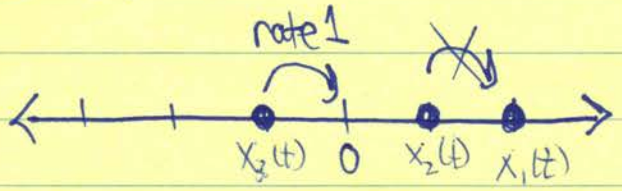
Schur processes:

TASEP and GUE

References: Lectures on integrable probability,
Borodin - Gorin.

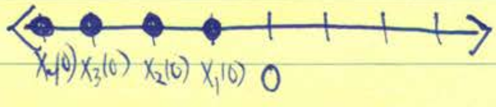
Reference from intro to sym. f^ns.

TASEP on $\mathbb{Z}$



$N_0(t)$ = # particles at or to left of origin

Step initial data



<u>Theorem</u> (Johansson '99)

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{N_0(t) - t/4}{2^{-1/3}\, t^{1/3}} \geq -s \right) = F_{GUE}(s)$$

An important result since it identifies the scaling $t^{1/3}$ and statistic F_{GUE} of what should be a large universality class of systems (both mathematical and physical)

- Proving said universality is important and far from done.

RMT

The Gaussian Unitary Ensemble (GUE_N) is a measure on Hermitian $N \times N$ matrices $H = (H_{ij})_{i,j=1}^{N}$, $H_{ij} = \overline{H_{ji}}$ given by $\mathbb{P}(H) = \frac{1}{Z} e^{-\frac{1}{2N} \mathrm{tr} H^2} \, dH$

Leb on alg. ind. entries of H.

Weyl Integration Formula shows that if $x_1 \geq x_2 \cdots \geq x_N$ are the eigenvalues of H, then

$$\mathbb{P}(x_1, \ldots, x_N) = \frac{1}{Z'} \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^{N} e^{-\frac{1}{2N} x_i^2} dx_i$$

(1-d coulomb gas, $\beta = 2$ ensemble)

<u>Theorem</u> (Tracy-Widom, Forrester, Nagao-Wadati)

$$\lim_{N \to \infty} \mathbb{P}\left(\frac{x_1 - 2N}{N^{1/3}} \leq s \right) =: F_{GUE}(s)$$

We will explore the connection between these two results via the lens of <u>Schur processes</u>.

We will show how the properties of Schur polynomials

(1) Gives rise to TASEP and GUE and accounts for their shared limit law.

(2) Provides for suitable expressions for probabilities via determinantal ~~Kern~~ point process kernel, to which we can perform asymptotic analysis

The approach of (1) will generalize up to the Macdonald process level, but (2) will not and hence requires new ideas and methods.

Recall $\Lambda_N = \mathbb{Q}[x_1,\ldots,x_N]^{S_N}$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ partition

Def For a partition μ, $\ell(\mu) \leq N$ the alternant a_μ is given by

$$a_\mu(x_1,\ldots,x_N) := \det\left[x_i^{\mu_j}\right]_{i,j=1}^N$$

"antisymmetrization"

$$= \sum_\sigma \operatorname{sgn}\sigma \; x_{\sigma(1)}^{\mu_1} x_{\sigma(2)}^{\mu_2} \cdots x_{\sigma(N)}^{\mu_N}$$

Note $a_\mu(x_{\sigma(1)},\ldots,x_{\sigma(N)}) = \operatorname{sgn}\sigma \; a_\mu(x_1,\ldots,x_N)$

so ratio's of alternants are symmetric.

Schur polynomial for $\delta = (N-1, N-2, \ldots, 0)$

$$s_\lambda(x_1,\ldots,x_N) = \frac{a_{\delta+\lambda}(x_1,\ldots,x_N)}{a_\delta(x_1,\ldots,x_N)}$$

- $a_\delta(x_1\ldots x_N) = \prod_{i<j}(x_i - x_j)$ Vandermonde det.
- A polynomial via degree counting and all roots of a_δ are roots of $a_{\delta+\lambda}$
- A basis for Λ_N since $a_{\delta+\lambda}$ span antisymmetric functions
- Clear from def^{n} that $s_\lambda(x_1,\ldots,x_N,0) = s_\lambda(x_1,\ldots,x_N)$ so

  we can extend s_λ to an element of Λ, and s_λ span Λ.
- If $\ell(\lambda) > N$, $s_\lambda(x_1,\ldots,x_N) = 0$
- Naturally arise from Weyl Character formula as Characters

  of $GL(N)$, $U(N)$.

