Macdonald processes:

Definition, dynamics and computations

References:
- Symmetric functions and hall polynomials, I.G. Macdonald
- Macdonald processes, Borodin - Corwin
\( g \)-TASEP on \( \mathbb{Z} \)

- Each particle \( x_i \) jumps right at rate \( 1 - q \cdot \text{gap} \), \( \text{gap} = x_{i-1}(t) - x_i(t) - 1 \)
- For \( g = 0 \) becomes TASEP
- For \( g \neq 1 \) particles become very far apart ... interesting limit.
- Step initial data \( x_i(0) = -i \) \( i \geq 1 \).

Goal: For step initial data, compute expression for distribution of \( x_i(t) \) which allows asymptotics in \( N, t, g \).

In fact, the real goal is to introduce Macdonald processes and observe how the integrable properties of Macdonald sym. polyn. gives rise to a large class of exactly solvable, probabilistic systems of phenomenological interest, \( g \)-TASEP being one example.

\( g \)-TASEP arises analogously to Mac: proc. as TASEP is to Schur, but the methods to compute are totally different!
A defining property of Schur sym. functions

Def: The dominance partial ordering on Young diagrams

\( \lambda \preceq \mu \iff \lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i \) and \( \lambda \neq \mu \)

and \( \lambda \preceq \mu \) if \( \lambda \preceq \mu \) and \( \lambda \neq \mu \).

Exercise: Show this is only a partial order (hint: requires size of diagrams \( \geq 6 \))

Recall: \( \Lambda \) has an \( \mathbb{F} \)-basis in the monomial symmetric functions.

The Schur sym. functions are defined by two properties

(1) \( s_{\lambda} = m_{\lambda} + \sum_{\mu \preceq \lambda} K_{\lambda \mu} m_{\mu} \) \( \quad \forall \lambda, \mu \in \mathbb{N} \) Kostka numbers.

(2) \( \langle s_{\lambda}, s_{\mu} \rangle = \delta_{\lambda,\mu} \).

Remarks: (1) is true via combinatorial formula

(2) is true by definition of \( \langle \cdot, \cdot \rangle \). However, we could

alternatively define \( \langle \cdot, \cdot \rangle \) via \( \langle p_{\lambda}, s_{\mu} \rangle = \mathbb{Z}_{\lambda} \delta_{\lambda,\mu} \)

with \( \mathbb{Z}_{\lambda} = \prod_{i=1}^{\infty} \frac{m_i!}{\lambda_i!} \quad \forall \lambda \in \mathbb{N} \).

In fact, the existence of such polynomials is amazing since

Gram-Schmidt won't produce the partial order in (1).
There are other choices of $\mathfrak{z}$ which produce interesting sym. func. 
(i.e. Hall-Littlewood, Jack, and on top Macdonald symmetric func. 

Macdonald Symmetric functions

- Coefficients in $\mathbb{Q}[q,t]$ with $q,t$ formal parameters

(analytic

(for our purposes we will take $q,t \in (0,1)$

- $\Delta = \mathbb{Q}[q,t][x_1, x_2, \ldots]^{-1}$

- Written $P_a(x; q,t) = R(x)$.

Uniquely defined by

1. $P_a^{(0)} = m_a^{(0)} + \sum R_{\alpha, \mu}^{(0)}$ $\mu$ are functions of $x_a, q,t$ only

2. $\langle P_a, P_a \rangle = \langle P_a, P_a \rangle_{\Delta, \mu}$ where the inner product $\langle \cdot, \cdot \rangle$ is defined via $\langle P_a, P_a \rangle_{\Delta, \mu} = \mathbb{Z}(q,t) S_{\lambda, \mu}$ and

$\mathbb{Z}(q,t) = \frac{\mathbb{Z}}{\prod_{i} (1 - z_i)}$, $\mathbb{Z}$ as before.

Define dual basis $Q_a = R / \langle P_a, P_a \rangle$ so that $\langle P_a, Q_a \rangle = S_{\lambda, \mu}$.

Setting $q = t$ recover $R_a = Q_a = S_a$.
Cauchy identity

Orthogonality implies that

\[ \sum_{x} P_{x}(x) Q_{y}(y) = \sum_{x} \frac{P_{x}(x) P_{y}(y)}{Z(q, \xi)} P_{x}(x) P_{y}(y) \]

(exercise)

\[ = \exp \left\{ \sum_{k=1}^{\infty} \frac{P_{x}(x) P_{y}(y)}{k} \frac{1 - \frac{x}{q^k}}{1 - q^k} \right\} \]

\[ =: T_{1}(x ; y) \quad \text{(replaces } H \text{ from Schur-case)} \]

Identity holds for specific replacing \( x, y \) by two specializations.

If all \( \Theta \) but finitely many \( x_{1}, \ldots, x_{n} \neq 0 \) \( x_{1}, \ldots, x_{n} \neq 0 \) (other \( 0 \)) then

(exercise)

\[ \exp \left\{ \sum_{k=1}^{\infty} \frac{P_{x_{1}}(x_{1}) \cdots P_{x_{n}}(x_{n})}{k} \frac{1 - \frac{x}{q^k}}{1 - q^k} \right\} \]

\[ = \prod_{i \neq j} \frac{(x_{i} x_{j}; q)_{\infty}}{(x_{i} x_{j} ; q)_{\infty}} \text{ where } \prod_{i \neq j} \left( a ; q \right)_{\infty} = \prod_{i \neq j} \left( 1 - q^{a-i} \right) \]

Note that setting \( k = q \) we recover the Schur-Cauchy identity.
Strew Macdonald functions

Def: For \( x, y \) define \( \textbf{P}_{\lambda, \mu}(x) \) via the coefficients in the expansion

\[
\textbf{P}_{\lambda, \mu}(x) = \sum_{\mu \in \mathbb{Y}} \textbf{P}_{x, \mu}(x) \textbf{P}_{\lambda, \mu}(y)
\]

and likewise \( \textbf{Q}_{\lambda, \mu}(x) \) via

\[
\textbf{Q}_{\lambda, \mu}(x, y) = \sum_{\mu \in \mathbb{Y}} \textbf{Q}_{x, \mu}(x) \textbf{Q}_{\lambda, \mu}(y)
\]

Note \( \textbf{P}_{x, \mu} = \frac{\langle \textbf{P}_{x, \mu} \rangle}{\langle \textbf{P}_{\lambda, \mu} \rangle} \textbf{Q}_{\lambda, \mu} \).

