Integrable probability:
Macdonald processes, quantum integrable systems and the Kardar-Parisi-Zhang universality class

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Defining the L-matrix

Goal: Construct and analyze interacting particle systems related to integrable higher spin vertex models. In so doing, unite all integrable members of the KPZ universality class in one 4-parameter family.

Vector spaces:
(H^J likewise)

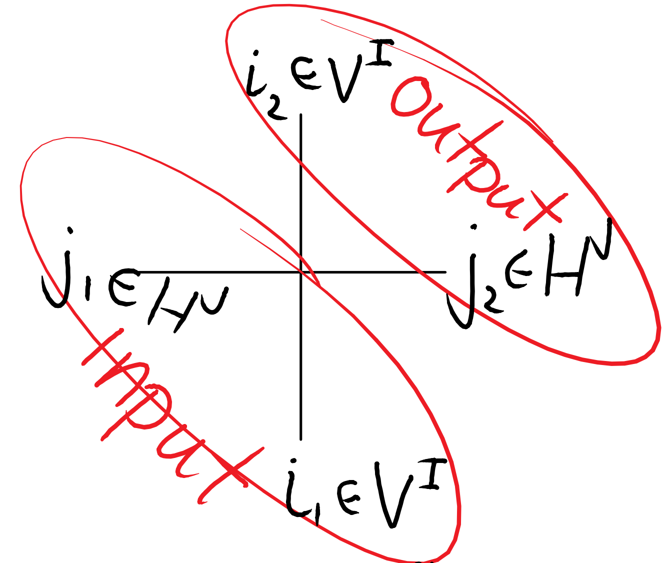
$$V^I = \begin{cases} \mathrm{span}(0,1,\dots,I) & , \ I \in \mathbb{Z}_{\geq 0} \\ \mathrm{span}(0,1,\dots) & , \ \text{else} \end{cases}$$

L-matrix: Indexed by complex parameters q, α, I, J such that

$$L : V^I \otimes H^J \to V^I \otimes H^J$$

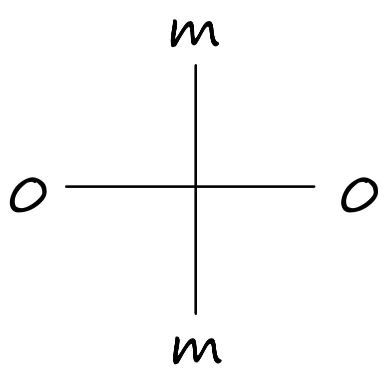
For most of the talk, we will set $J=1$, $\nu = q^{-I}$ and write $L_\alpha^{(J)}$.

L-matrix weights

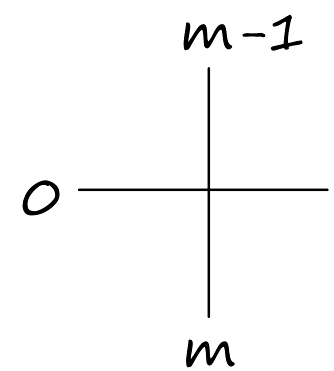


L-matrix elements: $L(i_1, j_1; i_2, j_2)$ indexed by $i_1, i_2 \in V^I$ and $j_1, j_2 \in H^\nu$.

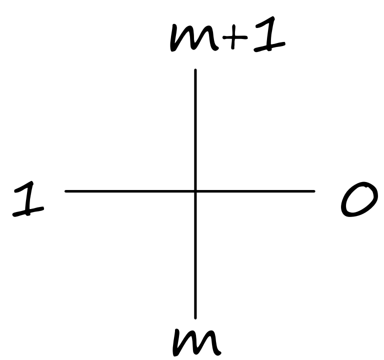
Definition: For $J=1$ and $m \geq 0$, the non-zero entries of $L^{(1)}_\alpha$ are:



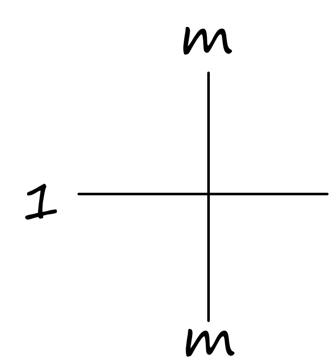
$$L^{(1)}_\alpha(m,0;m,0) = \frac{1+\alpha q^m}{1+\alpha}$$



$$L^{(1)}_\alpha(m,0;m-1,0) = \frac{\alpha(1-q^m)}{1+\alpha}$$



$$L^{(1)}_\alpha(m,1;m+1,0) = \frac{1-\nu q^m}{1+\alpha}$$



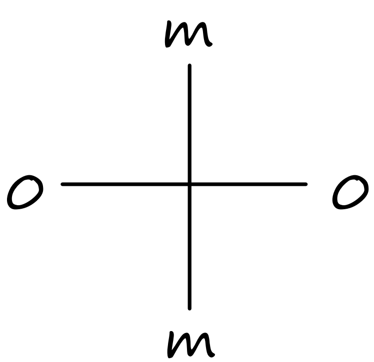
$$L^{(1)}_\alpha(m,1;m,1) = \frac{\alpha+\nu q^m}{1+\alpha}$$

L-matrix weights

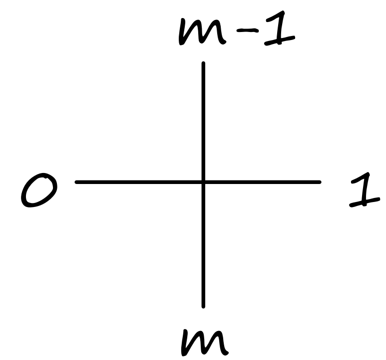
Particle conservation: sum of inputs $i_1 + j_1$ = sum of outputs $i_2 + j_2$

Stochasticity: Given i_1, j_1, sum over $i_2, j_2 = 1$; positive entries if:

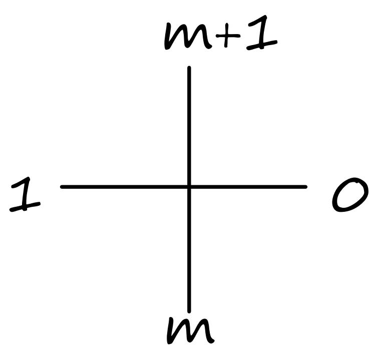
1) $q, \nu \in (0,1)$, $\alpha > 0$.
2) $q \in (1, \infty)$, $\nu = q^{-I}$ for $I \in \mathbb{Z}_{\geq 1}$, $\alpha \in (-\nu, 0)$.



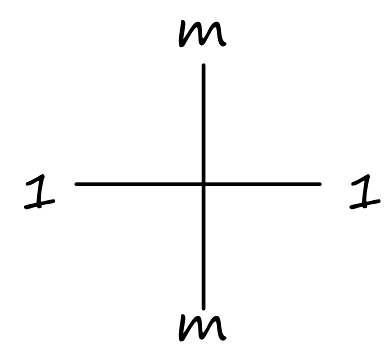
$$L^{(1)}_{\alpha}(m,0;m,0) = \frac{1+\alpha q^m}{1+\alpha}$$



$$L^{(1)}_{\alpha}(m,0;m-1,0) = \frac{\alpha(1-q^m)}{1+\alpha}$$



$$L^{(1)}_{\alpha}(m,1;m+1,0) = \frac{1-\nu q^m}{1+\alpha}$$



$$L^{(1)}_{\alpha}(m,1;m,1) = \frac{\alpha+\nu q^m}{1+\alpha}$$

Aside: Six vertex R-matrix

Define $$R(z) = \begin{pmatrix} zKq^{1/2} - z^{-1}K^{-1}q^{-1/2} & zq^{1/2}e \\ z^{-1}q^{-1/2}f & zK^{-1}q^{1/2} - z^{-1}Kq^{-1/2} \end{pmatrix}$$

