ASEP, q-TASEP and integrable many body systems

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Theorem [B-C11], [B-C-Sasamoto12] For g-TASEP with step initial data \( \{ u^{(0)} = -n \}_{n \geq 0} \).

\[ \frac{f_z}{f_v} \prod_{z \in \mathbb{Z}} \frac{1}{1 - q^{z-a}} \prod_{z > a} \frac{b - q^{A-a}}{q^A - q^a} = \left[ \left( x_{N+1} + x_N \right) + \cdots + \left( x_{N+1} + x_1 \right) \right] \]

\[ \prod_{n \geq 0} \left( 1 - b^q \right) \]

Example: \( g \)-TASEP [Borodin-Corwin, 2011]

Existence of a large family of observables whose averages are explicit.

Basic reason that all these models turned out to be accessible is the
Part 1: $q$-TASEP

Theorem [Borodin-C '11, Borodin-C-Sasamoto] For step initial data $\{X_n(0) = -n\}_{n=1}^{\infty}$

$$h(t) = E\left[ \prod_{i=1}^{k} q_{X_i(t)+n_i} \right] = (-1)^k \frac{q^k}{\pi^k} \int \prod_{1 \leq i < j \leq k} \frac{1}{z_i - z_j} \frac{1}{z_i - z_j} \frac{(g-1)z_j}{(1-z_j)z_i} \, dz_1 \cdots \, dz_k$$

\[ n_1 = n_2 = \cdots = n_k \]

(Quantum) many body system approach (for $q$-TASEP)

1. Find observables with closed whose expectations satisfy closed "true evolution equations".

2. Rewrite true evolution equation in "integrable form" as $k$ one body free evolution eqn with $k-1$ two body boundary cond.

3. Solve free system with bc via "nested contour integral ansatz" (a version of Bethe ansatz)

Generally, not (always) clear how to find systems which are amenable to this form of solvability (Macdonald processes and algebraic Bethe ansatz provide more structural approaches)
Step 1: True evolution eqn.

\[ \text{d} g_{n(t)+n} = (q_{n(t)+1+n} g_{n(t)+n}) (1-q_{n(t)-1}) \text{d} t + \text{d} W \]

\[ = (1-q) \nabla g_{n(t)+n} \text{d} t + \text{d} W \]

\[ \nabla f(n) = f(n-1) - f(n) \]

So:

\[ \frac{\text{d}}{\text{d} t} \mathbb{E}[g_{n(t)+n}] = (1-q) \nabla \mathbb{E}[g_{n(t)+n}] \]

\[ \mathbb{E}[g_{n(t)+n}] = 0 \]

K=2: Assume \( n_1 \geq n_2 \)

For \( n_1 > n_2 \):

\[ \frac{\text{d}}{\text{d} t} \mathbb{E}[g_{n_1(t)+n_1} g_{n_2(t)+n_2}] = (1-q) (\nabla + \nabla^T) \mathbb{E}[g_{n_1(t)+n_1} g_{n_2(t)+n_2}] \]

For \( n_1 = n_2 \):

\[ \text{d} g_{n(t)+n} = (q g_{n(t)+1+n} - q) (1-q \nabla g_{n(t)-1} - (1-q) \nabla g_{n(t)+n}) \text{d} t + \text{d} W \]

So:

\[ \frac{\text{d}}{\text{d} t} \mathbb{E} \left[ \prod_{i=1}^{n_1(t)+n_1} \right] = (1-q^2) \nabla_n \mathbb{E} \left[ \prod_{i=1}^{n_1(t)+n_1} \right] \]

K=3: For each cluster in \( \hat{n} = (n_1, \ldots, n_k) \), apply:

\[ \frac{\text{d}}{\text{d} t} \mathbb{E} \left[ \prod_{i=1}^{n_1(t)+n_1} \right] = \sum_{\text{clusters}} (1-q^{\text{cluster}^1}) \nabla_{\text{top label cluster}} \mathbb{E} \left[ \right] \]

(duality to a TAZRP)
Step 2:

The true evolution equation is not constant coefficient or separable.

- Bethe's idea (1931): Rewrite in terms of solution of a particle free evolution equation subject to k-1 two body boundary conditions. Advanced Reflection Principle!!

Usually not possible and one has to consider impose multiparticle boundary conditions. BUT, if possible $\Rightarrow$ "Integrable"!

- Idea in motion for $k=2$.

Consider $u : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}^2 \rightarrow \mathbb{R}$ solving

$$\frac{\partial}{\partial t} u(t, \vec{n}) = \sum_{i=1}^{2} (1-q) \nabla_i u(t, \vec{n})$$

Can violate 2nd order.

For $n_1 > n_2$, this matches true evolution equation.

But for $n_1 = n_2$, differs by

$$\sum_{i=1}^{2} (1-q) \nabla_i u(t, \vec{n}) - (1-q^2) \nabla_2 u(t, \vec{n})$$

($\ast$)

If $u$ such that ($\ast$) $\equiv 0$ then restricted to $\exists n_1 \neq n_2$

$u$ solves true evolution equation.

\[\nabla_1 - q \nabla_2 \big|_{n_1 = n_2} \equiv 0\]
For $k \geq 3$ a priori may have boundary conditions for all compositions of $(n_1 = n_2$, $n_2 = n_3$, also $n_1 = n_2 = n_3$).