Cauchy identity

Theorem Consider two sets of alg. ind^t variables $x_1, \ldots, y_1, \ldots$

$$\sum_{\lambda \in \mathbb{Y}} S_\lambda(x) S_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}$$

$$= \exp\left\{ \sum_{k \geq 1} \frac{p_k(x) p_k(y)}{k} \right\}$$

Remark For two specializations $\rho, \rho' : \Lambda \to \mathbb{C}$,

$$H(\rho;\rho') := \sum_{\lambda \in \mathbb{Y}} S_\lambda(\rho) S_\lambda(\rho') = \exp\left\{ \sum_{k \geq 1} \frac{p_k(\rho) p_k(\rho')}{k} \right\}$$

Proof: Suffices to prove for $x_1, \ldots, x_N$, $y_1, \ldots, y_N$

Recall Cauchy's determinant identity

$$\det\left(\frac{1}{1 - x_i x_j} \right) = \frac{\prod_{i<j} (x_i - x_j)(y_i - y_j)}{\prod_{i,j} (1 - x_i y_j)}$$

Exercise: Prove Cauchy's determinant identity

Since $s_\lambda(x_1,\dots,x_N) = \dfrac{a_{\delta+\lambda}(x_1,\dots x_N)}{a_\delta(x_1,\dots x_N)}$ and $a_\delta = \prod_{i<j}(x_i - x_j)$

we can reduce the desired identity to prove to

$$\sum_{\lambda : \ell(\lambda)\le N} a_{\delta+\lambda}(x_1,\dots,x_N)\, a_{\lambda+\delta}(y_1,\dots y_N) = \det\left(\frac{1}{1-x_iy_j}\right)_{i,j=1}^{N}$$

This identity follows by expanding both sides into series in x's and y's, and summing out one of the symmetric group summations on the LHS.

<u>Exercise</u>: Finish this proof.

The second line in the theorem uses the fact that

$$H(z) = \exp\left\{\sum_{k=1}^{\infty} \frac{p_k(x)z^k}{k}\right\} \text{ and } H(z) = \prod_i \frac{1}{1-x_iz}.$$

Note that $H(y_1)\cdots H(y_N) = \prod_{i,j} \frac{1}{1-x_iy_j}$.

but also equals $\exp\left\{\sum_{k\ge1} \frac{p_k(x)\,p_k(y)}{k}\right\}$ as desired.

Since s_λ restricted to Λ_N are characters, they are orthogonal wrt a natural "torus" inner product. In Λ we will simply define an inner product via

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda=\mu} \text{ and extend via linearity to } \Lambda.$$

<u>Theorem</u> $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda=\mu}$ where $z_\lambda = \prod_{i\geq 1} i^{m_i} m_i!$

where $m_i = m_i(\lambda) = \{ j : \lambda_j = i \}$

<u>Example</u> $\lambda = (4,2,1,1)$, $m_1 = 2, m_2 = 1, m_3 = 0, m_4 = 1$

$\lambda = 1^2 2^1 4^1 = 1^{m_1} 2^{m_2} \cdots$ is an alternative notation for partitions

<u>Proof</u> $\sum s_\lambda(x) s_\lambda(y) = \exp\left\{ \sum_{k\geq 1} \frac{p_k(x)p_k(y)}{k} \right\} = \sum_\lambda \frac{p_\lambda(x)p_\lambda(y)}{z_\lambda}$

Now observe a general fact: If f_λ, g_λ such that

$$\sum s_\lambda(x) s_\lambda(y) = \sum f_\lambda(x) g_\lambda(y)$$

then $\langle f_\lambda, g_\mu \rangle = \delta_{\lambda=\mu}$. This is because $\langle \sum s_\lambda(x) s_\lambda(y), s_\mu(y) \rangle = s_\lambda(x)$

hence by linearity $\langle \sum f_\lambda(x) f_\lambda(y), f_\mu(y) \rangle = f_\lambda(x)$, implies result.

Skew Schur functions

For two specializations ρ, ρ' define their union as the specialization taking $p_k(\rho,\rho') = p_k(\rho) + p_k(\rho')$

[If ρ, ρ' are finite substitutions this is clearly consistent with just taking the union of the two sets of variables.]