Clearly \( \textbf{P}_{x, \mu} \) is homogeneous sym. poly. of degree \( |\lambda| - |\mu| \)

and hence zero if \( |\lambda| < |\mu| \). More is true...
Stieltjes-Cauchy identity

\[ \sum_{x \in X} P_{x,x'}(x) Q_{x',y}(y) = \Pi(x:y) \sum_{x \in y} Q_{x,y'}(y) P_{x,x'}(x) \]

Chapman-Kolmogorov

\[ \sum_{x \in X} P_{x,x'}(x) P_{y,y'}(y) = P_{x,y'}(x,y) \quad \text{also holds with} \quad P \leftrightarrow Q. \]

Combinatorial formula

\[ P_{\lambda \mu}(x) = \sum_{T \in \mathbb{S}(T)^{\lambda \mu}} \Psi_T^{\lambda_1 \lambda_2 \cdots \lambda_n} X^T, \quad Q_{\lambda \mu}(x) = \sum_{T \in \mathbb{S}(T)^{\lambda \mu}} \Phi_T^{\lambda_1 \lambda_2 \cdots \lambda_n} X^T \]

- \( T \): semi std. Stieltjes Young Tableaux of shape \( \lambda \mu \) is a sequence of Young diagrams

\[ \mu = \lambda^{(0)} < \lambda^{(1)} < \lambda^{(2)} \ldots < \lambda^{(n)} = \lambda \quad \text{where} \quad \lambda^{(i)}/\lambda^{(i-1)} \text{ horizontal strip}, \ l \text{ is arbitrary} \]

- \( \Psi_T = \prod_{c=x} \Psi_{x^{(i)} y^{(i)}} (\text{same} \ P_T) \) and \( X^T = \begin{pmatrix} x^{(0)}_1 & x^{(0)}_2 & \cdots & x^{(0)}_n \end{pmatrix} \]

\( \Psi_{\lambda \mu} \) is independent of \( x \)'s and given via explicit (complicated) formula:

\[ \Psi_{\lambda \mu} = \prod_{1 \leq i < j \leq n} \frac{a(\mu_i - \mu_j, t - i - 1)}{a(\lambda_i - \lambda_j, t - i - 1)} \]

Similar definitions for \( \Theta_{\lambda \mu} \) and "dual" \( \Phi_{\lambda \mu} \).
**Def.** A specialization \( p: \Lambda \to \mathbb{C} \) is Macdonald positive if \( \Phi_{\lambda \mu}(p) \geq 0 \) for all \( \lambda, \mu \in \Lambda^+ \).

- Unlike Schur (or even Jack) case, the classification conjecture (of keru's) we now record has not been proved.

**Def.** The specialization \( p = (x; B; z) \) with \( \alpha = \frac{x}{\beta_1 z, \beta_2 z, \ldots, \beta_\gamma z} \), \( B = \frac{1}{\beta_1} \beta_2 \beta_3 \ldots 2^\gamma z \), \( \gamma \geq 0 \) and \( \sum (x_1 + \beta_i) \leq \infty \) is defined via generating series

\[
\exp \left( \sum_{k=1}^{\infty} \frac{z^k}{k} \frac{1-x^k}{1-q^k} \right) = \prod_{i=1}^{\infty} \frac{(1+Bz^i)}{(z^i)} \quad : \quad T(z; p).
\]

**Prop.** All such \( p \) are Macdonald non-negative.

**PF:** Pure \( \alpha \) case follows combinatorial formula, \( \beta \) follows a duality, \( \gamma \) from limits.

**Conj (keru):** This is all of the Macdonald non-negative specializations.

**Examples**
- Pure \( \alpha \), \( \alpha_i = 0 \) all else \( 0 \)
- Pure \( \alpha \) (all but \( \alpha_i \)'s zero) \( \Phi_{\lambda \mu}(p) = \sum_{\lambda' \mu'} c_{\lambda' \mu'}^\lambda \mu \) hor. strip else.
- Pure \( \beta \) (all but \( \beta_i \)'s zero) \( \Phi_{\lambda \mu}(p) = \sum_{\lambda' \mu'} c_{\lambda' \mu'}^\lambda \mu \) vert. strip else.
- Plancherell \( \gamma \) (all but \( \gamma \) zero) \( \Phi_{\lambda \mu}(p) = \sum_{\lambda' \mu'} c_{\lambda' \mu'}^\lambda \mu \) Plancherell more complicated, two directly opposite.
Macdonald measure / process

- (\beta, \varepsilon)-generalization of Schur case first mentioned in Forrester-Rains / with various degenerations also discussed in Vuletic '09, Fulman '02, Okounkov, O'Connell '12.