Amazingly: Only $k-1$ boundary conditions from $n_i = n_{i+1}$ must be imposed, all others follow from them!

Proposition [Borodin-C-Sasamoto "12] "Free evol. eq. with k-1 boundary cond."

If $u : \mathbb{R}_{\geq 0} \times \mathbb{Z}^k_{\geq 0} \to \mathbb{R}$ solves

1. $\forall \vec{n} \in \mathbb{Z}^k_{\geq 0}, t \in \mathbb{R}_{\geq 0}$
   
   \[ \frac{\partial}{\partial t} u(t; \vec{n}) = \sum_{i=1}^{k} (1-q_i) \nabla_i u(t; \vec{n}) \]

2. $\forall \vec{n} \in \mathbb{Z}^k_{\geq 0}$ s.t. $n_i = n_{i+1}$ ($1 \leq i \leq k-1$)
   
   $\left( \nabla_i - q_i \nabla_{i+1} \right) u(t; \vec{n}) \equiv 0$

3. $\forall \vec{n} \in \mathbb{Z}^k_{\geq 0}$ with $n_k = 0$, $u(t; \vec{n}) \equiv 0$

4. $\forall \vec{n} \in \mathbb{Z}^k_{\geq 0}$ s.t. $n_1 = n_2 = \ldots = n_k$, $\mathbb{W}^k_{\geq 0}$, $u(0; \vec{n}) = h(0; \vec{n})$

Then $\forall \vec{n} \in \mathbb{W}^k_{\geq 0}$, $h(t; \vec{n}) = u(t; \vec{n})$.

"Restriction solves true evolution equation"
Step 3: Check for step initial data, integral solves free system...

\( X_i(0) = -i \) means \( e^{X_i(0)+i} \equiv 1 \) so \( h(0;i,n) \equiv 1 \)

- Multiplicative term \( \prod_{j=1}^{k} \frac{e^{(g-1s)jz_j}}{(1-z_j)^{n_j}} \) solves (1)

- If \( n_i = n_{i+1} \) (eg \( i=1, i+1=2 \)) apply \( (D_i - gD_{i+1}) \) to integrand of integral. Introduces \((Z_i - gZ_{i+1})\) factor, cancels term in denominator, allows to deform \( Z_i, Z_{i+1} \) contours together.

\[ \oint (Z_i, Z_{i+1}) (Z_i - Z_{i+1}) = 0 \] hence (2)

\[ \text{contains other integrals} \]

- (3), (4) via residue calculus.

Hence integral satisfies (1), (2), (3), (4) \( \Rightarrow \) Theorem \( \Box \).

Note: Inserting other symmetric functions of \( z_i, \ldots, z_k \) (right now \( \prod \frac{1}{z_j} \)) will relate to other initial data of true evolution eqn.

The inverse map remains a challenge!

Could be solved if we can diagonalize the Hamiltonian.
A good way to check the solution, but how to produce it?

- Macdonald processes: Integrable properties of Macdonald polynomials led to $q$-TASEP (Pierce rules), the observables (eig values of $D_1$) and integral formulas (eig values + Cauchy identity).

  Many body system $\leftrightarrow$ Commutation relation involving $D_1$.

  Only useful for certain class of initial data.

  Coordinate

- Bethe ansatz [B-C-Detw-Sasamoto '03]: Diagonalize free evts.

  Eqts. arise via proving a new Plancherel theorem (gen. of $H_0$).

  Gives approach for general initial data.

  From "Boson"

- Algebraic Bethe ansatz [Sasamoto-Wadati '98]: Produces the time evts; eqts. and some indications of their integrability (without any reference to $q$-TASEP), check but not really studied or exploited yet.
One application of $q$-TASEP formulas.

Note: Expectation of observables completely determine distribution of $X_n(t)$.

- One point marginal via $q$-Laplace transform of $q^{X_n(t)+n}$:

$$E[s^\text{step}(\text{Hahn}_q g)] = E[\sum_{k=0}^{\infty} g^{k}(1-q)...(1-q^k)]$$

Justified for sufficiently small

$$= \sum_{k=0}^{\infty} E[s^\text{step}(kX_0(t)+n)] g^k/(1-q)...(1-q^k)$$

Deforming nested contours together (and keeping track of partial residue expansion)

\textbf{Theorem [B-C '11]}: For step initial data $q$-TASEP

$$E[s^\text{step}(\text{Hahn}_q g)] = \det(I + K_{s,n,t})$$

This result is suitable for asymptotics such as necessary to prove $t^{1/3}$ TW GUE asymptotics (i.e. KPT universality), as well as one-point statistics for things like KPT equation.
At \( q = 0 \rightarrow \) parallel geometric TASEP with blocking

\[
\begin{align*}
& \left( \left( 1 - \frac{1}{u} \right) \left( \frac{1}{u} \right) \right) = \left( \frac{1}{2} \right)
\end{align*}
\]

\( \text{jump} \)

Parallel Geometric discrete time \( q \)-TASEP [Borodin-C, 23]
At \( q = 0 \to \text{sequential Bernoulli TASER} \ 
(\text{Borodin-Ferrari}, 08) \)

\[ (\text{1) } \quad \rho_{q^+} \frac{1}{q}, \rho_{q^-} \frac{1}{q} \]

\[ (\text{2) } \quad \rho_{q^+} \cdot \frac{1}{q^+} \quad \rho_{q^-} \frac{1}{q^-} \]

\[ (\text{3) } \quad \rho_{q^+} \cdot \frac{1}{q^+} \quad \rho_{q^-} \frac{1}{q^-} \]