Def^n: For $\mu, \lambda \in \mathbb{Y}$ define $s_{\lambda/\mu}(\cancel{x})$ via

$$s_\lambda(\cancel{x},\cancel{y}) = \sum_\mu s_{\lambda/\mu}(\cancel{x})\, s_\mu(\cancel{y})$$

This decomposition is unique due to orthogonality of Schur f^n's.

~~Equivalently this says~~ $\cancel{\langle s_{\lambda/\mu}, s_\nu\rangle = \langle s_\lambda, s_\mu s_\nu\rangle}$

Equiv: $\langle s_{\lambda/\mu}, s_\nu\rangle = \langle s_\lambda, s_\mu s_\nu\rangle$

Clearly $s_{\lambda/\mu}$ is a homogeneous sym polynomial of degree $|\lambda| - |\mu|$ and hence zero if $|\lambda| < |\mu|$. More is true as we soon will see.

Skew Cauchy identity

For two specializations ρ, ρ'

$$\sum_{\mu \in \mathbb{Y}} s_{\mu/\lambda}(\rho)\, s_{\mu/\nu}(\rho') = H(\rho;\rho') \sum_{\kappa \in \mathbb{Y}} s_{\lambda/\kappa}(\rho')\, s_{\nu/\kappa}(\rho)$$

Follows from usual Cauchy, def^{n} of $s_{\lambda/\mu}$, inner product and the following identity

Chapmin-Kolmogorov identity

For two specializations ρ, ρ'

$$\sum_{\nu \in \mathbb{Y}} s_{\lambda/\nu}(\rho)\, s_{\nu/\mu}(\rho') = s_{\lambda/\mu}(\rho, \rho')$$

Proof is immediate from $s_{\lambda/\mu}$ def^{n} and orthogonality.

Jacobi Trudi formula

Theorem $$S_\lambda = \det\left[h_{\lambda_i - i + j} \right]_{i,j=1}^{\ell(\lambda)} \quad \text{with } h_m = 0 \quad m < 0$$

$$S_{\lambda/\mu} = \det\left[h_{\lambda_i - \mu_j - i + j} \right]_{i,j=1}^{\max(\ell(\lambda), \ell(\mu))}$$

Example Λ_2
$$S_{(4,1)} = \frac{\det\begin{pmatrix} x_1^5 & x_1^1 \\ x_2^5 & x_2^1 \end{pmatrix}}{\det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix}} = \frac{x_1^5 x_2 - x_2^5 x_1}{x_1 - x_2}$$

$$= x_1^4 x_2 + x_1^3 x_2^2 + x_1^2 x_2^3 + x_1 x_2^4$$
$$= x_1 x_2 \left(x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3 \right)$$

$$\det\begin{bmatrix} h_4(x_1,x_2) & h_5(x_1,x_2) \\ h_0(x_1,x_2) & h_1(x_1,x_2) \end{bmatrix}$$
$$= \left(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_1 x_2^4 \right)(x_1 + x_2) - \left(x_1^5 + x_1^4 x_2 + x_1^3 x_2^2 + x_1^2 x_2^3 + x_1 x_2^4 + x_2^5 \right)$$

same answer.

<u>Proof</u> General fact, if $f(u) = \sum_{m\geq 0} f_m u^m$ then

$$f(x_1)\cdots f(x_N) = \sum_{\lambda : \ell(\lambda)\leq N} \det\left(f_{\lambda_i - i + j}\right)_{i,j=1}^{N} s_\lambda(x_1,\dots,x_N)$$

To see this, multiply both sides by $a_\delta(x)$

$$a_\delta(x_1,\dots,x_N)\, f(x_1)\cdots f(x_N) = \sum_{\lambda : \ell(\lambda)\leq N} \det\left(f_{\lambda_i - i + j}\right) a_{\lambda+\delta}(x_1,\dots,x_N)$$

To obtain the coefficient of $a_{\lambda+\delta}$ must extract LHS coeff of $x_1^{\lambda_1+N-1}\cdots x_N^{\lambda_N}$ but LHS is just

$$\sum_{\sigma\in S_N} \operatorname{sgn}(\sigma)\, x_1^{N-\sigma(1)}\cdots x_N^{N-\sigma(N)} \left(\sum f_{m_1} x_1^{m_1}\right)\cdots\left(\sum f_{m_N} x_N^{m_N}\right)$$

which implies the fact.