**Def:** Macdonald process is measure on \( \lambda^{(i)} = \mu^{(i)} + \varepsilon^{(i)} \) \( 2 \lambda^{(2)} = \varepsilon^{(2)} \) \( 3 \lambda^{(3)} = \varepsilon^{(3)} \) \( \cdots \) \( n \lambda^{(n)} = \varepsilon^{(n)} \)

specifies by Macdonald positive \( P^+, P^- P^+ \ldots P^- \) \( \cdots P^- \) \( P^+ \).

\[
P(\lambda, \mu) = \frac{W(\lambda, \mu)}{\sum W(\lambda, \mu)} \text{ with } W(\lambda, \mu) = P_{x^{(1)} P^+_{x^{(1)}} \mu_{x^{(1)}}} \cdots P_{x^{(n)} P^+_{x^{(n)}} \mu_{x^{(n)}}}
\]

Write as \( \phi \)

- Normalization calculated by skew Cauchy and Chaplin-Kolmogorov as

\[
\sum W(0, \mu) = \Pi P(\lambda^+, \lambda^-)
\]

\[
\Pi(\lambda, \mu) = \sum P(\lambda, \mu) Q_\lambda(\mu) = \exp \left( \sum_{k=1}^{\infty} \frac{P(\lambda, \mu) P(\mu, \lambda)}{1 - t^k} \right)
\]
We will focus here on a special case in which all \( p_i = 0 \) except \( p_n = p \) and all \( p_k = (a_k); 0; 0 \). This implies that the process is supported on seq. \( 0 \leq x^{(0)} \leq x^{(2)} \leq \ldots \leq x^{(n)} \) of partitions with \( x^{(k)} \geq (-1) \) hor. and \( |x^{(k)}| \leq i \) or equiv an interlacing triangular array:

\[
\begin{align*}
\lambda_1 &\quad \lambda_2 &\quad \ldots &\quad \lambda_n \\
\vdots &\quad \vdots &\quad \ddots &\quad \vdots \\
\lambda_{n-1} &\quad \lambda_{n-2} &\quad \ldots &\quad \lambda_1 \\
\lambda_n &\quad \lambda_n &\quad \ldots &\quad \lambda_n
\end{align*}
\]

Def.: Ascending Macdonald process on \( 0 \leq x^{(0)} \leq \ldots \leq x^{(n)} \)

\[
M_{\text{asc}}(a_1, \ldots, a_n; p) (x^{(0)}, \ldots, x^{(n)}) := \frac{P_{\lambda^{(0)}}(x^{(0)}) P_{\lambda^{(1)}}(a_2) \ldots P_{\lambda^{(n)}}(a_n) \Omega_{\lambda^{(n)}}(p)}{\Pi(a_1, \ldots, a_n; p)}
\]

where (by def.) \( \Pi(a_1, \ldots, a_n; p) = \Pi(a_2; p) \ldots \Pi(a_n; p) \).

- Projection onto level \( k \) is simple

\[
\text{Def. : Macdonald measure on } x^{(k)}: \quad \text{MM}(a_1, a_k; p)(x^{(k)}) = \frac{P_{\lambda^{(k)}}(x^{(k)}) \Omega_{\lambda^{(k)}}(p)}{\Pi(a_1, a_k; p)}
\]

Remarks:

When \( q = \ell \) these degenerate the Schur case. Other (less trivial) degenerations include that to O’Connell’s Whittaker measure/process and a hierarchically slightly higher measure of COS\( \ell \). Natural relation to directed polymers via tRSK and recently \( q \)-tRSK connection to \( q \)-TASEP (more later).

\( \text{Def. : Macdonald measure on } x^{(k)}: \quad \text{MM}(a_1, a_k; p)(x^{(k)}) = \frac{P_{\lambda^{(k)}}(x^{(k)}) \Omega_{\lambda^{(k)}}(p)}{\Pi(a_1, a_k; p)} \)
Theorem If \( p = (0; 0; 2) \) and \( a_i = 1 \) then
\[
P_{M\mu}(\binom{X_N}{-N} = x) = P_{\mu, 1}(X_N^0 = x)
\]

- We show via analogous approach as gave rise to TASEP/Schur relation.

**Just the highlights**

- For \( p, p' \) Mac. pos. spec. define matrices \( P_{\mu, \mu}(p; p') \), \( P_{\mu, \mu}(p; p') \)

\[
P_{\mu, \mu}(p; p') := \frac{P_{\mu}(p)}{P_{\mu}(p; p')}
\]

- \( \mu, \mu' \) stochastic and act well on Macdonald measures

\[
M\mu(p_1; p_2) P_{\mu}(p_2; p_3) = M\mu(p_1, p_2; p_3)
\]

\[
M\mu(p_1; p_2, p_3) P_{\mu}(p_2; p_3) = M\mu(p_1, p_2)
\]

- Commute \( \mu(p_1, p_2; p_3) P_{\mu}(p_1; p_2) = P_{\mu}(p_1; p_2) \mu(p_1, p_2; p_3) \)

- \( \lambda \approx (\lambda^1, \ldots, \lambda^m) \)

\[
M\mu(p_1, p_2; p_3)(\lambda^1, \ldots, \lambda^m) = \frac{P_{\mu}(p_1, p_2; p_3)}{P_{\mu}(p_1, p_2; p_3)}
\]
Thus: We may apply the multivariate Markov Chain construction (similar to Schur case) to construct dynamics on interacting triangrams.

$S_k = \text{Young diagrams with at most } k \text{ rows}$

$S^{(\omega)} = \{ \lambda^{(1)} \preceq \lambda^{(2)} \preceq \ldots \preceq \lambda^{(n)} \} \text{ with } \lambda^{(i)} \in S_i \}$

$P_k = P^\uparrow (a_1, a_k; p')$

$\Lambda^k_{k-1} = P^\downarrow (a_1, \ldots, a_{k-1}; a_k)$

The subsequential update rule is that given $\mu \in S_k$ and an update $\lambda \in S_{k-1}$, the $\mu$ updates to $\nu \in S_k$ with probability equal to $P(\nu|\mu)$ constant $\nu \cdot Q_{\mu}(p') \cdot P_{\nu \lambda}(a_k)$.