(Sequential Bernoulli discrete time \( q \)-TASEP \ (Borodin-C, 13))
$f(z) = \begin{cases} 
\frac{e^{-tz}}{(1+z)^t}, & \text{Geometric discrete } q\text{-TASEP} \\
\int_{(1+z)^t}, & \text{Bernoulli discrete } q\text{-TASEP} 
\end{cases}$

$E_{\text{step}} \left[ T \left. g_{y_i}(z_{n-i}) \right| i \right] = \frac{(-z_{n-i})^{k-1/2}}{(a(t))^k} \int \cdots \int \frac{e^{-2a(t)}}{f(1)^2 ; \nu_i \cdots \nu_k} d\nu_k d\nu_{k-1} \cdots d\nu_1$

Theorem [Borodin-C '13]: For $n_1 \geq n_2 \geq \cdots \geq n_k > 0$

$q\text{-TASEP joint moments satisfy various many body systems}$
\[(a; q)_n = (1 - a)(1 - qa) \cdots (1 - q^{n-1}a)\]

\[
E \left[ \frac{1}{(a q^{X_n(t)} + a)} \right] = E \left[ \sum_{k=0}^{\infty} \frac{g_k}{(1 - q) \cdots (1 - q^k)} q^{k(X_n(t) + n)} \right]
\]

\[\text{justified for small } q\]

LHS: "q-Laplace transform of } g^{X_n(t) + n} \text{" identifies dist.

RHS: we have explicit formulas.

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**Slide**

Theorem (Baratin-Corwin '11): q-TASEP step initial data, \(SFC \setminus \mathbb{R}_+\)

\[
E \left[ \frac{1}{(a q^{X_n(t)} + a)} \right] = \det(I + K_{5, n, t}) L^2(C_1)
\]

\[K_{5, n, t}(w, w') = \frac{1}{2\pi i} \int_{i\mathbb{R} + \frac{1}{2}} \frac{\pi}{\sin(\pi\tau)} (-5)^\tau g(\tau) \frac{ds}{q^{\tau w - w'}} \]

\[g(w) = \frac{e^{-\tau w}}{(w; q)^n} \]
$g$-Laplace trans

\[ E_q(x) := \frac{1}{(1-q)x + q) ; \infty} \]

\[ E_q(x) := (1-q)x + q) ; \infty \]

\[ (a : q) \infty = (1-aq)(1-q^2) \ldots (1-q^{n-1}a) \]

HW As \( q \to 1 \), \( E_q(x) = e^x \)

Prop: \( f \in l^2(\mathbb{Z}_q) \), for \( z \in \mathbb{C} / \mathbb{Z}_q \) \( \mathbb{Z}_q^{-m} \), define

\[ \hat{f}_q(z) := \sum_{n=0}^{\infty} \frac{f(n)}{(zq^n + q) ; \infty} \]

Then \( f(n) = -q^n \frac{1}{\text{ar} n} \int (zq^n + q) ; \infty \hat{f}_q(z) \, dz \)

contains \( z = q^{-m} : 0 \leq m \leq n \)

- Linearity, scaling, shift, trans. under \( g \)-der. /int.
  - \( g \)-prod /convolution, useful in solving \( g \)-diff. equations.
Vested contours to
Fredholm determinants.

\[ \mu_k = \frac{e^{ik}}{(2\pi i)^k} \prod_{A \leq B} \frac{Z_A - Z_B}{Z_A - q Z_B} \prod_{j=1}^{k} \frac{i}{(1 - z_j) \Phi(z_j)} \]

\[ \mu_{\{N_1, \ldots, N_k\}} \]

Contours become cumbersome as \( k \) grows.

Crosses all poles from

\[ \prod_{A \leq B} \frac{X_A + X_B}{X_A - q X_B} \]

term

Crosses pole at \( z_j = 0 \).
Proposition [Borodin-C '11]: For "nice" \( f(x) \)

\[
M_K(\vec{n}) := \frac{(-1)^k \, i^{k(k-1)/2}}{(2\pi i)^k} \int \cdots \int \prod_{1 \leq A < B \leq K} \frac{Z_A - Z_B}{Z_A - g Z_B} \prod_{j=1}^k \frac{1}{(1 - z_j)^n} \frac{f(z_j)}{f(\bar{z}_j)} \frac{dz_j}{z_j}
\]

\[
= \sum_{\lambda \vdash K} \frac{(1-q)^k}{m_1! m_2! \cdots} \cdot \frac{1}{(2\pi i)^{2n}} \int \cdots \int \operatorname{det} \left[ \frac{1}{\mu_i - \lambda_j^*} \right]_{i,j=1}^n E_{\vec{n}}(w_1, g w_1, \ldots, g^{\lambda_1 - 1} w_1, w_2, \ldots, g^{\lambda_2 - 1} w_2, \ldots, w_{2n}, \ldots, g^{\lambda_{2n} - 1} w_{2n}) \, dw
\]

where

\[
E_{\vec{n}}(z_1, \ldots, z_k) = \prod_{j=1}^k \frac{f(z_j)}{f(\bar{z}_j)} \cdot \sum_{\sigma \in S_k} \prod_{k \geq A \geq 1} \frac{Z_{\sigma(A)} - g Z_{\sigma(B)}}{Z_{\sigma(A)} - Z_{\sigma(B)}} \prod_{j=1}^k \frac{1}{(1 - z_{\sigma(j)})^{n_j}}
\]
Prop: Assume $\hat{H}(z)$ holomorphic ($\hat{H}(z) = e^{z}$) then