To use this fact, consider

$$f(u) = \sum_{m\geq 0} h_m(y_1,\dots,y_N)\, u^m = \prod \frac{1}{1-y_i u}.$$

~~By Cauchy~~ Hence $f(x_1)\cdots f(x_N) = \prod_{i,j} \frac{1}{1-x_i y_j}$

$$\| $$

$$\sum \det\left(h_{\lambda_i - i + j}^{(x)}\right) s_\lambda(y) = \sum s_\lambda(x) s_\lambda(y)$$

hence we are done.

The Lindström-Gessel-Viennot lemma and the Jacobi Trudi formula implies the following (see Wikipedia)

<u>Combinatorial formula</u>

$$S_{\lambda/\mu}(x) = \sum_{T : sh(T) = \lambda/\mu} x^T$$

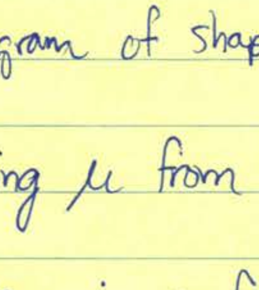
where T is a semi-std Young Tableaux of shape λ/μ

and $x^T = x_1^{\#1\text{s} \in T} \, x_2^{\#2\text{s} \in T} \cdots$

~~Example~~ • A skew Young diagram of shape λ/μ is what remains after removing μ from λ (assuming $\mu \subset \lambda$)

• A semi-std Young Tableaux is a filling of a diagram with $1, 2, \ldots$ such that ~~so~~ the numbers are weakly increasing in each row and strictly ~~incr~~ increasing in columns.

Example: $\lambda = (4, 2, 2, 1)$
$\mu = (2, 1)$

T 

$x^T = x_1^2 x_2^3 x_3^1$

Example In Λ_2 $\quad S_{(4,1)} = x_1x_2\left(x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3\right)$

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array} \quad \text{or } 1\,1\,2 \text{ or } 1\,2\,2 \text{ or } 2\,2\,2$$

As a corollary of the combinatorial formula if we are in N variables $x_i \geq 0$ then

$$S_\lambda(x_1,\dots,x_N) \geq 0 \quad \text{and} \quad S_{\lambda/\mu}(x_1,\dots,x_N) \geq 0$$

- A specialization $\rho: \Lambda \to \mathbb{C}$ such that

$$S_\lambda(\rho) \geq 0 \quad \forall \lambda \in \mathbb{Y} \quad \text{is called } \underline{\text{Schur positive}}$$

and it follows too that $S_{\lambda/\mu}(\rho) \geq 0 \quad \forall \lambda \supseteq \mu$.

Question: Classify all Schur-positive specializations

Equivalent question: Classify all upper triang. toeplitz matrices which are totally positive (i.e. det of all minors non-negative)

$$\begin{pmatrix} 1 & c_1 & c_2 & c_3 & \cdots \\ & 1 & c_1 & c_2 & \\ & & \ddots & \ddots & \ddots \\ & 0 & & \ddots & \ddots \end{pmatrix}$$

The second question was answered by Edrei in 1953 and the first by Thoma in 1964, with the connection only realized much later.

<u>Theorem</u> The Schur positive specializations are parameterized by non-negative reals $\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0)$ $\beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0)$ and $\gamma \geq 0$ where $\sum(\alpha_i + \beta_i) < \infty$.

The specialization $\rho = (\alpha ; \beta ; \gamma)$ is defined via its value on power sums

$$p_1(\alpha;\beta;\gamma) = \gamma + \sum(\alpha_i + \beta_i)$$

$$k \geq 2, \quad p_k(\alpha;\beta;\gamma) = \sum\left[(\alpha_i)^k + (-1)^{k-1}(\beta_i)^k\right]$$

or via generating function

$$e^{\sum_{k \geq 1} \frac{p_k(\alpha;\beta;\gamma)\, z^k}{k}} = e^{\gamma z} \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}$$

Remarks

- α specialization is simply substitution ~~and if all β's $\gamma = 0$~~

~~Then the generating function says~~ $\cancel{\sum S_\lambda(\alpha;0;0) S_\lambda(z;0;0) = \prod \frac{1}{1-z\alpha_i}}$

- β specialization is given by $S_\lambda(0;\beta;0) = S_{\lambda'}(\beta;0;0)$
  and is called the "super-symmetric" specialization

- γ specialization is a limit of infinite α or β specializations
  and is called the "Plancherel" specialization

$$S_\lambda(0;0;\gamma) = \lim_{M\to\infty} S_\lambda\Big(0;\underbrace{\big(\tfrac{\gamma}{M},\tfrac{\gamma}{M},\dots,\tfrac{\gamma}{M},0,\dots\big)}_{M \text{ times}};0\Big)$$