If we apply these dynamics to an ascending Schur process with $(a_\lambda, a_\omega; p)$ then after the time step $p$ has updated to $p \cup p'$.

If $p$ is initially $(0; 0; 0)$ then the measure is initially supported on $\emptyset \leq \emptyset \ldots \leq \emptyset$. 
Consider when $\beta' = (0; (c); 0)$ then
\[
P(v|x, \mu) = \begin{cases} \text{const.} & \beta' = (v; \mu; v); \frac{c}{v^2}, v' \mu \text{ hor-strips} \\ 0 & \text{else.} \end{cases}
\]

Take $c = \varepsilon$ and repeat this $\varepsilon^{-1} T$ times. Then the

$\beta$ part in the ascending Max. proc. specialization is
\[
(0; (e^k, ... e^s); 0) \rightarrow (0; 0; T) \quad \text{Plancherel}
\]

The update rule corresponds to a Poisson limit from $0$ at part of $P(v|x, \mu)$.

- Each coordinate $\lambda^{(m)}_k$ has int exp. clock with rate

\[
\text{Am.} \frac{\Psi(X^{(m)}_k, t^{(m)}_k)/X^{(m-1)}_k}{\Psi(X^{(m)}_k, t^{(m)}_k)/X^{(m)}_k} \Psi'(X^{(m)}_k, t^{(m)}_k)/X^{(m)}_k \quad (\lambda U \Delta_k \text{ means increase})
\]

- When its clock rings, $\lambda^{(m)}_k$ increases in value by 1 (add box to $X^{(m)}_k$)

- By virtue of $\Psi, \Psi'$, jump rate goes to 0 if jump would violate

interlacing with $\lambda^{(m-1)}$, and after a jump of $\lambda^{(m)}_k$ occurs, and

interlacing violations are resolved with rate $\infty$.

- Unfortunately, $\Psi, \Psi'$ are still pretty complicated and non-local

- Note: When $\beta = t$ this reduces to Cts time Schur-dynamics from earlier
When \( t = 0 \), \( \gamma' \) and \( \gamma'' \) simplify and we find a nice 2\textsuperscript{nd} interacting particle system with local update rule and \( q \)-TASEP as a marginal.

**Def:** The \( q \)-deformed 2\textsuperscript{nd} dynamics with rates \( \alpha_1, \ldots, \alpha_k > 0 \) has

The coordinate \( \lambda_k^{(m)} \) of an interlacing triangular array jump right by

\[
1 \quad \text{at rate} \quad \lambda_k^{(m)} = a_m \cdot \frac{1 - q^{\lambda_{k-1}^{(m)} - \lambda_k^{(m)}}(1 - q^{\lambda_k^{(m)} - \lambda_{k+1}^{(m)} + 1})}{1 - q^{\lambda_k^{(m)} - \lambda_{k+1}^{(m)} + 1}}
\]

with terms which don't make sense omitted.

\[
\lambda_{kn}^{(m)} \quad \rightarrow \quad \lambda_k^{(m)} \quad \quad \text{rate} \quad \lambda_k^{(m)} \quad \rightarrow \quad \lambda_k^{(m)}
\]

**Simulation:** \( \lambda_k^{(m)} = \lambda_k^{(m)} - k \)

- These dynamics have many interesting limits, properties and explicit formulas.
- Set \( X_k = \lambda_k^{(k)} - k \) then \( X_1 > X_2 > \ldots \) evolves marginally as a Markov process.
  - General nature
  - Which is a \( q \)-TASEP: Given time interacting particle system on \( \mathbb{Z} \)
    - \( x_i \) jumps one to the right at rate \( \alpha_{i,j}(1 - q^{X_{i-1}^{(k)} - X_i^{(k)} - 1}) \)
    - Step initial data \( X_n(0) = -n \), \( n^{\geq 1} \) corresponds with \( p \) initially 0.
- **Proves Theorem**
  - If \( p = (0, 0, 2) \) and \( a_i = 1 \) then \( \mathbb{P}^{q \text{-TASEP}}(X_n(t) = x) = \mathbb{P}^{\text{MM}}(X_n^{(m)} - N = x) \)
Goal: Compute exact formula for distribution of $x_n(z)$ which does not grow in complexity as $N, z, g$ scale. More generally, develop methods of exact solvability for Macdonald processes/measures allowing us to compute a robust set of expectations for interesting observables.

We will focus on the Macdonald measure on $x^\infty = \lambda$

$$P(\lambda) = \frac{P_\lambda(a_1, \ldots, a_N) Q_\lambda(p)}{\prod (a_1, \ldots, a_N, p)}$$

- Big idea: A linear operator on $\Lambda_N$ such that for all $\lambda_l(\lambda) \leq N$,

$$\Delta P_{\lambda}(x_1, \ldots, x_N) = d_{\lambda} P_{\lambda}(a_1, \ldots, a_N) \text{ then}$$

$$\Delta \frac{\prod (a_1, \ldots, a_N, p)}{\prod (a_1, \ldots, a_N, p)} = \frac{1}{\prod (a_1, \ldots, a_N, p)} \sum_{\lambda} \Delta(\lambda) P_{\lambda}(a_1, \ldots, a_N, p)$$

$$= \Delta x \sum_{\lambda} d_{\lambda} P_{\lambda}(a_1, \ldots, a_N, p) \prod (a_1, \ldots, a_N, p) = E \left[ d x \right]$$

Applying products of such operators leads to various expectations.