$$\mu_n(\alpha, \beta) = \sum_{n_1, n_2, \ldots, n_k} \frac{(1-\theta^n)^k}{(2\pi i)^{n_k} n_1! \ldots n_k!} \prod_{j=1}^{n_k} \frac{1}{n_j!} \det \left[ \frac{1}{w_{i_1}^{n_j} w_{i_2}^{n_j} \ldots w_{i_{n_j}}^{n_j}} \right]_{i,j=1}^{n_j}$$

$$\times E(w_1^{a_1} w_1^{b_1}, w_2^{a_2} w_2^{b_2}, \ldots, w_{n_k}^{a_{n_k}} w_{n_k}^{b_{n_k}})$$

$$E(z_1, \ldots, z_n) = \prod_{j=1}^{n} \frac{f(z_j, \alpha)}{f(z_j, \beta)} \sum_{A>B} \frac{1}{Z_{\alpha(B)} - Z_{\alpha(A)}} \prod_{j=1}^{n} \frac{1}{1 - z_j}$$

An important object - eigenfunction for many body system, and the integration is related to completeness.

If $N_j = N$, $E$ simplifies

$$E(z_1, \ldots, z_n) = \prod_{j=1}^{n} \frac{f(z_j, \alpha)}{f(z_j, \beta)} \sum_{A>B} \frac{1}{Z_{\alpha(B)} - Z_{\alpha(A)}} \prod_{j=1}^{n} \frac{1}{1 - z_j}$$

**Lemma:** $C_k = \frac{(1-\theta)(1-\theta^2) \cdots (1-\theta^k)}{1-\theta^k} = k^k!$

**Lemma:** $k^k! \to k^k$ as $\theta \to 1.$
So \( \mu_k(N) = (1-q^N-1-q^k) \sum_{l+k} \frac{1}{m_1 m_2 \ldots} \frac{1}{(2\pi i)^d} \int \ldots \int \prod_{j=1}^{l+k} \frac{1}{\#(w_j)} (1 + \frac{1}{(w_j; q)_{\lambda_j^j}})^N \). 

\[
\frac{\mu_k(N) \cdot b^k}{(1-q^N-1-q^k)} = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\lambda_1 + \ldots + \lambda_k = k} \sum_{\lambda_1 = 1}^{\infty} \sum_{\lambda_2 = 1}^{\infty} \ldots \sum_{\lambda_l = 1}^{\infty} 1 \cdot \sum_{i=1}^{\infty} \sum_{w_1, w_2, \ldots, w_l} \det[\tilde{K}_{S,\lambda}(w_1, w_2, \ldots, w_l)]_{i,j=1}^l
\]

\[
\tilde{K}_{S,\lambda}(w_1, w_2, \ldots, w_l) = \sum_{\lambda_1 + \ldots + \lambda_k = k} \frac{1}{\#(w_1, w_2, \ldots, w_l)} (1 - \frac{1}{(w_1; q)_{\lambda_1}})^N \cdot \frac{1}{w_{i_2}^3 - w_{i_1}}
\]

\[
G(s) = \sum_{k=0}^{\infty} \frac{\mu_k(N) \cdot b^k}{(1-q^N-1-q^k)} = 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{\lambda_1 + \ldots + \lambda_k = k} \sum_{w_1, w_2, \ldots, w_l} \det[\tilde{K}_{S,\lambda}(w_1, w_2, \ldots, w_l)]_{i,j=1}^l
\]

\[
= \det(I + \tilde{K}_{S,\lambda})_{L^2(\mathbb{Z}_0 \times C_1)}
\]

\[
= 1 + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{w_1, w_2, \ldots, w_l} \det(K(w_1, w_2, \ldots, w_l))_{S_{n,t}}
\]

\[
= \det(I + K)_{L^2(C_1)}
\]
$\mathcal{G}(w)$ with $K_0(w, w') = \sum_{x=1}^{\infty} e^x \cdot \mathcal{G}(w, w') 
abla_{w, w'}(q^x)$

$\nabla_{w, w'}(q^x) = \frac{\pi(q^w)(q^w; q)_{\infty}^N}{\pi(w)(q^w; q)_{\infty}^N} \cdot \frac{1}{w^{x-1}}$

This is not yet ready for $q = 1$ asymptotics since the series for $K$ above has termwise limit which is divergent as $q = 1$.

Instead, should sum the series and then take $q = 1$.

Identity "Mellin-Barnes representation"

$$\sum_{x=1}^{\infty} g(q^x) s^x = \frac{1}{2\pi i} \int_{\text{Contour}} \frac{\pi(q^s)(q^s; q)_{\infty}^N}{\pi(s)(q^s; q)_{\infty}^N} g(q^s) ds$$

Valid under some conditions on \( g \) and contour.