(can be seen from $\left(1+\frac{\gamma}{M}\right)^M \to e^\gamma$ in generating function

- The union of $(\alpha;\beta;\gamma)$, (α',β',γ')

  is $(\alpha\cup\alpha' ; \beta\cup\beta' ; \gamma+\gamma')$

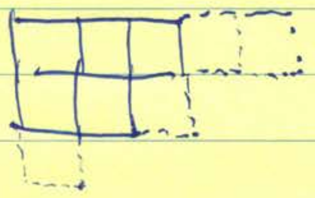
  where $\alpha\cup\alpha'$, $\beta\cup\beta'$ are disjoint unions, and sorted to decrease.

Three important examples

Def$^{\underline{n}}$ λ/μ is a <u>horizontal strip</u> if $0 \le \lambda_i' - \mu_i' \le 1 \ \forall i$

<u>vertical strip</u> if $0 \le \lambda_i - \mu_i \le 1 \ \forall i$

Example



Horizontal strip

vertical strip.

1) If $\alpha_1 = C$ and $\alpha_i = 0 \ i>1$, $\beta_i \equiv 0$, $\gamma = 0$

$$S_{\lambda/\mu}(\alpha;\beta;\gamma) = \begin{cases} C^{|\lambda|-|\mu|} & \lambda/\mu \text{ horizontal strip} \\ 0 & \text{else.} \end{cases}$$

2) If $\alpha_i \equiv 0$, $\beta_1 = C$, $\beta_i = 0 \ i>1$, $\gamma = 0$

$$S_{\lambda/\mu}(\alpha;\beta;\gamma) = \begin{cases} C^{|\lambda|-|\mu|} & \lambda/\mu \text{ vertical strip} \\ 0 & \text{else.} \end{cases}$$

3) If $\alpha_i \equiv 0$, $\beta_i \equiv 0$, $\gamma = C$

$$S_\lambda(\alpha;\beta;\gamma) = \frac{C^{|\lambda|}}{|\lambda|!} \dim \lambda$$

$\cancel{\text{If } \gamma=1, \text{ Cauchy implies } \sum_{|\lambda|=N} (\dim\lambda)^2 = N!}$

$\dim \lambda = \#\,\text{SYT of shape } \lambda = \dim$ irred. rep of $S_{|\lambda|}$ indexed by λ.

Schur measure Okounkov 2001

Def^{n}: For any two Schur-positive specializations ρ_1, ρ_2

$S_{\rho_1;\rho_2}$ is a measure on $\mathbb{Y}$ with probability

$$\mathbb{P}_{\rho_1;\rho_2}(\lambda) := \frac{s_\lambda(\rho_1)s_\lambda(\rho_2)}{H(\rho_1;\rho_2)}$$

where $H(\rho_1;\rho_2) := \sum_{\lambda\in\mathbb{Y}} s_\lambda(\rho_1)s_\lambda(\rho_2) = \exp\left\{\sum_{k\geq 1}\frac{p_k(\rho_1)p_k(\rho_2)}{k}\right\}$

Theorem If $\rho_1 = ((\overbrace{1,1,\ldots 1}^{N}),0,0)$ and $\rho_2 = (0;0;t)$ then

$$\mathbb{P}_{\rho_1;\rho_2}(\lambda_N = x) = \mathbb{P}^{\text{TASEP, step}}(X_N(t)+N = x)$$

pf: Next section.

Theorem With ρ_1, ρ_2 as above ~~...~~ write $\lambda(t) = (\lambda_1(t), \ldots \lambda_N(t))$

then ~~...~~ $\varepsilon^{1/2}\left[\lambda(\varepsilon^{-1}t) - \varepsilon^{-1}t\right] \xrightarrow[\varepsilon\to 0]{(d)} \tilde{y}(t)$ ~~...~~

$\tilde{y}(t) = (\tilde{y}_1(t) \geq \cdots \geq \tilde{y}_N(t))$

where $\mathbb{P}(\tilde{y}(t)=y) = \frac{1}{Z}\prod_{i<j}|y_i - y_j|^2 \prod_{i=1}^{N} e^{-\frac{1}{2t}x_i^2}dy_i$

which is the $GUE_N(t)$ ensemble. [earlier looked ~~at~~ $GUE_N(N)$]

Pf: From def^{n} of $s_\lambda(1,\ldots 1)$ and $\dim\lambda$ formulas