where e, f, k generate C_q, related to $U_q(\widehat{sl2})$, via the relations:

$$Ke = qeK, \quad Kf = q^{-1}fK, \quad ef - fe = (q - q^{-1})(K^2 - K^{-2}).$$

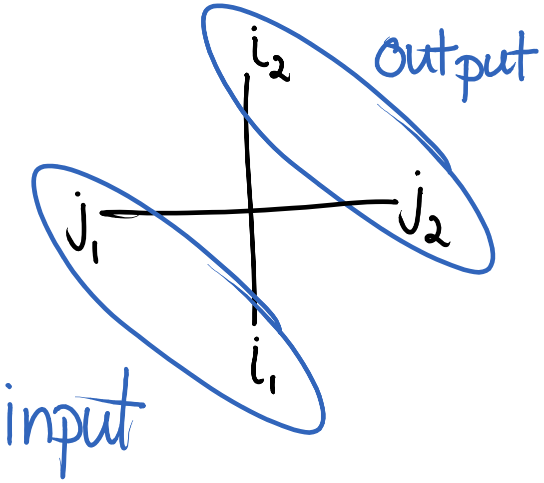
$V^{(I)} \cong \mathbb{C}^{I+1}$ is $I+1$ dim (spin $I/2$) irreducible rep'n with action

$$K V_i^{(I)} = q^{\frac{I}{2}-i} V_i^{(I)}, \quad f V_i^{(I)} = (q^{I-i} - q^{-I+i}) V_{i+1}^{(I)}, \quad e V_i^{(I)} = (q^i - q^{-i}) V_{i-1}^{(I)}$$

and basis $V^{(I)} = \mathrm{Span}\{V_0^{(I)}, V_1^{(I)}, \ldots V_I^{(I)}\}$. Identify $V_i^{(I)}$ as i up-spins / particles

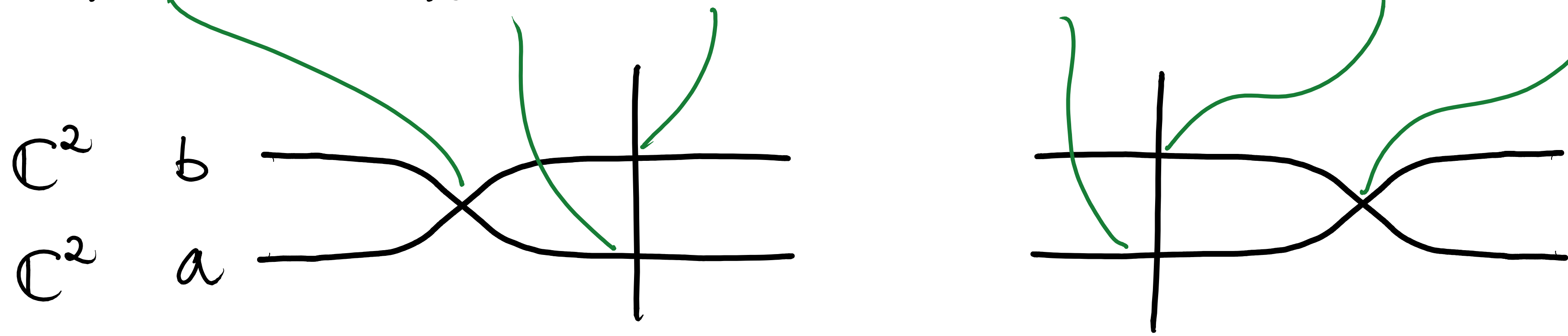
Aside: Six vertex R-matrix

With C_q acting on V, R(z) maps $\mathbb{C}^2 \otimes V$ to itself with matrix elements indexed by $i_1, i_2 \in V$, $j_1, j_2 \in \mathbb{C}^2$



The R-matrix satisfies the Yang-Baxter relation

$$R_{a,b}(z/w)\, R_{1,b}(z)\, R_{1,a}(w) = R_{1,a}(w)\, R_{1,b}(z)\, R_{a,b}(z/w)$$

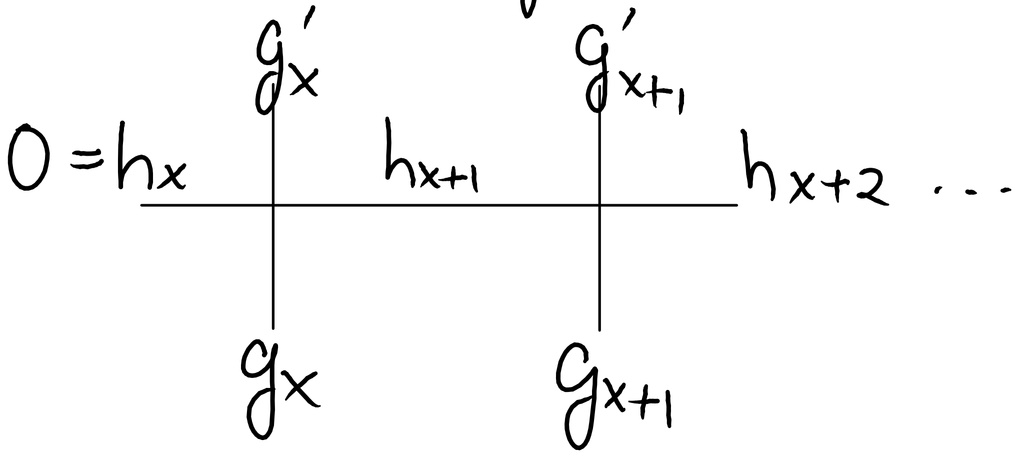


Algebraic Bethe ansatz diagonalizes associated transfer matrix.

Our L-matrix is a modification of R-matrix, so as to be stochastic.

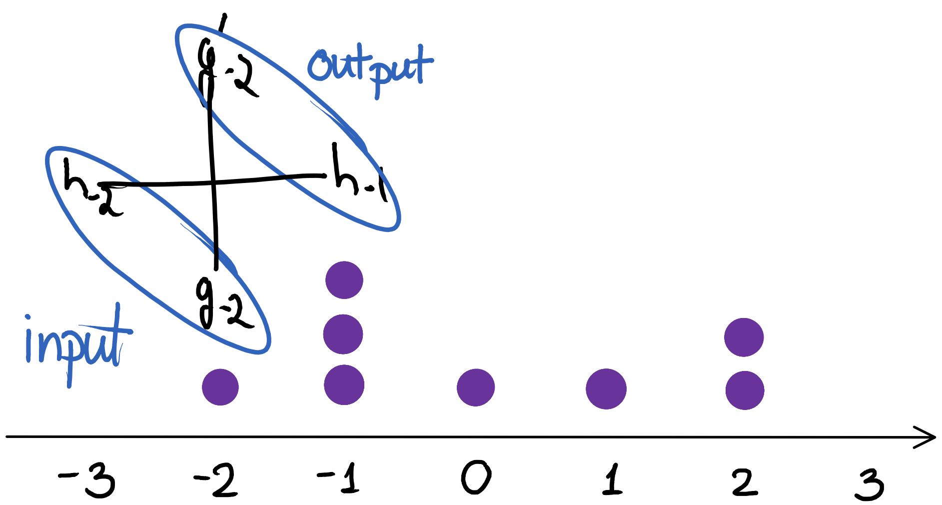
Zero range process (stochastic transfer matrix)

ZRP: State $\vec{g} = (g_i)_{i\in\mathbb{Z}}$, $g_i \in \mathbb{Z}_{\geq 0}$, $\exists x$ s.t. $g_x > 0$, $g_y = 0$ $\forall y < x$.



Seq'n (left->right) update via $L^{(1)}_\alpha$ Markov chain so given g_x, h_x choose g'_x, h_{x+1} with probability $L^{(1)}_\alpha(g_x, h_x; g'_x, h_{x+1})$.