Taking contour away from \( 1, 2, \ldots \), this is now amenable to asymptotic analysis.
Good for asymptotics (Patrick's talk)

9-Laplace transform [Haun, 49] identifies \( X_n(t) \) distribution.

\[ \ldots \left ( \zeta_b \cdot 1 \right ) \left ( \gamma_b \cdot 1 \right ) \left ( \alpha_b \cdot \infty \right ) = \left ( b \cdot 0 \right ) \]

\[ \frac{\infty (b \cdot m)}{m^2 - \alpha} = (m)^b \]

\[ \mathcal{L} \left \{ \frac{m - m_b}{b} \frac{b}{(m)^b} \right \} \cdot \frac{\sin(\lambda b)}{\pi} \int \frac{d\omega}{\lambda} = (m')^b \]

Where

\[ \mathcal{L} \left \{ I + K \right \} \cdot D = \left [ \frac{\infty (b \cdot \chi_n)}{1} \right ] \]

Theorem (Borodin, 11); 9-TASEP step initial data

\[ \chi_{x_n} \]
Proof of nested contour expansion result.

Example $k = 2$: \[ \frac{z_1 - z_2}{z_1 - q^2 z_2} \]

Pole $z_1 = q^2 z_2$ so pick residue at $z_1 = q^2 z_2$ and remaining integral.

Example $k = 3$: \[ \frac{z_1 - z_2}{z_1 - q^2 z_2} \cdot \frac{z_1 - z_3}{z_1 - q^2 z_3} \cdot \frac{z_2 - z_3}{z_2 - q^2 z_3} \]

Shrink $z_2$, cross pole at $z_2 = q^2 z_3$.

Residue term and integral term

\[ \frac{z_1 - q^2 z_3}{z_1 - q^2 z_3} \cdot \frac{z_2 - z_3}{z_2 - q^2 z_3} \cdot \frac{q^2 z_3 - z_1}{z_2 - q^2 z_3} \]

Apparent pole at $z_2 = q^2 z_3$ not present: only pole at $z_1 = q^2 z_3$.

We decompose into strings of residues at geometric
Generally shrink $Z_{k-1}, \ldots, Z_1$ and express integral as sum over integral in which some variables are along $C_1(\pm \theta)$ and others have taken residues on geo.

Strings of example, $l(\lambda), \lambda=1^m1^m2^m$...

Index strings by partitions $\lambda+k$ and

$Z_{i_1} = q Z_{i_2} = q^2 Z_{i_3} = \ldots = q^{\lambda_1-1} Z_{i_1}, \quad i_1 < i_2 < \ldots < i_{\lambda_1}$

$Z_{j_1} = \ldots = q^{\lambda_2-1} Z_{j_1}, \quad j_1 < j_2 < \ldots < j_{\lambda_2}$

Ratter complex... rewrite in more canonical way:

$(Z_{i_1}, \ldots, Z_{i_{\lambda_1}}) \mapsto (Y_{\lambda_1}, \ldots, Y_1)$

$(Z_{j_1}, \ldots, Z_{j_{\lambda_2}}) \mapsto (Y_{\lambda_1+2}, \ldots, Y_{\lambda_1+1})$

If A has some elements of multiplicity, we can interchange levels. Let's count them all, but divide each by multiplicity.

Let $\mathfrak{R}_{S_k} : (Z_1, \ldots, Z_k) = (Y_{\alpha(1)}, \ldots, Y_{\alpha(k)})$

-- not all $\alpha$ arise --
Call \( Y_1 = w_1, Y_{\lambda+1} = w_2, \ldots \)

\[
M_K(n) = \sum_{\lambda+k} \frac{1}{m_{\lambda+k}(\lambda+k)! \prod \mathfrak{d}w,} \quad \mathfrak{d}w = \prod \mathfrak{d}w
\]

\[
\text{Res } \sum \prod_{A \neq B} \frac{Y_{\alpha}(\lambda)}{Y_{\alpha} - q Y_{\beta}} = \frac{1}{\prod_{j=1}^n \left( 1 - Y_{\alpha j} \right)^{Y_{\beta j}}} \frac{f(q Y_{\alpha j})}{f(Y_{\beta j})}
\]

Initially not all of \( S_k \), but other terms have zero residue.

\[
\prod_{A \neq B} \frac{Y_{\alpha} - Y_{\beta}}{Y_{\alpha} - q Y_{\beta}} = \prod_{A \neq B} \frac{Y_{A} - Y_{B}}{Y_{A} - q Y_{B}} \cdot \prod_{A \neq B} \frac{Y_{\alpha} - q Y_{\beta}}{Y_{\alpha} - Y_{\beta}}
\]

Can see has residue has no residue.

\[
\text{Res } \left( \prod_{A \neq B} \right) \Rightarrow \det \left( \frac{1}{w_{ij}} \right)
\]

Rest gives \( E(w_1, \ldots, q^{\lambda_1} \cdot w_1, \ldots) \)
q-TASEP satisfies:

\[
\begin{align*}
\frac{d}{dt} q^{X_n(t)+n} &= (1 - q) \nabla q^{X_n(t)+n} dt + q^{X_n(t)+n} dM_n(t) \\
q^{X_n(0)+n} &\equiv 1 \quad \text{(step)}, \quad q^{X_0(t)+0} \equiv 0 \quad (X_0 = \infty)
\end{align*}
\]