Example: Ising model $Z(\beta, h) = \sum_{\sigma} e^{\beta \sum \sigma_{xy} + h \sum \sigma_x}$ then

$$E \left[ \sum \sigma_x \right] = \frac{\partial_h Z(\beta, h)}{Z(\beta, h)} \quad , \quad E \left[ \sum \sigma_{xy} \right] = \frac{\partial_\beta Z(\beta, h)}{Z(\beta, h)} \text{ etc.}$$

$\log Z \to$ Free energy contains much of the physics.
There is a whole integrable system of $N$ commuting operators diagonalized by $P_x$.

**Def**: $(T_n;x,t)(x_1,\ldots,x_N) := f(x_1,\ldots,ux_i,\ldots,x_N)$

**Def**: Macdonald $r$-th difference operator $D^r_N$, $1 \leq r \leq N$

$$
D^r_N := \sum_{I \subset \{1,\ldots,N\}, |I| = r} \prod_{i \in I} A_i(x; t) \prod_{j \not\in I} T_j x_i,
\quad A_i(x; t) := t^{\lambda_i - 1} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j}.
$$

- Act taking $\Delta_N \to \Delta_N$
- Self adjoint operator with $\langle \cdot, \cdot \rangle$ inner product
- All diagonalized by $P_x$, $x; r(x) \leq N$ (i.e. mutually commuting) with

$$
D^r_N P_x(x_1,\ldots,x_N) = e_r(g_1^N + g_2^N + \ldots + g_N^N) P_x(x_1,\ldots,x_N)
$$

where $e_r$ is $r$-th elementary symmetric polynomial.

In particular $D^1_N P_x = (g_1^N + g_2^N + \ldots + g_N^N) P_x$

and when $t = 0$ $D^N_N P_x = g_1^N P_x$.

Remark: When $q = t$, $P_x = s_x$ indep. of $q,t$. But $D^r_N$ still depend on $q,t$ value. This degeneracy can be used to provide a new route to Schur-measure/process correlation function.
Recall that we wish to apply these operators to $\Pi(a_1, \ldots, a_N; \mathcal{D}) = \Pi(a; \mathcal{D}) \cdot \Pi(a_1)$. 

so we need only consider the action of the operators on $F(x_1, x_N) = f(x_1) \cdots f(x_N)$.

**Fact:** $D_n$ action on such $F$'s can be encoded via contour integrals. 

**Why useful?**

- *Like going to a generating function*
- Complex analysis (especially residue calculations) can yield non-trivial combinatorial identities. (Analytic continuation powerful tool).
- Good for asymptotics and organizing combinatorial information analytically.

Preliminary example $N = r = 1$ then $(D_1^r F)(x) = F(qx)$

**Note that** $F(qx) = f(x) \cdot \int_{\mathbb{C}} \frac{f(qz)}{z-x} \frac{dz}{2\pi i}$.

**Pick residue at $z=x$.**

$N=2, n=1 \quad (D_2^1 F)(x_1, x_2) = \frac{\pm x_1 - x_2}{x_1 - x_2} f(qx_1) f(x_2) + \frac{\pm x_2 - x_1}{x_2 - x_1} f(x_1) f(qx_2)$

**How is this encoded?**
Prop: Assume \( F(x_1, \ldots, x_N) = f(x_1) f(x_2) \ldots f(x_N) \). Consider \( a_1, \ldots, a_N > 0 \) and assume \( f(x) \) is holomorphic and non-zero in a complex neighborhood containing an interval of \( \mathbb{R} \) containing \( \{ a_j, g, g_j, f_j \} \). Then

\[
(D_r^N F)(a_1, \ldots, a_N) = \frac{1}{(2\pi i)^r} \oint \cdots \oint \frac{1}{\prod_{k=1}^N (z_k - a_k)} \prod_{j=1}^N \left( \prod_{m=1}^N \frac{z_j - a_m}{z_j - a_k} \right) f(g_k) \, dz
\]

Where each of the \( r \) integrals is over a positively oriented contour containing \( \{ a_j, g \} \) and no other singularities of the integrand. (as long as \( \varepsilon \) small such contour exists)

Proof: (For \( r=1 \)) Note \( (D_1^N F)(a_1, a_2, \ldots, a_N) = F(a_1, a_2) \sum_{i=1}^N \prod_{j \neq i} \frac{z_i - a_j}{a_i - a_j} \frac{f(g_i)}{f(a_i)} \)

Claim: \( (x) = \frac{1}{2\pi i} \oint \frac{1}{\prod_{m=1}^N (z - a_m)} \frac{f(g(z))}{f(z)} \, dz \) over \( \mathbb{C} \)

Assume WLOG all \( a \)'s diff. Then integral is sum over residues at \( z = a_m \)

which equals \( \sum_{m=0}^N \frac{z_i - a_i}{z_i - a_i} \prod_{j \neq i} \frac{z_i - a_j}{a_i - a_j} \frac{f(g_i)}{f(a_i)} \)

For \( r > 1 \) similar residue calculus plus Cauchy determinant identity yields proof.
Products of $D_n F$ can be similarly encoded. For example, notice

$$\left(D_n F(\alpha_1, \ldots, \alpha_n) = \frac{1}{2\pi i} \int \frac{1}{\prod_{m=1}^{n} (f(\alpha_m) \frac{z - \alpha_m}{z - a_m})} \cdot \frac{f(z)}{f(z)} \, dz \right.$$  

(all $g(\alpha_m)$)

If we apply $D_n$ again, and use linearity of integral we find that we can use the proposition again to get a two-fold nested integral.