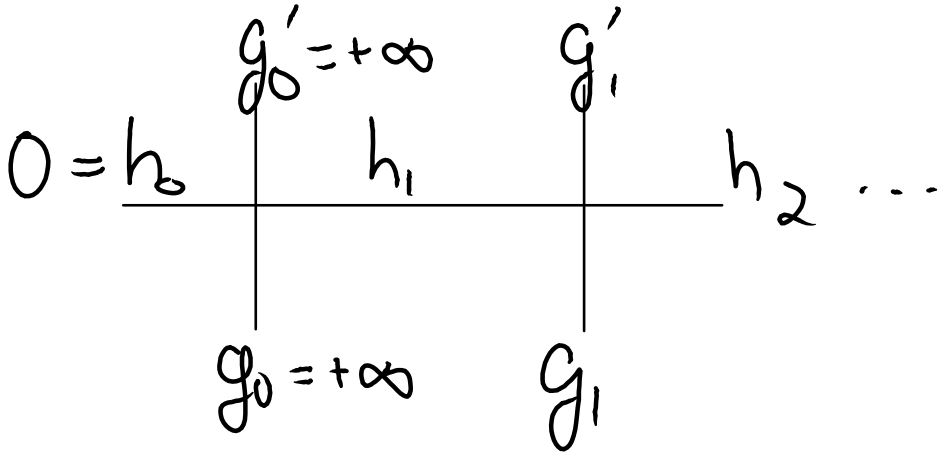
(dynamics conserve sum of g's, and h's $=0$ or 1)



Call $B^{\alpha, q\alpha}(\vec{g}, \vec{g}')$ transition probability / matrix and define its space reversal $\tilde{B}^{\alpha, q\alpha}(\vec{y}, \vec{y}')$ with state variables $\vec{y}$.

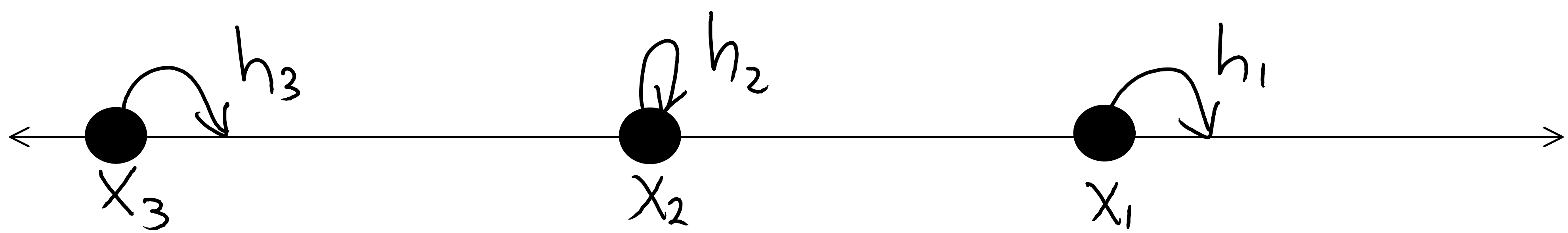
Asymmetric exclusion process

<u>AEP</u>: State $\vec{x} = (x_1 > x_2 > \cdots)$, $x_i \in \mathbb{Z}$, $x_i \equiv +\infty$, $i \le 0$ (need V^I inf. dim)



ZRP on holes: Let $g_i = x_i - x_{i+1} - 1$ and update $\vec{g} \to \vec{g}'$ via ZRP. Set $x_i' = x_i + h_i$

Call $T^{\alpha, q\alpha}(\vec{x}, \vec{x}')$ transition probability / matrix for the AEP.



What do we do?

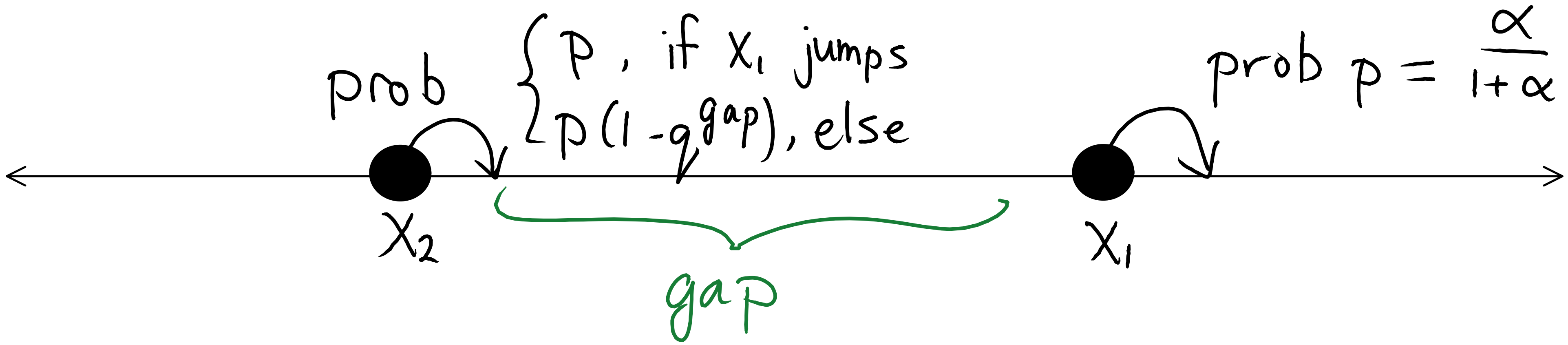
- **Diagonalize** transfer matrices (B's) via the q-Hahn Boson eigenfunctions on the line (uses completeness / Plancherel theory)
- Demonstrate Markov **dualities** between $B \leftrightarrow \widetilde{B}$ and $T \longleftrightarrow \widetilde{B}$ which enables the computation of moments / distribution functions.
- Generalize to J>1 via probabilistic version of '**fusion**'.

The resulting 4-parameter q, α, ν, J family of processes is on the top of the (known) ***integrable KPZ class hierarchy*** and we gain tools to study all of the models simultaneously.

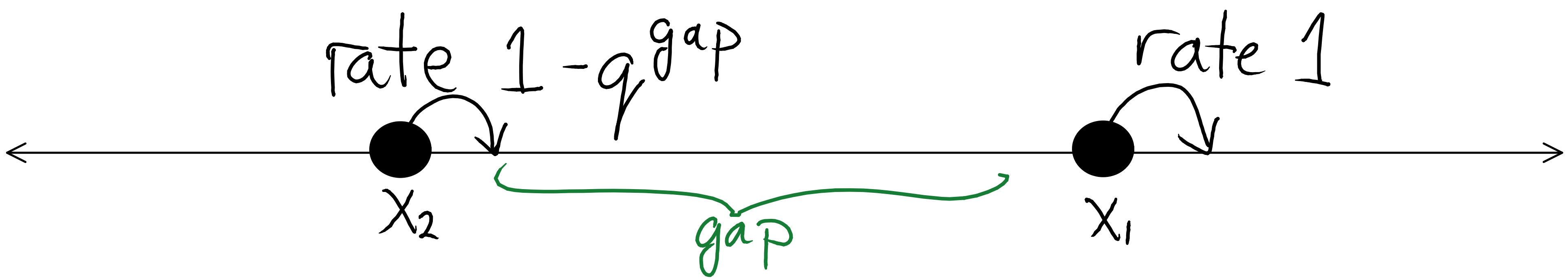
But first, let's explore two special degenerations of the processes.