Theorem [Borodin-C 11]: For q-TASEP with step init. cond. scale \( q = e^{-\varepsilon}, \quad t = \varepsilon^{-2} \tau, \quad X_n(t) = \varepsilon^{-2} \tau - (n-1)\varepsilon^{-1} \log \varepsilon^{-1} - \varepsilon^{-1} f_{\varepsilon}(\tau, n) \)
and call \( \mathcal{Z}_\varepsilon(\tau, n) = \exp \left\{ -\frac{3\tau}{2} + f_{\varepsilon}(\tau, n) \right\} \). Then as \( \varepsilon \downarrow 0, \mathcal{Z}_\varepsilon(\cdot, \cdot) \Rightarrow \mathcal{Z}(\cdot, \cdot) \) where \( \mathcal{Z} \) solves the semi-discrete SHE:

\[
\begin{align*}
\frac{d\mathcal{Z}(\tau, n)}{d\tau} &= \nabla \mathcal{Z}(\tau, n) d\tau + \mathcal{Z}(\tau, n) dB_n(\tau) \quad \text{(ind. BM's)} \\
\mathcal{Z}(0, n) &= 1_{n=0}, \quad \mathcal{Z}(\tau, 0) \equiv 0
\end{align*}
\]
**Vignette 1: Polymer replica method limit**

(From duality) we know $q$-TASEP dynamics imply

$$d q^n(x(t),n) = (1-q) \nabla q^n(x(t),n) \, dt + q^n(x(t),n) \, dM_n(t)$$

martingale

$q^n(x_0,0) = 0$ \quad zero b/c $x_0 = \infty$

$q^n(x_0,n) = 1$ \quad step initial data

**Def.** $Z: \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ solves the semi-discrete stochastic heat equation (SHE) with $Z_0(n)$ initial data if:

$$d Z(z;n) = \nabla Z(z;n) \, dZ + Z(z;n) \, dB_n(t)$$

where

$Z(z;0) = 0$

$Z(0;n) = Z_0(n)$

**Theorem [Borodin-C]:** For $q$-TASEP with step initial condition set

$q = e^{-\epsilon}$, \quad $t = \epsilon^{-2} \tau$, \quad $x_n(t) = \epsilon^{-2} \tau - (n-1) \epsilon^{-1} \log \tau - \epsilon^{-1} f_n(\tau)$

and call $Z^\epsilon(z;n) = e^{\frac{-3\tau}{2}} e^{f_n(\tau)}$. Then, ass a spacet ime process, $Z^\epsilon(z;n)$ converges weakly to $Z(\tau;n)$ with $Z_0(n) = 1_{n=1}$.
Sketch of stochastic analysis proof:

- **Initial data:** \( Z_{e}(0; n) = e^{n-1}e^{en} \to 1_{n=1} \)

- **Dynamics:** \( df_{e}(\tau; n) = f_{e}(\tau; n) - f_{e}(\tau - e^{-2}\tau_{1}; n) \)

\[
= e^{-3}d\tau - 3e\left[ x_{n}(e^{-2}\tau) - x_{n}(e^{-2}\tau - e^{2}\tau_{1}) \right] 2d\tau
\]

- **q-TASEP jump rate in rescaled time variables is**

\[
1 - q_{n-1}(\tau) - q_{n}(\tau) = 1 = e^{\frac{1}{e}f_{e}(\tau; n-1) - f_{e}(\tau; n) + O(\epsilon^{2})}
\]

so in time \( e^{-2}\tau_{1} \) (by convergence of poisson pt.proc. to BM)

\[
E\left[ x_{n}(e^{-2}\tau) - x_{n}(e^{-2}\tau_{1} - e^{2}\tau_{1}) \right] \approx e^{\frac{1}{e}} - \int_{B_{n}(\tau_{1})}^{B_{n}(\tau_{1})} d\tau_{1} = dbn(\tau) - dbn(\tau_{1})
\]

Thus we see \( df_{e}(\tau; n) = e^{\frac{1}{e}f_{e}(\tau; n-1) - f_{e}(\tau; n)} d\tau + dbn(\tau) + O(\epsilon)
\]

Exponentiating and applying Itô's Lemma gives

\[
df_{e}(\tau; n) = \left( \frac{1}{2} e^{f_{e}(\tau; n)} + e^{f_{e}(\tau; n) - f_{e}(\tau; n)} \right) d\tau + e^{f_{e}(\tau; n)} dbn + O(\epsilon)
\]

Or in terms of \( Z_{e} \) as in statement of theorem

\[
dZ_{e}(\tau; n) = \nabla Z_{e}(\tau; n) d\tau + Z_{e}(\tau; n) dbn + O(\epsilon)
\]

And as \( \epsilon \to 0 \) we recover the claimed formula \( \square \)
Start building a picture

\begin{align*}
\text{q-TASEP} \left( {q^{\times_{\text{int}}} + n} \right) \\
\text{Implications}
\end{align*}

- Weak convergence as \( q \to 1 \) implies

\text{q-Laplace transforms of} \( q^{\times_{\text{int}}} + n \Rightarrow \text{Laplace transforms of} \ Z(\text{\texttt{int}}) \)

hence we rigorously prove formula:

\[
\mathbb{E}^{1_{n\times_{\text{int}}}} \left[ e^{-\beta Z(n, \text{\texttt{int}})} \right] = \text{det}(I + \tilde{K}_{g, \tau, n})
\]

which identifies distribution of \( \text{Mall} \ Z(\text{\texttt{int}}) \).

