(1) Recall \( h_k, e_k, p_k \) are the homogeneous, elementary and power-sum symmetric functions in variables \( x_1, x_2, \ldots \). Let \( H(z) = \sum_{k=0}^{\infty} h_k z^k \), \( E(z) = \sum_{k=0}^{\infty} e_k z^k \) and \( P(z) = \sum_{k=1}^{\infty} p_k z^{k-1} \). Prove that these formal power series satisfy

(a) \[ H(z) = \prod_i \frac{1}{1-x_i z}, \]

(b) \[ E(z) = \prod_i (1 + x_i z), \]

(c) \[ P(z) = \frac{d}{dz} \sum \log \left( \frac{1}{1-x_i z} \right), \]

(d) \[ H(z) = \frac{1}{1-E(-z)} = \exp \left\{ \sum_{k=1}^{\infty} \frac{p_k z^k}{k} \right\}. \]

(2) The monomial symmetric functions \( m_{\lambda} \) and the power-sum symmetric functions \( p_{\lambda} = p_{\lambda_1} p_{\lambda_2} \cdots \) form linear bases of \( \Lambda \). The space spanned by these bases is \( \Lambda \) which is naturally graded with respect to the degree. Work out the transition matrix between these two linear bases when restricted to symmetric functions of degree \( \leq 3 \). (Hint: The matrix has block form based on the degree. For example when degree is 2, the block should correspond to the transition matrix between \( (m_{(2)}, m_{(1,1)}) \) and \( (p_{(2)}, p_{(1,1)}) \).

(3) The symmetric functions \( \{e_k\}_{k \geq 0}, \{h_k\}_{k \geq 0} \) and \( \{p_k\}_{k \geq 0} \) all constitute algebraically independent sets of generators of \( \Lambda \). This means that every element in \( \Lambda \) can be written in terms of these generating sets via the \((+, \times)\) operators.

(a) Write \( p_2 \) in terms of \( h_k \)'s.

(b) Write \( e_3 \) in terms of \( p_k \)'s.

(c) Write \( h_3 \) in terms of \( e_k \)'s.