Bernoulli q-TASEP

Take $V = 0$, $q \in (0,1)$, $\alpha > 0$ then the AEP becomes

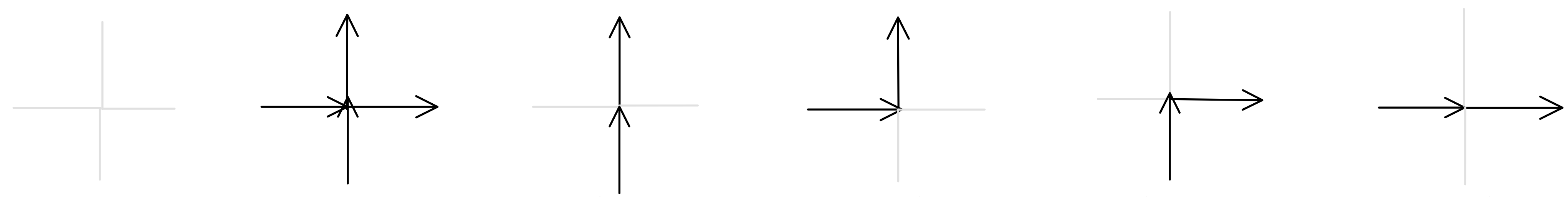


Taking $p \to 0$, jumps become seldom and speeding up by $1/p$ we recover the continuous time q-TASEP



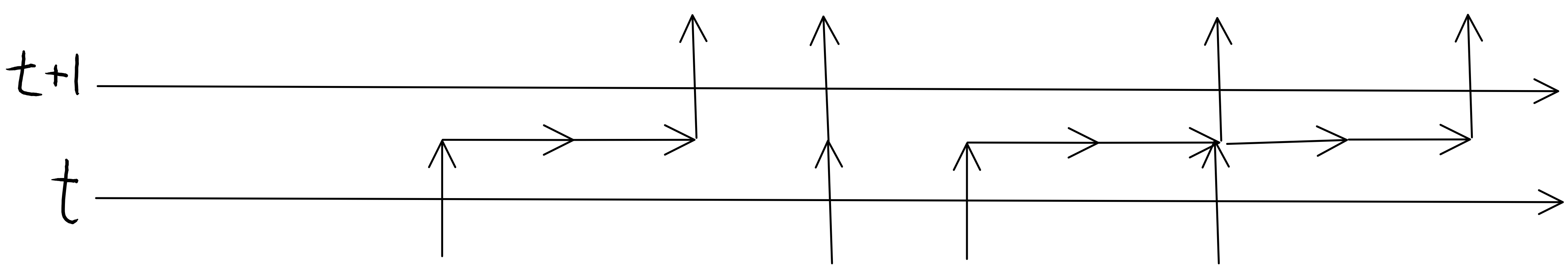
Stochastic six vertex model

Take $\nu = q^{-1}$ $(I=1)$, $q \in (1,\infty)$, $\alpha \in (-\nu, 0)$. The six non-zero weights depend on q, α and can be reparameterized via $b_1, b_2 \in (0,1)$ as

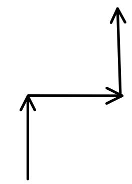


weight	1	1	b_1	$1-b_1$	b_2	$1-b_2$

ZRP obeys exclusion rule [Gwa-Spohn '92], [Borodin-C-Gorin '14].



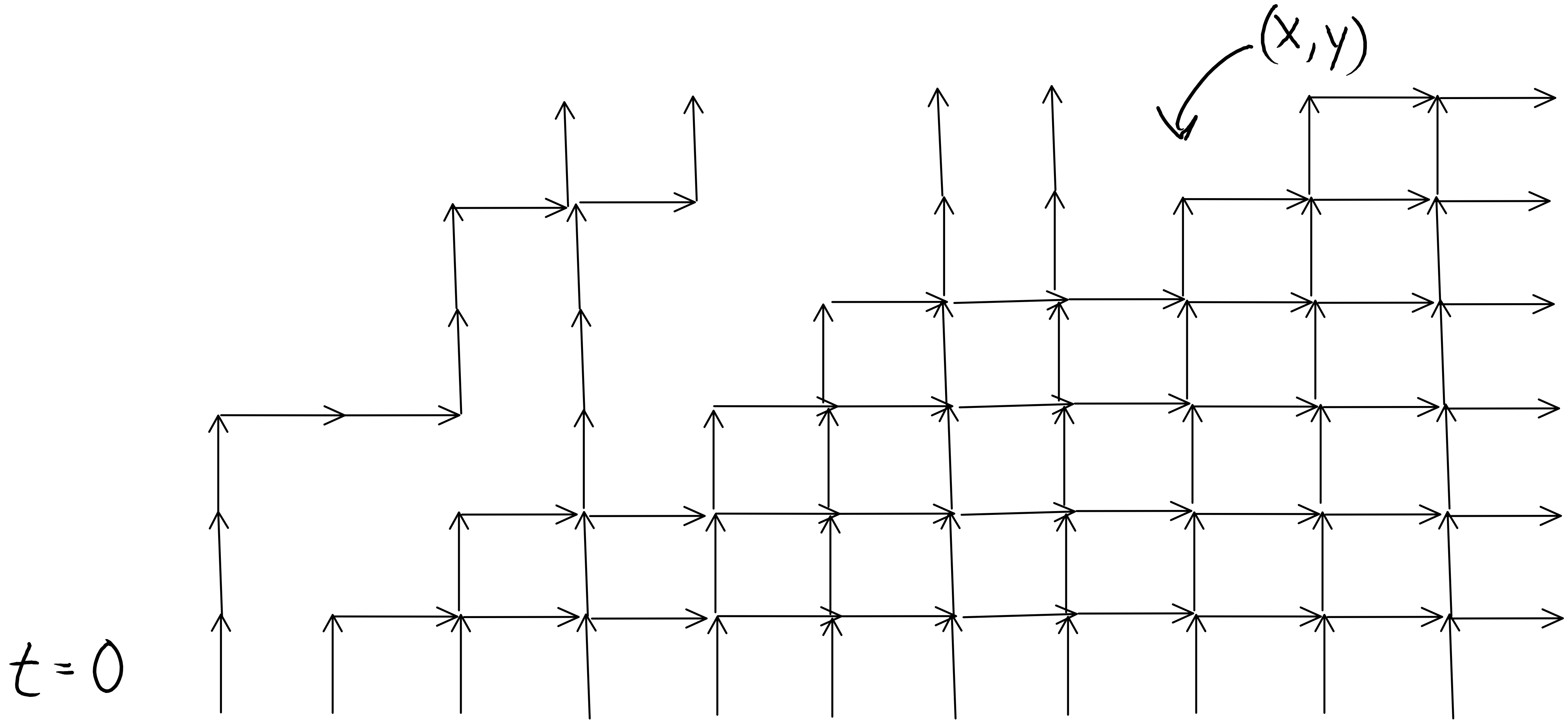
ASEP limits

The ratio $\frac{b_2}{b_1} = \nu$. Fixing this and taking $b_1, b_2 \searrow 0$ $(\alpha \searrow -\nu)$
Particles almost always follow a  trajectory. Subtracting this diagonal motion and speeding up time by $1/b$ we arrive at ASEP with left jump rate ℓ and right jump rate r having ratio $\frac{r}{\ell} = \nu$.

Thus we see already that we have **united q-TASEP and ASEP as processes** (earlier we saw their spectral theory united).

Note: q-TASEP has extra structure from Macdonald processes.
No generalization of Macdonald processes is known when $\nu \neq 0$.

Half domain wall boundary conditions (step initial data)



Start stochastic six-vertex with $g_i(0) = 1_{i \geq 0}$ and define a height function: $H(x,y) = \#$ lines left of (x,y).