Prop: Assume $F$ factors, $a_1, \ldots, a_N > 0$ and $f(x)$ holomorphic/ nonzero around $a_i$. Then

$$\left(\left(D_n \right)^k F(\alpha_1, \ldots, \alpha_N) = \frac{(t-1)^k}{(2\pi i)^k} \int \frac{t z^j \prod z_{\alpha} \prod (z_{\alpha} - z_{B})}{\prod (z_{\alpha} - z_{A})} \cdot \frac{f(z)}{f(z)} \, dz \right.$$  

where the $\Gamma_j$ contour contains $\alpha_i$, $\Gamma_i$,$\ldots$,$\Gamma_k$ contours $\alpha_i$, and no other poles.

Example: if $a_i \equiv 1$, $t$ small enough compared to $g$, then
Let's apply this result to $q$-TASEP (set $t=0$, $a_i=1$).

Since $D'_n P_n = q^{\lambda_n} P_n$ we find that

$$
(D'_n)^k \frac{\prod(a_{n-k}, a_n; (0:0;\tau))}{\prod(a_{n-k}, a_n; (0:0;\tau))} = E \left[ g^{K_{\lambda_n}} \right]_{MM(a_1, \ldots, (0:0;\tau))} = E_{q\text{-TASEP}} \left[ g^{K(x_n(t)+n)} \right]
$$

On the other hand,

$$
\prod(a_{n-k}, a_n; (0:0;\tau)) = \prod(a_i; (0:0;\tau)) \cdots \prod(a_N; (0:0;\tau)), \quad \prod(x_i; (0:0;\tau)) = e^{xt}
$$

So

$$
E \left[ g^{K_{\lambda_n}} \right] = (-1)^k g^{\frac{1}{2\pi i} \sum \prod \frac{Z_i - Z_j}{Z_i - Z_j} \frac{z_j}{j!} (1 - z_j)^N e^{(g-1)z_j z_j} dZ_j}
$$

(\text{contour contains } g Z_j, B > A) and 1, but not zero.

This is one of many nice formulas for expectations w.r.t. MacDonald measure.

It is also possible to compute joint level expectations of the ascending MacDonald process.

Later we will see how this formula arises for $q$-TASEP in a different (though related) manner using duality/replica method.
The nesting structure of contours becomes a bit cumbersome as $k$ grows. Can use complex analysis/residue theorem and Cauchy theorem to deform until all contours match. Two choices:

Crosses all poles coming
from $\prod_{j=1}^{\infty} \frac{z - z_j}{z - 3} \text{ term.}$

Crosses pole at $z = 0$ $\frac{1}{f(z)}$.

Both yield nice formulas. The large contour formula is easier, but not as useful as the small one, which we now state.
Prop: Assume \( f(x) \) is holomorphic and non-zero in a neighborhood of the real interval containing \( 0 \leq \xi \leq k \), then

\[
\begin{align*}
&(-1)^k \frac{1}{2 \pi i} \sum_{1 \leq a < b \leq k} \prod_{1 \leq j \leq k} \frac{1}{2 \pi i} \int_{|z_j|=1} \frac{f(z_j)}{f(z_j^*)} \frac{1}{z_j - z} \, dz_j \\
&= \sum_{\lambda=1}^{k} \frac{(1-g^{m_\lambda})}{m_1 \cdots m_\lambda} \prod_{1 \leq j \leq \lambda} \left( \text{small circle containing } 1 \right) \frac{1}{m_1 \cdots m_\lambda} \prod_{i \neq j} (1-z_i-z_j) \prod_{j=1}^{\lambda} \frac{f(z_j)}{f(z_j^*)} \prod_{1 \leq a < b \leq k} \frac{z_{\lambda a} - z_{\lambda b}}{z_{\lambda a} - z_{\lambda b}} \\
\end{align*}
\]

where

\[
E(z_1, \ldots, z_k) = \prod_{j=1}^{k} \frac{f(z_j)}{f(z_j^*)} \cdot \sum_{\sigma \in S_k} \prod_{1 \leq a < b \leq k} \frac{z_{\sigma a} - z_{\sigma b}}{z_{\sigma a} - z_{\sigma b}}
\]

If all \( N_j = N \) then \( E \) simplifies to

\[
E(z_1, \ldots, z_k) = \prod_{j=1}^{k} \frac{f(z_j)}{f(z_j^*)} \cdot \frac{1}{(1-g^N)^N} \cdot \sum_{\sigma \in S_k} \prod_{1 \leq a < b \leq k} \frac{z_{\sigma a} - z_{\sigma b}}{z_{\sigma a} - z_{\sigma b}}
\]

Notice \( C_k = \frac{1}{a_k(z_1, z_k)} \sum_{\sigma \in S_k} \prod_{1 \leq a < b \leq k} \frac{z_{\sigma a} - z_{\sigma b}}{z_{\sigma a} - z_{\sigma b}} \) must be a constant (by degree considerations).

In fact \( C_k = \frac{(1-g)(1-g^2) \cdots (1-g^k)}{(1-g)(1-g) \cdots (1-g)} = K_k \phi \)

Exercise: As \( g \to 1 \), \( K_k^\phi \to K_k^\phi \).
Hence we conclude that (equiv for q-TASEP $X_N(0+N)$)

$$E\left[q^{K^N}\right] = k^q \sum_{m_1, m_2} \frac{(1-q^k)}{m_1, m_2} \prod_{i=1}^{m_1} f(z_i) \prod_{j=1}^{m_2} f(w_j)$$

where $f(z) = e^{z^2}$ and the final terms can be telescoping and $(a; q)_k = (1-a) (1-aq) \cdots (1-aq^{k-1})$

We will use the above expression to uncover a Fredholm determinant.