**Remark:**

Corday [Borodin - C, Borodin - C - Ferrari]

set \( F(n, \text{\texttt{int}}) = \frac{3}{2} \tau + \log Z(n, \text{\texttt{int}}) \) \( (\Rightarrow \lim_{\text{\texttt{int}}} F(n, \text{\texttt{int}})) \) then for all \( \gamma > 0 \)

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{F(n, \text{\texttt{int}}) - n \overline{f}_k}{n^{\frac{1}{3}}} \leq \gamma \right) = F_{\text{GUE}}(\frac{\overline{f}_k - \frac{1}{2}}{\sqrt{2}} \gamma)
\]

\( \overline{f}_k = \inf_{\beta > 0} (\beta t - \psi'(t)) \)

\( \overline{f}_k \rightarrow \inf \overline{f}_k \quad \psi(t) = (\log t)''(t) \)

\( \overline{g}_k = -\psi''(\overline{f}_k) \).

- \( \overline{f}_k \) conj O'Connell - Yor, proved O'Connell - Novikov

- Seppäläinen - Valko proved upper-bound of \( O(t^{2/3}) \) on variance of \( Z(n, \text{\texttt{int}}) \)

- Similar result should hold for q - TASEP directly
Limit of q-TASEP duality, many body systems and moment formulas
(either take limit or use replica method)

Aside of Feynman-Kac representation:

- Consider homogeneous Markov process generator $L$ and deterministic potential $V$

- Goal: Solve $\frac{d}{dt} Z(t,x) = (LZ)(t,x) + V(t,x)Z(t,x) : Z(0,x) = Z_0(x)$

- Probabilistic interpretation
  
  for $L = V$

- For potential $V \equiv 0$ by superposition/linearity of expectation
  
  $Z(t,x) = \mathbb{E}^{\varphi(t) = x} \left[ Z_0(\varphi(0)) \right]$

- When $V$ is turned on, Duhamel's principle shows
  
  $Z(t,x) = \int_{-\infty}^{\infty} p(t,x-y) Z_0(y) dy + \int_0^t ds \int dy p(t-s,y-x) Z(s,y) V(s,y)$

- Apply identity multiple times yields series which sums to

  $Z(t,x) = \mathbb{E}^{\varphi(t) = x} \left[ \mathbb{E}^{\varphi(0) = 0} \left[ Z_0(\varphi(0)) \right] \right]$
This $V$ is random, care is needed in defining stochastic integrals.

Generally leads to correction in exponential which goes by Wick or Gaussian correction.

- For $\mathcal{Z}(t; n)$, $V(t; n) = dB_n(t)$, $L = \nabla$ so

$$
\mathcal{Z}(t; n) = \mathbb{E}^{\Phi(t) = \eta} \left[ \exp \left\{ \int_0^t (dB_n(s)) - \frac{ds}{2} \right\} \mathcal{Z}_0(\Phi(t)) \right]
$$

Using these path integrals we can show (like last week) that $\bar{\mathcal{Z}} = \mathbb{E} \left[ \prod_{i=1}^k \mathcal{Z}(t; n_i) \right]$ solves stochastic gas

$$
\frac{d}{dt} \bar{\mathcal{Z}}(t, \eta) = -\bar{\mathcal{Z}}(t, \eta) \quad , \quad H = \sum_{i \in \eta} \Delta_i + \sum_{i < j \in \eta} \eta_{ij}
$$
• This semi-discrete delta Bose gas is integrable and equivalent to solving free evolution equations with \( k-1 \) two-body boundary condition of form

\[
(\nabla_i - \nabla_{i+1} - 1) U_{n_i = n_{i+1}} = 0.
\]

• Either as limit of \( q \)-TASEP moments or directly from above system we get nested contour integral formulas for \( E(t; \mathfrak{n}) \).

Could we have found Laplace transform from these moments?

• No! They grow like \( E[\mathfrak{z}(\mathfrak{n})^k] \approx e^{\lambda r^2} \).

So RHS of below "equality" is divergent

\[
E[ e^{-S E(t; \mathfrak{n})} ] \overset{?}{=} \sum_{k=0}^\infty \frac{(-\lambda)^k}{k!} E[\mathfrak{z}(\mathfrak{n})^k]
\]

Could proceed formally (shadowing \( q \)-version) and, after "summing" the divergent series would get right answer.

Thus polymer replica method is shadow of rigorous approach explained earlier.
There is a further limit to continuum SHE/polymer and we get rigorous proof of Laplace transform formula there (earlier proof by [Amr-Ganesh]).

\[ g - \text{TASEP} \rightarrow g^\times_G(t) + n \]

\[ \downarrow \]

semi discrete SHE \[ Z(\mathbb{Z} : n) \]

\[ \downarrow \]

continuum SHE \[ Z(\mathbb{R} : x) \]
Two versions of ASEP

1. Occupation process $\gamma \in \{0,1\}^Z$

   \[
   (L^{\text{occ}}f)(\gamma) = \sum_{y \in \mathbb{Z}} \left( p\gamma_y(1-\gamma_{y+1}) + q(1-\gamma_y)\gamma_{y+1} \right) \left[ f(y,\gamma_{y+1}) - f(y) \right]
   \]

   \[
   N_y(\gamma) = \sum_{x \geq y} \gamma_x
   \]

2. K-particle process $\{\gamma_1 < \ldots < \gamma_k\}$

   \[
   (L^{\text{k-part}}f)(\gamma) = \sum_{\text{clusters}} \left( p\gamma_i(1-\gamma_{i+1}) + q(1-\gamma_i)\gamma_{i+1} \right)
   \]