Asymptotics

Theorem [Borodin-C-Gorin '14]: For $0 < b_2 < b_1 < 1$, $\kappa := \frac{1-b_1}{1-b_2}$ we have

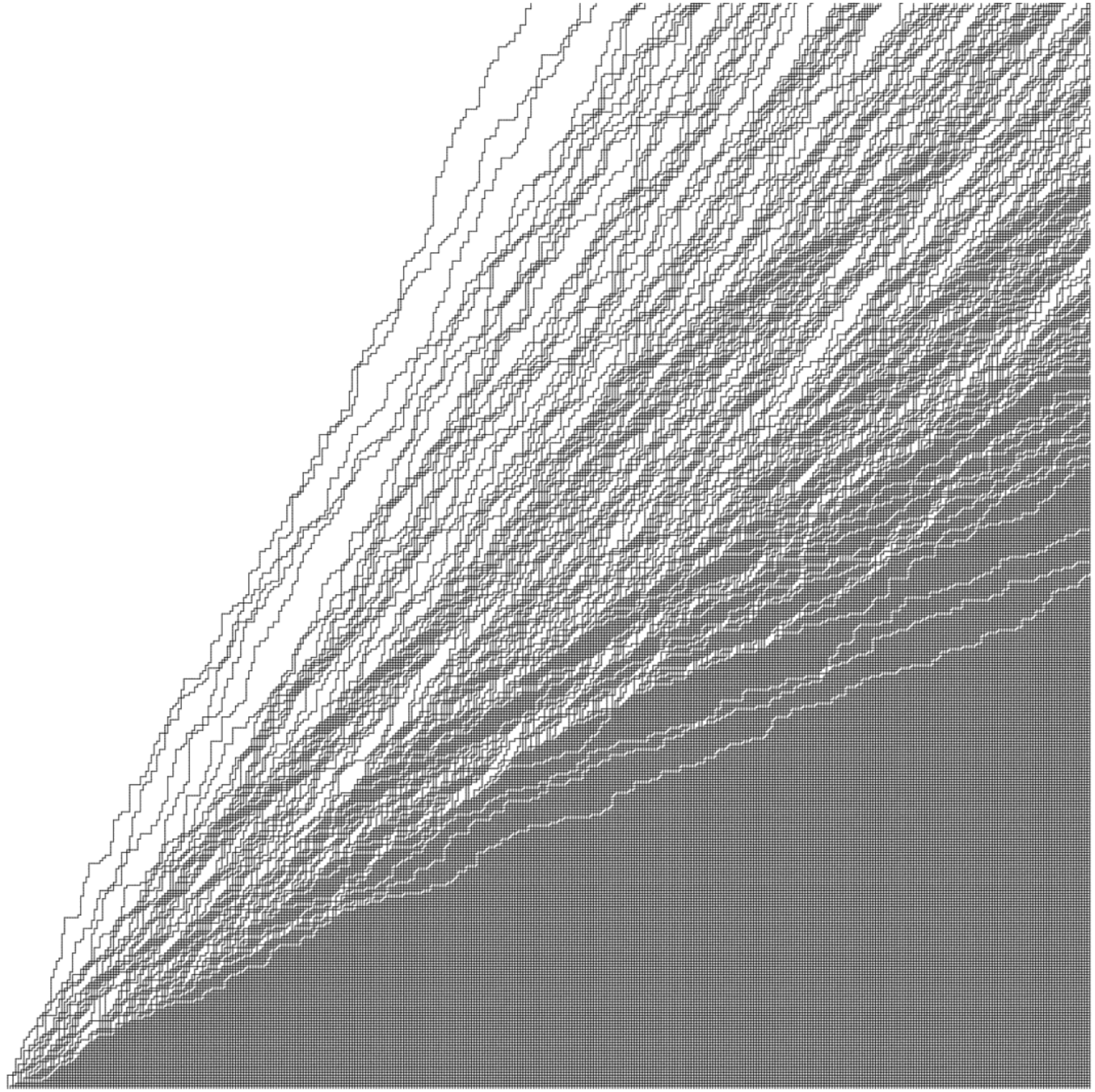
Law of large numbers:

$$\lim_{L\to\infty} \frac{H(Lx,Ly)}{L} = \mathcal{H}(x,y) := \begin{cases} 0 & , \frac{x}{y} < \kappa \\ \left(\sqrt{y(1-b_1)} - \sqrt{x(1-b_2)}\right)^2 & , \kappa < \frac{x}{y} < \frac{1}{\kappa} \\ x-y & , \frac{1}{\kappa} < \frac{x}{y} \end{cases}$$

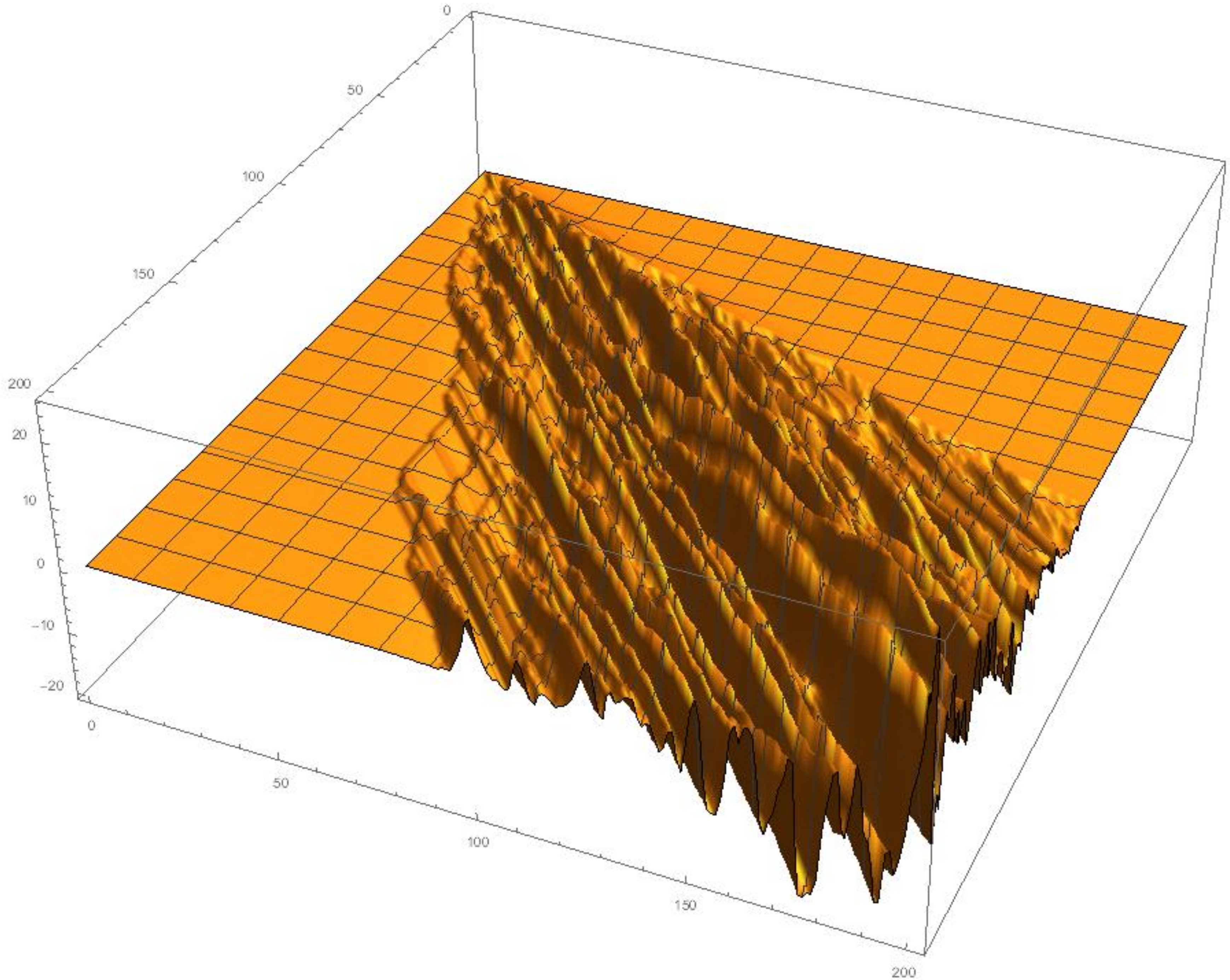
Central limit theorem: For $\kappa < \frac{x}{y} < \frac{1}{\kappa}$,

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{\mathcal{H}(x,y)L - H(Lx,Ly)}{\sigma_{x,y} L^{1/3}} \le s \right) = F_{GUE}(s)$$

Asymptotics



Asymptotics



Bethe ansatz diagonalization

Consider the space-reverse ZRP with k particles ($\sum y_i = k$) and label stated by $(n_1 \geq n_2 \geq \cdots \geq n_k) = \vec{n}$. Recall q-Hahn left eigenfunction:

$$\Psi^{\ell}_{\vec{z}}(\vec{n}) = \sum_{\sigma \in S(k)} \prod_{a>b} \frac{z_{\sigma(a)} - q z_{\sigma(b)}}{z_{\sigma(a)} - z_{\sigma(b)}} \prod_{j=1}^{k} \left(\frac{1 - \nu z_{\sigma(j)}}{1 - z_{\sigma(j)}} \right)^{n_j}$$

indexed by $z_1, \ldots, z_k \in \mathbb{C} \setminus \{1, \nu^{-1}\}$ and depending on q, ν only.

