps by 1 to the right independently of the others with



$$\mathrm{rate}\left(\lambda^{(m)}_k\right) = \frac{\left(1-q^{\lambda^{(m-1)}_{k-1}-\lambda^{(m)}_k}\right)\left(1-q^{\lambda^{(m)}_k-\lambda^{(m)}_{k+1}+1}\right)}{\left(1-q^{\lambda^{(m)}_k-\lambda^{(m-1)}_k}\right)}.$$

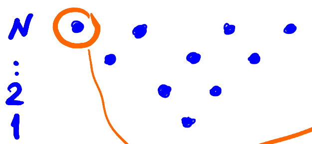
The set of coordinates $\left\{\lambda^{(m)}_m - m\right\}_{m\geq 1}$ forms q-TASEP

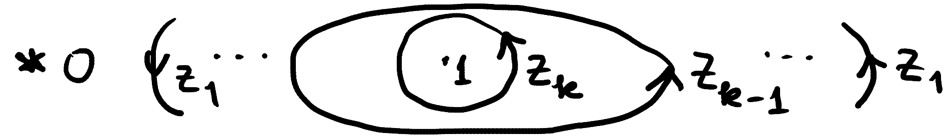




Taking the observables corresponding to powers of the first Macdonald operator yields

$$\mathbb{E}\left[\left(q^{\lambda_N^{(N)}(\tau)}\right)^k\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint\cdots\oint \prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k} \frac{e^{(q-1)\tau z_j}}{(1-z_j)^N}\,\frac{dz_j}{z_j}$$





$$\mathbb{E}\Bigg[\underbrace{\sum_{k=0}^{\infty} \frac{\left(q^{\lambda_N^{(N)}}\right)^k \zeta^k}{(1-q)\cdots(1-q^k)}}_{q\text{-Laplace transform of } q^{\lambda_N^{(N)}}}\Bigg] = \mathbb{E}\left[\frac{1}{\left(\zeta q^{\lambda_N^{(N)}};q\right)_\infty}\right] = \det(\mathbb{1}+K)_{L^2(\mathbb{N}\times \circlearrowleft_1)}$$

with

$$K(n_1,w_1;n_2,w_2) = \frac{f(w_1)\cdots f(q^{n_1-1}w_1)\,\zeta^{n_1}}{q^{n_1}w_1-w_2}, \qquad f(w) = \frac{e^{(q-1)\tau w}}{(1-w)^N}$$

A rigorous version of the physics "replica trick" q-TASEP $\xRightarrow{\quad}$ Semi-discrete SHE $\xRightarrow{\quad}$ Continuum SHE

$q^{x_n(t)+n}$ $\qquad Z(t,n)$ $\qquad Z(t,x)$

[Molchanov '86] [Kardar '87] observe $\bar{Z}(t;x_1,\ldots,x_k) := \mathbb{E}\left[\prod_{i=1}^{k} Z(t,x_i)\right]$

satisfies

$$\partial_t \bar{Z} = \frac{1}{2}\Big(\sum_{i=1}^{k} \partial_{x_i}^2 + \underset{=1}{c} \sum_{i\neq j} \delta_{x_i=x_j}\Big)\bar{Z}$$

"delta Bose gas"

Bethe ansatz [c<0 Lieb-Liniger '63, c>0 McGuire '64] gives eigenbasis

"Replica trick" [Dotsenko '10, Calabrese-Le Doussal-Rosso '10]

$$\mathbb{E}\left[e^{sZ}\right] \text{ ``}=\text{''} \sum_{k=0}^{\infty} \frac{\mathbb{E}[Z^k]\, s^k}{k!}$$

Divergent series! Risky to draw conclusions (originally obtained incorrect answer)

<u>Theorem (Borodin-Corwin, '11)</u>: For $x_1 < x_2 < \dots < x_k$, the integral

$$\mathcal{U}(t; x_1, \dots, x_k) := \int \cdots \int \prod_{A<B} \frac{z_A - z_B}{z_A - z_B - c} \prod_{j=1}^{k} e^{\frac{t}{2} z_j^2 + x_j z_j} \frac{dz_j}{2\pi i}$$

solves the delta Bose gas for all $c \in \mathbb{R}$ and for $\mathcal{U}(0; x_1, \dots, x_k) = \prod_{i=1}^{k} \delta_{x_i = 0}$

(Here the z_j-integration is over $\alpha_j + i\mathbb{R}$ with $\alpha_1 > \alpha_2 + c > \alpha_3 + 2c > \dots$)

- *Clear symmetry between attractive (c>0) and repulsive (c<0) cases*
- *Bethe eigenstates are very different in attractive/repulsive cases*
- *Formula can be found in [Heckman-Opdam '97] Plancherel theorem for delta Bose gas; ideas trace back to [Harish Chandra, Helgason]*

For semi-discrete SHE, $\mathbb{E}\left[\prod_{i=1}^{k} Z(t,n_i)\right]$ satisfies [Borodin-C '11]

$$\partial_t V(t;n_1,\ldots,n_k) = \left(\sum_{i=1}^{k} \nabla_i + \sum_{i<j} \mathbb{1}_{n_i=n_j}\right) V(t;n_1,\ldots,n_k)$$

∇_i acts as $(\nabla f)(n) = f(n-1) - f(n)$
in the n_i coordinate

For q-TASEP, $\mathbb{E}\left[\prod_{i=1}^{k} q^{X_{n_i}(t)+n_i}\right]$ satisfies [Borodin-C-Sasamoto '12]

$$\partial_t V(t;n_1,\ldots,n_k) = (1-q)\left(\sum_{i=1}^{k} \nabla_i + (1-q^{-1})\sum_{i<j} \mathbb{1}_{n_i=n_j}\, q^{j-i}\,\nabla_i\right) V(t;n_1,\ldots,n_k)$$

In all cases, the "nested contour integral ansatz" solves Bose gas

ASEP is not solved by Macdonald process. However,

- Self-duality of ASEP [Schutz '97] -> moments satisfy Bose gas
- Nested contour integral ansatz applies [Borodin-C-Sasamoto '12]
- Leads to two Fredholm determinants (one new and one TWs)
- TW compute ASEP k-particle Green's function via Bethe ansatz

Formulas suggest search for new structure:

- For q-TASEP: Nested contour integral formulas and Bose gas are consequences of structural properties of the Macdonald polynomials
- For ASEP: No structure to predict existence of nested contour integral formulas (duality is from $\mathcal{U}_q(sl_2)$ symmetry)

To summarize:

- ASEP and q-TASEP are important systems in the KPZ universality class, which can be scaled to the KPZ equation
- Macdonald processes are a source of integrable probabilistic models
- Generalize Schur processes but are not determinantal
- Integrability from structural properties of Macdonald polynomials (lead to nice Markov dynamics and concise formulas for averages)
- Turning averages into asymptotics remains challenging
- Rigorous replica trick developed for q-TASEP and ASEP
- Nested contour integral ansatz formulas for ASEP moments suggest search for new structure parallel to Macdonald processes

Lecture 1: Overview and intro to symmetric functions.

Lecture 2: Schur processes

Lecture 3: Macdonald processes I

Lecture 4: Macdonald processes II

Lecture 5: Duality and Bose gas methods

Lecture 6: Analysis of ASEP, conjectures and open problems

Exercise handout and office hours (Wed., Thur. 3-5pm 3.040)

Lectures times Wed. 10-12 and Thur. 9-11.

Website: http://math.mit.edu/~icorwin/Lipschitz.html