But first, let us sketch the proof of the proposition. First consider the possible residues

**Example:** $K=2$

\[z_i - z_2, \quad z_1 - z_2\]

As $z_i$ shrinks to $z_2$ contour we cross $z_i - g z_2$ pole at $z_1 = g z_2$. Can either pick the residue, or the integral.

Hence our integral decomposes into a double integral and a single integral with $z_1 = g z_2$.

$K=3$

\[\frac{z_1 - z_2}{z_1 - g z_2}, \frac{z_1 - z_3}{z_1 - g z_3}, \frac{z_2 - z_3}{z_2 - g z_3}\]

Shrink $z_2$. Cross pole at $z_2 = g z_3$.

If we pick residue then

\[\frac{z_1 - g z_2}{z_1 - g z_3}, \frac{z_1 - z_3}{z_1 - g z_3}, \frac{z_1 - z_3}{z_1 - g z_3}\]

The apparent pole at $z_1 = g z_3$ is actually not present due to numerator.

Hence we only have pole at $z_1 = g^2 z_3$.

This shows how we get geometric strings of residues.
If (for general $k$) we shrink $Z_k$, then $Z_{k-1}$, ... then $Z_1$ we find the integral is equal to a sum of integrals with free integration variables $Z_{i_1}, Z_{i_2}, ...$

and all other $Z$'s fixed according to the following residue subspaces:

For $\lambda-k$

$$Z_i = gZ_{i_2} = g^2Z_{i_3} = \cdots g^{i-1}Z_{i_1} \text{ with } i_1 < i_2 < \cdots < i_{\lambda}$$

$$Z_j = gZ_{j_2} = g^2Z_{j_3} = \cdots g^{j-1}Z_{j_1} \text{ with } j_1 < j_2 < \cdots < j_{\lambda}$$

etc.

Then we may reverse order and permute the $\lambda Z$'s such that

$$(Z_{i_1}, \ldots, Z_{i_{\lambda}}) \mapsto (Y_{i_1}, Y_{i_2}, \ldots, Y_{i_{\lambda}})$$

$$(Z_{j_1}, \ldots, Z_{j_{\lambda}}) \mapsto (Y_{j_1}, Y_{j_2}, \ldots, Y_{j_{\lambda+1}})$$

etc.

There is some possible freedom in choice of permutation coming from clusters of residues with same size. If $\lambda = 1^m 2^m \cdots$ then there is a total multiplicity of $m_1!m_2! \cdots$ such permutations which suffice. Hence we can write the sum of all residues that correspond to a given partition $\lambda$ as follows: (X)
\[ x = \frac{1}{m_1! m_1! \ldots} \sum_{\text{res}} \operatorname{Res} \prod_{i < A < B < k} Y_{\text{var}(i)} - Y_{\text{var}(B)} \prod_{j=1}^k \frac{f(x_{ij})}{1 - Y_0(x_j)} \]

Note \[ \prod_{i < A < B < k} Y_{\text{var}(i)} - Y_{\text{var}(B)} = \prod_{A \neq B} Y_{\text{var}(A)} - Y_{\text{var}(B)} \]

So, introducing \( w_j = Y_{x_i} - x_{i+1} \) as the remaining integration variables,

\[ x = \frac{1}{m_1! m_1! \ldots} \operatorname{Res} \left( \prod_{A \neq B} Y_{\text{var}(A)} - Y_{\text{var}(B)} \right) \cdot \mathcal{E}(w_1, \ldots, q^{1/2}, w_2, \ldots, q^{1/2}) \cdot \prod_{j=1}^k w_j^{-1/2} - x_{i+1} \]

It remains to prove that

\[ \operatorname{Res} \left( \prod_{A \neq B} Y_{\text{var}(A)} - Y_{\text{var}(B)} \right) = (-1)^k (1 - q^k) \prod_{j=1}^k w_j^{1/2} \cdot \det(w_j q^{1/2} - w_j) \]

This relies on a careful calculation and the Cauchy det. identity.

Combining this provides the \( \pi \) term in the proposition and summing over all \( \pi \)-rk yields the proposition \( \square \).
Now return to \( \mathbb{E}[g^{k_{w}}] \), notice that we can write (for \( k=1 \))

\[
\mathbb{E}[g^{k_{w}}] = k! \sum_{l=1}^{\infty} \frac{1}{x!} \sum_{\lambda=1}^{\infty} \cdots \sum_{\lambda_{l}=1}^{\infty} \frac{1}{\lambda_{1}! \cdots \lambda_{l}!} \int \cdots \int \frac{\text{det}[K^{(r)}_{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{l}}(x_{1}, x_{2}, \ldots, x_{l})]}{C_{1}} \cdot g^{(q-1)}(w; q)^{\infty} \cdot wq^{x_{1} - w_{1}}.
\]

where \( K^{(r)}_{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{l}}(x_{1}, x_{2}, \ldots, x_{l}) = \sum_{r} g^{(q-1)}(w; q)^{\infty} \cdot wq^{x_{1} - w_{1}}. \)

This suggests a generating function \( G(s) \)

\[
G(s) := \sum_{k=0}^{\infty} \frac{\mathbb{E}[g^{k_{w}}] s^{k}}{k!} = 1 + \sum_{l=1}^{\infty} \frac{1}{x!} \sum_{\lambda=1}^{\infty} \cdots \sum_{\lambda_{l}=1}^{\infty} \frac{1}{\lambda_{1}! \cdots \lambda_{l}!} \int \cdots \int \frac{\text{det}[K^{(r)}_{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{l}}(x_{1}, x_{2}, \ldots, x_{l})]}{C_{1}} \cdot g^{(q-1)}(w; q)^{\infty} \cdot wq^{x_{1} - w_{1}}.
\]