<u>Theorem</u>: For $z_i : \left| \frac{1 - z_i}{1 - \nu z_i} \cdot \frac{\alpha + \nu}{\alpha + 1} \right| < 1, \quad i = 1, \ldots, k$

$$\left(\tilde{B}^{\alpha, q\alpha} \Psi^{\ell}_{\vec{z}} \right)(\vec{n}) = \prod_{i=1}^{k} \frac{1 + q\alpha z_i}{1 + \alpha z_i} \Psi^{\ell}_{\vec{z}}(\vec{n}).$$

Follows Algebraic Bethe Ansatz, or recent work of [Borodin '14].

Relation q-Hahn Boson transition operator

Recall that for the q-Hahn Boson process with parameters q, μ, ν

$$P^{\text{Boson}}_{q,\mu,\nu} \Psi^{\ell}_{\vec{z}} = \prod_{j=1}^{k} \frac{1-\mu z_j}{1-\nu z_j} \Psi^{\ell}_{\vec{z}} .$$

This, along with the Plancherel theory implies that we can write

$$\tilde{B}^{\alpha, q\alpha} = P^{\text{Boson}}_{q,-q\alpha,\nu} \left(P^{\text{Boson}}_{q,-\alpha,\nu}\right)^{-1} .$$

Note: While q-Hahn Boson process can be factored into free evolution equation and 2-body boundary conditions, the general ZRP considered here does not admit such a factorization.

Direct and inverse Fourier type transforms

Let
$$\mathcal{W}^k = \{ f : \{ n_1 \geq \ldots \geq n_k \mid n_j \in \mathbb{Z} \} \to \mathbb{C} \text{ of compact support} \}$$
$$\mathcal{C}^k = \mathbb{C}\left[\left(\frac{1-\nu z_1}{1-z_1}\right)^{\pm 1}, \ldots, \left(\frac{1-\nu z_k}{1-z_k}\right)^{\pm 1} \right]^{S(k)} = \text{symmetric Laurent poly's in } \left(\frac{1-\nu z_j}{1-z_j}\right),\ 1 \leq j \leq k.$$

Direct tranform: $\mathcal{F} : \mathcal{W}^k \to \mathcal{C}^k$

$$\mathcal{F} : f \longmapsto \sum_{n_1 \geq \ldots \geq n_k} f(\vec{n}) \cdot \Psi^r_{\vec{z}}(\vec{n}) =: \langle f, \Psi^r_{\vec{z}} \rangle_{\mathcal{W}}$$

Inverse transform: $\mathcal{J} : \mathcal{C}^k \to \mathcal{W}^k$

$$\mathcal{J} : G \longmapsto (q-1)^k q^{-\frac{k(k-1)}{2}} \frac{1}{(2\pi i)^k k!} \underset{\substack{|w_j| = R \in (1, \nu^{-1}) \\ j = 1, \ldots, k}}{\oint \cdots \oint} \det \left[\frac{1}{q w_i - w_j} \right]_{i,j=1}^k \prod_{j=1}^k \frac{w_j}{(1-w_j)(1-\nu w_j)} \Psi^\ell_{\vec{w}}(\vec{n}) G(\vec{w}) \, d\vec{w}$$
$$=: \langle \Psi^\ell(\vec{n}), G \rangle_{\mathcal{C}}$$

Plancherel isomorphism theorem

Theorem [Borodin-C-Petrov-Sasamoto '14] On spaces $\mathcal{W}^k$ and $\mathcal{C}^k$, operators $\mathcal{F}$ and $\mathcal{J}$ are mutual inverses of each other.

Isometry:

$$\langle f, g\rangle_{\mathcal{W}} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{\mathcal{C}} \quad \text{for } f, g \in \mathcal{W}^k$$

$$\langle F, G\rangle_{\mathcal{C}} = \langle \mathcal{J}F, \mathcal{J}G\rangle_{\mathcal{W}} \quad \text{for } F, G \in \mathcal{C}^k$$

Biorthogonality:

$$\langle \Psi^{\ell}_{\bullet}(\vec{m}), \Psi^{r}_{\bullet}(\vec{n})\rangle_{\mathcal{C}} = \delta_{\vec{m},\vec{n}}$$

in a certain weak sense

$$\langle \Psi^{\ell}_{\vec{z}}(\bullet), \Psi^{r}_{\vec{w}}(\bullet)\rangle_{\mathcal{W}} = \prod_{a\neq b} \frac{z_a - q z_b}{z_a - z_b} \prod_{j=1}^{k} (1-z_j)(1-\nu z_j) \det\left[\delta(z_i - w_j)\right]_{i,j=1}^{k}$$

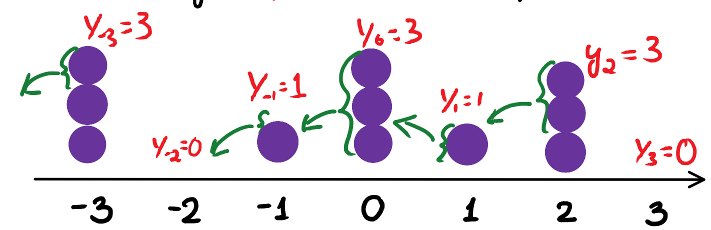
Proof of $\mathcal{J}\mathcal{F} = \mathrm{Id}$ uses residue calculus in nested contour version of $\mathcal{J}$, while proof of $\mathcal{F}\mathcal{J} = \mathrm{Id}$ uses existence of simultaneously diagonalized family of matrices

Back to the q-Hahn Boson particle system

<u>Corollary</u> The (unique) solution of the ZRP evolution equation

$$f(t+1,\vec{y}) = \tilde{\mathcal{B}}^{\alpha,q\alpha} f(t,\vec{y}) \text{ with } f(0,\vec{y}) = f_0(\vec{y}) \text{ is}$$

particles at $\vec{n}=(2,2,2,1,0,0,0,-1,-3,-3)$
heights $\vec{y}(\vec{n})$, $k=11$ particles



$$f(t,\vec{y}(\vec{n})) = \mathcal{J}\Bigg(\Bigg(\underbrace{\prod_{j=1}^{k}\frac{1+q\alpha z_i}{1+\alpha z_i}}_{\text{eigenvalue of } H \text{ corr. to } \Psi_{\vec{z}}}\Bigg)^t \mathcal{F} f_0\Bigg)$$

$$= \frac{1}{(2\pi i)^k}\underbrace{\oint\cdots\oint}_{\text{nested}} \prod_{a<b}\frac{z_a-z_b}{z_a-qz_b}\prod_{j=1}^{k}\left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j}\frac{1}{(1-z_j)(1-\nu z_j)}\left(\prod_{j=1}^{k}\frac{1+q\alpha z_i}{1+\alpha z_i}\right)^t \langle f_0, \Psi^r_{\vec{z}}\rangle_W \, d\vec{z}$$

Already know $\mathcal{F} f_0$ for step initial data $f_0(\vec{n}) = \mathbb{1}_{\{n_i \geq 1,\ 1\leq i\leq k\}}$.

