(Brief) Introduction to Symmetric function theory.

References:
- Symmetric Functions and Hall Polynomials, Macdonald
- The Symmetric Group, Sagan
- Macdonald processes 2.1, Borodin - C.

- A <u>multivariate polynomial</u> $f(x_1, \dots, x_N)$ is symmetric if $\forall \sigma \in S_N$, $f(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = f(x_1, \dots, x_N)$

Write space of all such f with coefficients in a field $\mathbb{F}$ as $\mathbb{F}[x_1, \dots, x_N]^{S_N}$ or Λ_N more compactly

The monomial sym. functions $m_\lambda(x_1, \dots, x_N)$ form a linear basis on Λ_N. Need "partition" / "Young diagram" to define them

- Partitions/ Young diagram.

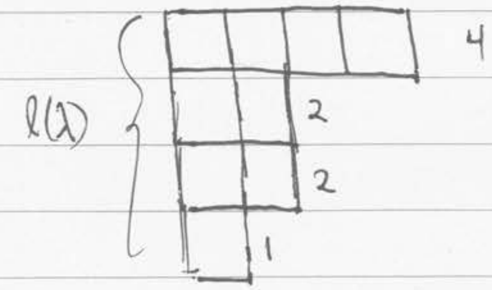
$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ $\lambda_i \in \mathbb{Z}_{\geq 0}$

$|\lambda| = \sum \lambda_i$ is its <u>size</u> and if $|\lambda| = n$, one writes $\lambda \vdash n$.

$\ell(\lambda) = \# \{ i : \lambda_i > 0 \}$ is the <u>length</u> of λ

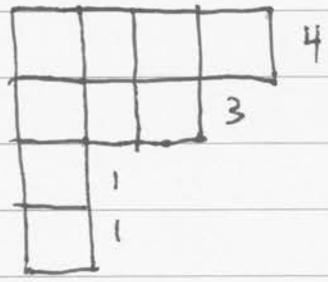
e.g. $\lambda = (4, 2, 2, 1)$ then $|\lambda| = 9$, $\ell(\lambda) = 4$

can graphically represent λ as a <u>Young diagram</u>



Will use partition and Young diagram interchangably.

The transpose of ~~α~~ λ is denoted λ' and it is ~~the~~ corresponds to transposing the Young diagram across the diagonal



$\lambda' = (4,3,1,1)$

We denote $\mathbb{Y}$ as the set of all Young diagrams and $\mathbb{Y}_N$ as those with $\ell(\lambda) \leq N$.

Note $\emptyset = (0,0,\ldots) \in \mathbb{Y}, \mathbb{Y}_N \ \forall N$.

Monomial symmetric polynomials $m_\lambda(x_1,\dots,x_N)$ $\lambda : \ell(\lambda) \le N$

$$m_\lambda(x_1,\dots,x_N) = \sum_\pi x^{\pi(\lambda)}$$

where $\pi(\lambda) = (\lambda_{\pi(1)}, \lambda_{\pi(2)}, \dots)$

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots$$

and the summation is over all $\pi \in S_N$ yielding unique $x^{\pi(\lambda)}$ terms.

<u>Example</u> $N=3$, $\lambda = (3,1,1)$

$$m_\lambda(x_1,x_2,x_3) = x_1^3 x_2 x_3 + x_1 x_2^3 x_3 + x_1 x_2 x_3^3$$

Λ_N = linear$^{\mathbb{F}}$ span of $\{m_\lambda\}$ with $\ell(\lambda) \le N$.

Λ_N is a ring (i.e. closed under multiplication too)

Symmetric functions are symmetric polynomials in infinite variables and of bounded degree. Restricting all but finitely many variables to 0 returns symmetric polynomials

$$\Lambda = \text{"}\mathbb{F}[x_1, x_2, \ldots]^{S_\infty}\text{"}$$

<u>Example</u> $m_{(1)} = x_1 + x_2 + x_3 + \cdots \in \Lambda$

but $(1+x_1)(1+x_2)(1+x_3)\cdots \notin \Lambda$ b/c degree is unbounded

<u>Elementary Sym. f^n</u>: $\displaystyle e_k = \sum_{i_1 < i_2 < \cdots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$

<u>Ex</u>: Restricting to Λ_3

$$e_0 = 1$$
$$e_1 = x_1 + x_2 + x_3$$
$$e_2 = x_1x_2 + x_1x_3 + x_2x_3$$
$$e_3 = x_1x_2x_3$$
$$e_4 = 0 \ \ldots$$

<u>Complete sym f^n</u>: $\displaystyle h_k = \sum_{i_1 \le i_2 \le \cdots \le i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$

$$h_1 = x_1 + x_2 + x_3$$
$$h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2$$
$$\vdots$$

<u>Power Sums</u> $\displaystyle p_k = \sum_i x_i^k$, $P_\lambda = P_{\lambda_1} P_{\lambda_2} \cdots P_{\lambda_{\ell(\lambda)}}$

$$p_1 = x_1 + x_2 + x_3,$$
$$p_2 = x_1^2 + x_2^2 + x_3^2$$

<u>Fundamental Theorem</u> $\{e_k\}$, $\{h_k\}$ and $\{p_k\}$ are

~~all~~ each algebraically independent sets of generators of Λ.

(i.e. $\Lambda = \mathbb{F}[e_1, e_2, \ldots] = \mathbb{F}[h_1, h_2, \ldots] = \mathbb{F}[p_1, p_2, \ldots]$

Example: Λ_3 $\quad p_2 = 2h_2 - (h_1)^2$.

<u>Exercises:</u> Let $H(z) = \sum_{k\geq 0} h_k z^k$, $E(z) = \sum_{k \geq 0} e_k z^k$, $P(z) = \sum_{k\geq 1} p_k z^{k-1}$

Then

1) $H(z) = \prod_i \frac{1}{1-x_i z}$

2) $E(z) = \prod_i (1 + x_i z)$

3) $P(z) = \frac{d}{dz} \sum_i \log\left(\frac{1}{1-x_i z}\right)$

4) $H(z) = \frac{1}{E(-z)} = \exp\left\{\sum_{k\geq 1} \frac{p_k z^k}{k}\right\}$

A <u>specialization</u> of Λ is an algebra homomorphism

$\rho : \Lambda \to \mathbb{C}$ write as $f \mapsto f(\rho)$

(i.e. $(f+g)(\rho) = f(\rho) + g(\rho)$, $(fg)(\rho) = f(\rho)g(\rho)$, $(\theta f)(\rho) = \theta f(\rho)$, $\theta \in \mathbb{C}$

A specialization can be defined uniquely via its value

on an algebraically ind^t generating set of Λ, such as $\{p_k\}$.

When restricted to Λ_N, specializations correspond to

substituting complex ~~numbers~~ numbers into the variables $x_1, \dots x_N$.

For Λ there are additional one's.

~~Example~~

We will next introduce another set of

symmetric f^n's known as Schur functions and relate

their remarkable properties and uses.