This is the Fredholm expansion of \( I + K_{s} \) on \( L^{2}(\mathbb{Z}_{\geq 0} \times C_{1}) \), but since the kernel is independent of \( \lambda \) we can take the \( \lambda \) summations inside the determinant so that

\[
G(s) = \text{det}(I + K_{s})_{L^{2}(C_{1})} = 1 + \sum_{l=1}^{\infty} \frac{1}{x!} \sum_{\lambda=1}^{\infty} \cdots \sum_{\lambda_{l}=1}^{\infty} \frac{1}{\lambda_{1}! \cdots \lambda_{l}!} \int \cdots \int \frac{\text{det}[K^{(r)}_{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{l}}(x_{1}, x_{2}, \ldots, x_{l})]}{C_{1}} \cdot g^{(q-1)}(w; q)^{\infty} \cdot wq^{x_{1} - w_{1}}.
\]

with kernel \( K_{s}(w, w') = \sum_{r} s^{r} (1 - q)^{r} g^{(q-1)}(w; q)^{\infty} \cdot wq^{x_{1} - w_{1}} = \sum_{\lambda} s^{\lambda} \cdot [1 - q^{\lambda}] \)

with \( g(q^{r}) = \frac{e^{(q-1)T_{w}}}{{[q^{\lambda}]}} \cdot \frac{[w; q]^{\infty}}{[w_{1}; q]^{\infty}} \) (Note \( g(\cdot) \) is analytic away from its poles)
We may finally replace the summation by a contour integral via the identity

\[ \sum_{n=1}^{\infty} g(n^s) \delta^s = \frac{1}{2\pi i} \int_{\gamma} \frac{\pi}{\sin(\pi s)} (-\pi s) g(q^s) \, dq \]  

"Meijer-Bomer" type representation

which makes analytic sense for suitable \( q \) and choices of contours.

**Thm**

\[ G(s) = \sum_{k=0}^{\infty} \frac{E(k^w)}{k!} s^k = \text{det}(I + KS)_{[2]}(s) \]

\[ K_S(w,w') = \frac{1}{2\pi i} \int_{\gamma} \frac{\pi}{\sin(\pi s)} (-1)q^s \left( \frac{q^w}{q^w} \right)^n e^{-q^w} \, dq \]

**Question:** How does this help us compute the distribution of \( \lambda w (\xi^0, w) \)?

Note that since \( q \in (0,1) \) and \( w > 0 \), \( q^w \leq 1 \), hence all its moments finite.

This means that (by Carleman's condition) the moments of \( q^w \) determine its distribution, and hence since \( G(s) \) determines these moments, it determines the distribution.

The question is to find an inverse of the transform from distribution to \( G(s) \).
In 1949 Hahn introduced two $q$-deformed exponential functions

$$e_q(x) := \frac{1}{(1-q)x + q}, \quad E_q(x) := \frac{-(1-q)x + q}{(1-q)x + q}$$

Exercise: Show pointwise convergence of $e_q(x), E_q(x) \to e^x$ as $q \to 1$.

Focusing on $e_q$, there is a "Taylor series" which is consequence of $q$-Binomial theorem

$$e_q(x) = \sum_{k=0}^{\infty} \frac{x^k}{q^k}$$

This, along with the fact that $q^m \leq 1$, implies that for $s$ small enough

$$E\left[ e_q(sq^m) \right] = \sum_{k=0}^{\infty} \frac{E[q^{k/m}]}{q^k} = \text{det} \left[ I + K_s \right]^{(1/2)}$$

The left and right are analytic in $s$ away from poles, hence identity holds.

This is a $q$-Laplace transform with spectral variable $s$.

It is completely surprising that this is such a simple Fred det.

In the Schur/RMT/TW limits this recovers the fact that the dist. of the top eigenvalue is given by such a Fred det.

That was a consequence of det. p.p. structure - what has taken that place?
Def: For a function $f \in l'(\mathbb{Z}_{30})$ define for $z \in \mathbb{C}/5^{m=3}m_{30}$

$$\hat{f}(z) = \sum_{n=0}^{\infty} \frac{f(n)}{(z q^n : q)_{\infty}}$$

Prop: Can recover $f$ from $\hat{f}$ via

$$f(n) = q^{-\frac{1}{2\pi i}} \int_{\text{contour}} (q^{n+1} z : q)_{\infty} \hat{f}(z) dz$$

with $z$ contour containing only $z = q^m$, $0 \leq m \leq n$ poles.

This is the $q$-deformed Laplace transform and has many nice properties such as linearity, scaling, shift, transformation under $q$-derivative/integral, $q$-product/convolution. (see Gaspard Bongerezko manuscript for uses in solving $q$-Difference equations)

Hence we have found expression for distribution of $\lambda_{N} (\text{eq. } x_{[x]} + x)$ in which complexity does not grow with $(N, z, q)$. In practice we will generally deal with $q$-Laplace transforms since convergence of such transform suffices for weak convergence of distribution, and ultimately this transform converges to a probability (in a certain scaling)
Final note: Had we pursued here the "large contour" residue formula we are also led to a simpler (but less useful) Fred. det.

\[ \text{Thm } \mathbb{E} \left[ e_g(\delta g \omega) \right] = \frac{1}{(1-\rho \lambda)^2} \det \left( I + K_{\rho} \right) \]

\[ K_{\rho}(w, w') = 5 \frac{(1-\rho) (1-w)^{-2} e^{(\rho-1) e^t \omega}}{q w' - w} \]

Notes: good for asymptotics because separately the det and prefactor has (in the scalings we will take) no clear limit, but rather there must be some intrinsic cancelation which is a priori not obvious.