Can solve Kolmogorov forward equation for transition probabilities

AEP – ZRP duality

Define a duality functional $H(\vec{x},\vec{y}) := \prod_{i\in\mathbb{Z}} q^{(x_i+i)y_i}$ $\left(=0 \text{ if } y_i>0 \text{ for any } i\le 0\right)$

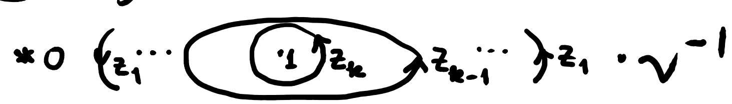
Theorem [C-Petrov '15]: $T^{\alpha,q\alpha} H = H\,(\tilde{B}^{\alpha,q\alpha})^T$

Corollary: $\mathbb{E}\left[H(\vec{x}(t),\vec{y})\right] = (\tilde{B}^{\alpha,q\alpha})^t\,\mathbb{E}\left[H(\vec{x}(0),\vec{y})\right]$

Corollary: For the AEP with step initial data $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\left[q^{(x_{n_1}(t)+n_1)+\dots+(x_{n_k}(t)+n_k)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\oint\cdots\oint \prod_{A<B}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}\left(\frac{1-\nu z_j}{1-z_j}\right)^{n_j}\left(\frac{1+q\alpha z_j}{1+\alpha z_j}\right)^t\frac{dz_j}{z_j}$$

$(n_1\geq n_2\geq\dots\geq n_k)$



This is the starting point for distributional formulas and asymptotics.

ZRP self-duality

Define a duality functional $G(\vec{g},\vec{y}) := \prod_{i>j} q^{g_i y_j}$

Theorem [C-Petrov '15]: $B^{\alpha, q\alpha} G = G(\tilde{B}^{\alpha, q\alpha})^T$

- There are other derivative self-dualities which comes from this.
- This generalizes the ASEP self-duality [Schutz '97], [Borodin-C-Sasamoto '12] and yields a self-duality for stochastic six-vertex which enables the computation of $\mathbb{E}[\nu^{k N_y(\vec{g}(t))}]$, where we have that $\nu = q^{-1}$, $N_y(\vec{g})$ = # particles in $\vec{g}$ left of y.
- AEP-ZRP duality generalizes that for q-Hahn TASEP/Boson.

J>1 via fusion

Define the higher horizontal spin ZRP transition operator as

$$B^{\alpha, q^J\alpha} := B^{\alpha, q\alpha} B^{q\alpha, q^2\alpha} \cdots B^{q^{J-1}\alpha, q^J\alpha}.$$

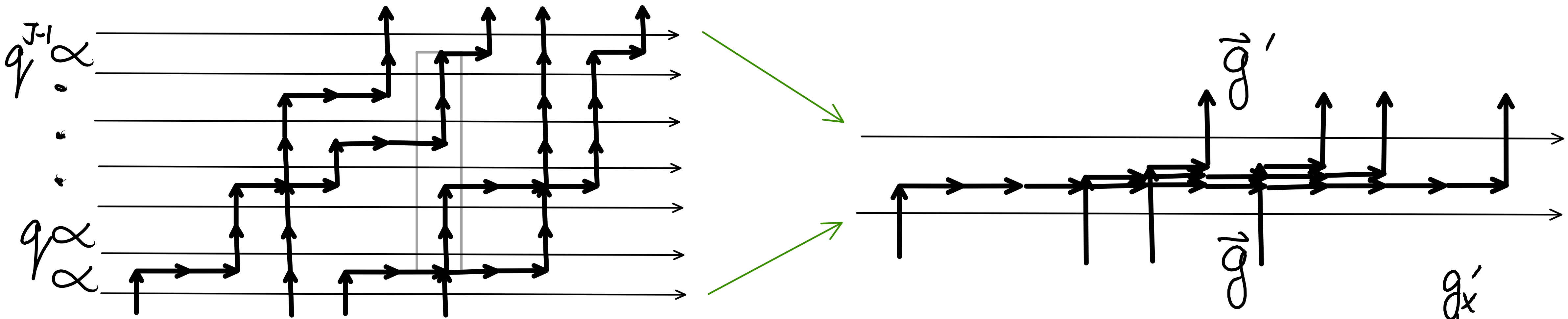
Clearly this is still stochastic (if each B was) and it is diagonalized via the same eigenfunctions with eigenvalue $\prod_{j=1}^{k} \frac{1+q^J\alpha z_j}{1+\alpha z_j}$.

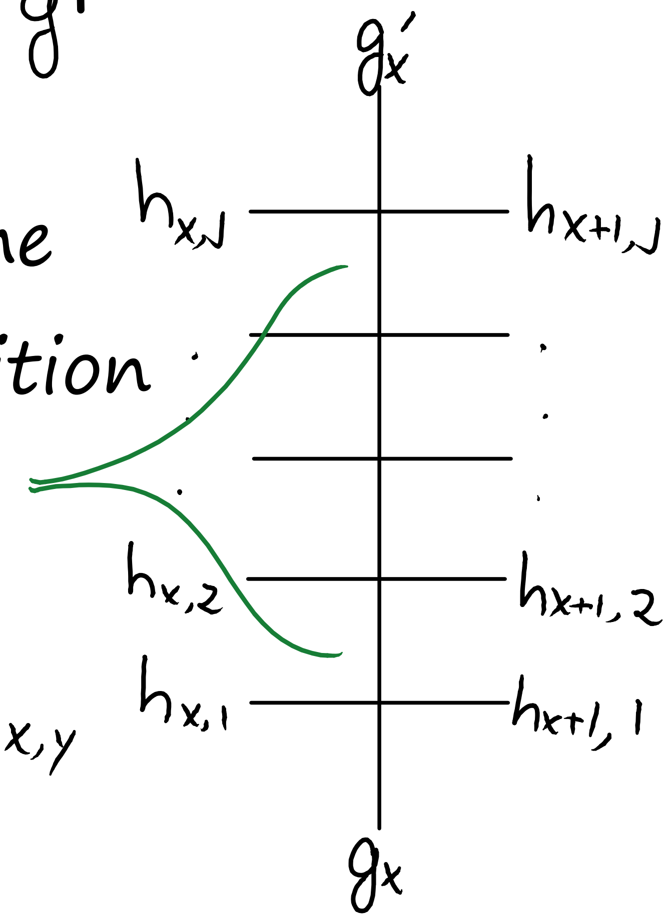
<u>Question</u>: Can this be realized via a sequential update Markov chain using some $L_\alpha^{(J)} : V^I \otimes H^J \to V^I \otimes H^J$?

<u>Answer</u>: Yes, due to [Kirillov-Reshetikhin '87] fusion procedure.

This simplifies on the line and we provide a probabilistic proof.

J>1 via fusion





- Call $S=(H^1)^{\otimes J}$, $S'=H^J$. Given $\vec{g}$ define the x-dependent horizontal Markov chain with transition matrix $P_x : S \to S$ given by the product of weights with $h_{x,1} \otimes \cdots \otimes h_{x,J} \in S$.
- Define $\Phi : S \to S'$ as $\Phi(h_{x,1} \otimes \cdots \otimes h_{x,J}) = \sum_{y=1}^{J} h_{x,y}$

Is Φ applied to the P_x Markov chain still Markov?