Definition: $\eta(\cdot) \in \mathcal{X}$, $\gamma(\cdot) \in \mathcal{Y}$ and morph dual wrt $H: \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$ if for all $(\gamma,y,t)$,

\[
E^\eta H(\gamma(t),y) = E^y H(\gamma,y(t)).
\]

Theorem: Duality for

\[
H(\gamma,y) = \prod_{i=1}^k \gamma Y_i(\gamma) = \prod_{i=1}^k \gamma N_{Y_i}(\gamma)
\]

also

\[
G(\gamma,y) = \prod_{i=1}^k \gamma N_{Y_i}(\gamma)
\]
Remarks

- $p = q$, H duality $\Rightarrow$ SSEP cor. $F^m$.
- H duality $\approx$ Schütz '97 (F generalization).
- G duality at $k=1$ is Garder's transform.

Proof directly from showing $L_{\text{occ}} H(n,y) = L_{k,\text{part}} H(n,y)$.

Proof of formula from earlier much like $q$-TASEP.

1. True evol: eqs. (Uniqueness more subtle than $q$-TASEP)
   Duality implies $h(t;\tilde{y}) = \left[ E^{\text{free}} \right] [H(n(t);\tilde{y})]$.
   Solves $\frac{d}{dt} h(t;\tilde{y}) = L_{k,\text{part}} h(t;\tilde{y})$.

2. Can rewrite as free evolution eqs. wth $k=1$ free-boundary cond.

3. Check directly.

One hopes to the complete asymptotics. Suitable combinations of these observables gives formula for $\left[ E^{k\text{N}_x(t)} \right]$ and this leads to det. formula.
Basic reason that all these models turned out to be accessible is the existence of a large family of observables whose averages are explicit.

Example 2: ASEP

Set \( \tau = \frac{p}{q} < 1 \), \( N_y(t) = \sum_{x \leq y} \eta_x(t) \), \( \tilde{Q}_y = \frac{\tau^{N_y} - \tau^{N_y-1}}{\tau - 1} \).

Theorem [B-C-Sasamoto, 2012] For ASEP with step initial data \( \{X_n(0) = -n\} \) \( n \geq 1 \)

\[
E \left[ \tilde{Q}_{y_1}(t) \cdots \tilde{Q}_{y_k}(t) \right] = \frac{\tau^{k(k-1)/2}}{(2\pi i)^k} \int \cdots \int \prod_{A < B} \frac{z_A - z_B}{z_A - \tau z_B} \\
\times \prod_{j=1}^{k} e^{-\frac{z_j(p-q) t}{(1+z_j)(p+q z_j)}} \left( 1 + \frac{z_j}{1+z_j} \right)^{y_j+1} \frac{dz_j}{\tau + z_j}
\]
\[
\frac{(m+2)(m+1)}{2m-1} \cdot \frac{b}{(m+3)\theta^2} = (m,\theta^2)^K \quad \therefore \quad \frac{m^2-2}{m^2} \cdot \frac{(m+2)b}{(m+3)^2} \frac{\sin\left(\frac{m+2}{m+3}\right)}{\sin\left(\frac{m+1}{m+3}\right)} \frac{\text{such that}}{\text{such that}} \quad \frac{1}{\text{such that}} = (m,\theta^2)^K
\]

\[
\left(\begin{array}{c}
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K \\
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K
\end{array}\right)
\]

\[
\left(\begin{array}{c}
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K \\
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K
\end{array}\right) = \left(\begin{array}{c}
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K \\
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K
\end{array}\right)
\]

\[
\left(\begin{array}{c}
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K \\
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K
\end{array}\right) = \left(\begin{array}{c}
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K \\
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K
\end{array}\right)
\]

ASEP with \( a \leq 0, b \geq 0 \) and \( p > b \) hence \( c = 2 \). For step initial condition

**Theorem [Borodin–C-Sasamoto, 12]**: Suitable combinatorial yields

\[
\left(\begin{array}{c}
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K \\
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K
\end{array}\right) = \left(\begin{array}{c}
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K \\
\frac{1}{2} \int \frac{d}{dT} (\theta^2)^K
\end{array}\right)
\]
Corollary [Tracy-Widom ’09, Borodin-C-Sasamoto ’12]:

\[ \lim_{t \to \infty} \mathbb{P}^{\text{step}} \left( \frac{N_0(t/\delta)}{t^{1/3}} - \frac{t/4}{\gamma} \geq -r \right) = F_{\text{GUE}} \left( 2^{4/3} r \right) \]

Recovering the celebrate Tracy-Widom / Johansson result.

Remarks:

- Mellin Barnes Fredhold det. new and easy for asymptotics
- Inversion of Cauchy Fredholm det. equivalent to initial det. in [Tracy-Widom ’09]
- Completely parallel to q-TASEP formulas
Coordinate approach of [Tracy-Widom '08-'09]:

- Study k particle ASEP and use coordinate Bethe ansatz (cf. [Schutz '97] for k=2) to compute Green's functions.
- Manipulate formulas to extra one-point marginal.
- Approach step initial condition by taking k to infinity and observe an integral transform of Cauchy type Fredholm det.
- Functional analysis to rework for asymptotic analysis.

Using k-particle Green's functions can write solution of duality ODEs as k! k-fold contour integrals [Imamura-Sasamoto '11]. Equivalence to nested formula is non-trivial.