Markov function theory

Theorem [Pitman-Rogers '80]: If there exists a Markov kernel $\Lambda : S' \to S$ such that

- $\Lambda \Phi =$ Identity on S'
- $Q_x := \Lambda P_x \Phi : S' \to S'$ satisfies $\Lambda P_x = Q_x \Lambda$

Then starting the P_x chain with measure $\Lambda(y; \bullet)$, its image under Φ coincides in law with the Q_x chain start at y.

$$
\begin{tikzcd}
S \arrow[r, "P_x"] \arrow[d, shift right, "\Phi"'] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[r, "P_x"] \arrow[d, shift right, "\Phi"'] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[r, "P_x"] \arrow[d, shift right, "\Phi"'] \arrow[dr, phantom, "\circlearrowright"] & \cdots \\
S' \arrow[r, "Q_x"'] \arrow[u, shift right, "\Lambda"'] & S' \arrow[r, "Q_x"'] \arrow[u, shift right, "\Lambda"'] & S' \arrow[r, "Q_x"'] \arrow[u, shift right, "\Lambda"'] & {}
\end{tikzcd}
$$

Applying markov function theory

Define $\Lambda(h_x ; h_{x,1} \otimes \cdots \otimes h_{x,J}) = Z^{-1} \mathbb{1}_{\Phi(h_{x,1} \otimes \cdots \otimes h_{x,J}) = h_x} \prod_{y : h_{x,y} = 1} q_y$

This is the conditional distribution of $h_{x,1} \otimes \cdots \otimes h_{x,J}$ given its sum.

Theorem [C-Petrov '15]: Both conditions of Pitman-Rogers are satisfied and thus the Q_x chain started with $h_x = 0$ provides a way to sequentially update $\vec{g} \to \vec{g}'$ so as to agree with $B^{\alpha, q^J \alpha}$.

In particular, we may define $L^{(J)}_{\alpha}(g_x, h_x ; g_x', h_{x+1}) := Q_x(h_x, h_{x+1})$.

Can also develop a recusion in J for the L-matrix elements by decomposing J vertical steps in to 1 followed by J-1.

Explicit formula for higher spin L-matrix

Based on [Mangazeev '14] we solve the recursion explicitly ($\beta := \alpha q^J$)

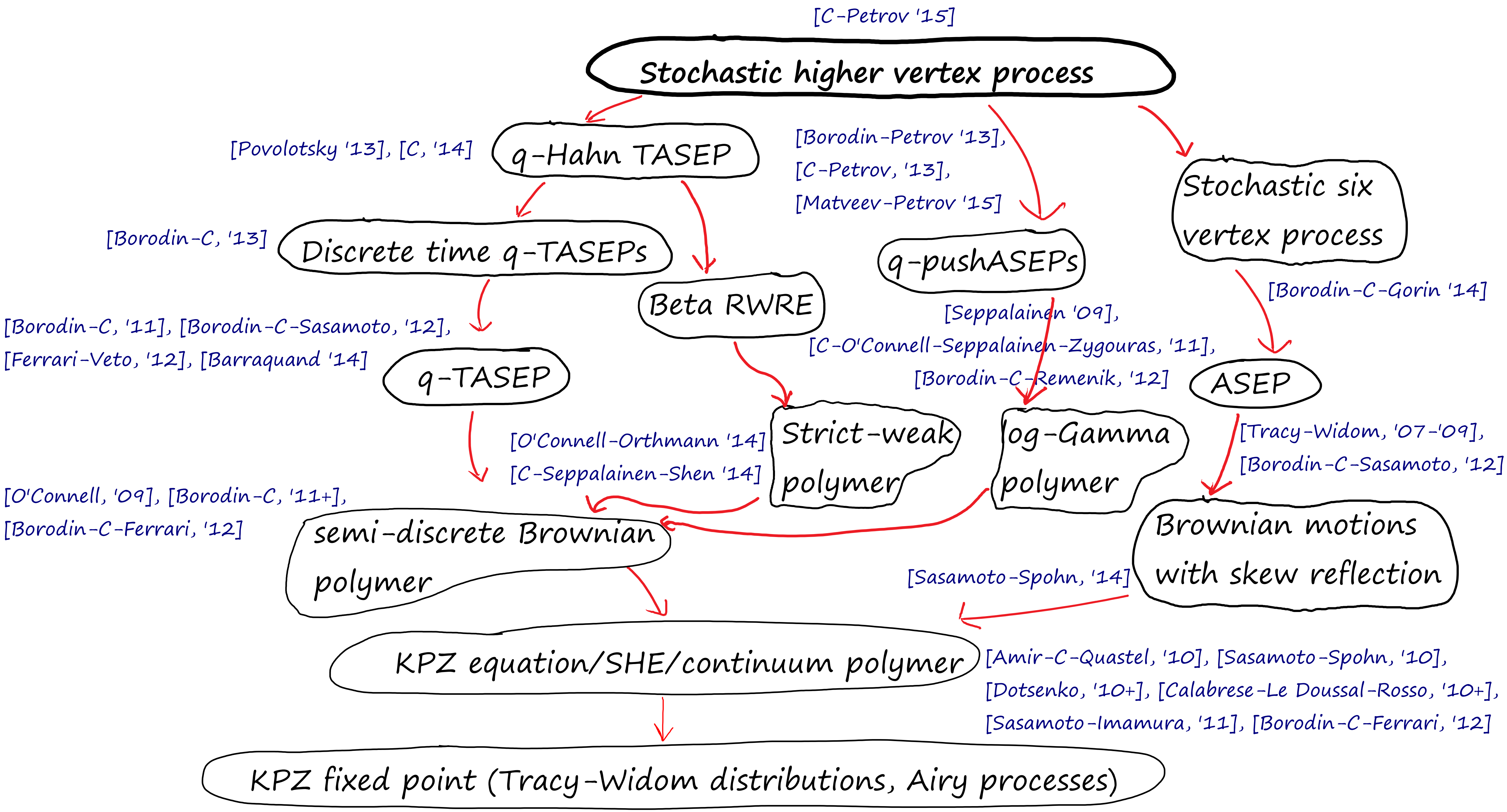
$$L^{(J)}_{\alpha}(i_1,j_1;i_2,j_2) = \mathbb{1}_{i_1+j_1=i_2+j_2}\; q^{\frac{2j_1-j_1^2}{4}-\frac{2j_2-j_2^2}{4}+\frac{i_1^2+i_2^2}{2}+\frac{i_2(j_2-1)+i_1j_1}{2}}$$

terminating basic hypergeometric series

$$\frac{\nu^{j_1-j_2}\,\alpha^{j_2-j_1+i_2}\,(-\alpha\nu^{-1};q)_{j_2-i_1}}{(q;q)_{i_2}\,(-\alpha;q)_{i_2+j_2}\,(\beta\alpha^{-1}q^{1-j_1};q)_{j_1-j_2}}\;{}_4\bar{\phi}_3\left(\begin{matrix} q^{-i_2};\ q^{-i_1},\ -\beta,\ -q\nu\alpha^{-1} \\ \nu,\ q^{1+j_2-i_1},\ \beta\alpha^{-1}q^{1-i_2-j_2}\end{matrix}\,\middle|\, q,q\right).$$

We can analytically continue in β. Positivity is generally lost, though taking $\beta=-\mu$ and setting $\alpha=-\nu$ we recover it. This specialization corresponds to the q-Hahn Boson process update.

Degenerations to known integrable stochastic systems in KPZ class



Summary

- Found **stochastic L-matrix** and constructed **ZRP/AEP** from it.
- **Diagonalized** via complete Bethe ansatz basis.
- Described two **Markov dualities** (AEP-ZRP and ZRP-ZRP).
- Combined duality and diagonalization to find **moment formulas**.
- Gave Markov functions proof of **fusion** to get $J>1$ L-matrices.
- Provided explicit formula for 4-parameter (q, ν, α, J) family of processes encompassing **all known integrable KPZ class models**.
- Many directions: asymptotics, new degenerations, other initial data, product matrix ansatz, higher rank groups, boundary conditions, connections to Macdonald-like processes…