(1) Prove that for any anti-symmetric polynomial $P(x) = P(x_1, x_2, \ldots, x_N)$ (i.e. $\forall \sigma \in S_N$, $P(\sigma x) = \operatorname{sgn}(\sigma)P(x)$), the ratio $P(x)/a_{\delta}(x)$ is a symmetric polynomial (here $a_{\delta}$ is the Vandermonde determinant).

(2) Prove Cauchy’s determinant identity
\[ \det \left( \frac{1}{1 - x_i y_j} \right)_{i,j=1}^{N} = \prod_{1 \leq i < j \leq N} (x_i - x_j)(y_i - y_j) / \prod_{1 \leq i,j \leq N} (1 - x_i y_j). \]

(3) Finish the proof of the Cauchy identity for Schur functions by proving that
\[ \sum_{\lambda, \ell(\lambda) \leq N} a_{\delta+\lambda}(x_1, \ldots, x_N) a_{\delta+\lambda}(y_1, \ldots, y_N) = \det \left( \frac{1}{1 - x_i y_j} \right)_{i,j=1}^{N}. \]

(Hint: Expand both sides into series in $x$’s and $y$’s and perform one of the summations over the symmetric group on the left-hand side to match with the right-hand side.)

(4) Restrict attention to $A_2$ (symmetric polynomials in $x_1$ and $x_2$. For $\lambda = (4, 1)$ and $\mu = (2)$ compute $s_\lambda$ and $s_{\lambda/\mu}$.

(5) Let us prove the Jacobi Trudi formula which states that
\[ s_\lambda(x) = \det (h_{\lambda_i-i+j}(x))_{i,j=1}^{\ell(\lambda)} \]
with the convention $h_m(x) = 0$ for $m < 0$.

(a) Prove the following general fact: If $f(u) = \sum_{m=0}^{\infty} f_m u^m$ then
\[ f(x_1) \cdots f(x_N) = \sum_{\lambda, \ell(\lambda) \leq N} \det (f_{\lambda_i-i+j})_{i,j=1}^{\ell(\lambda)} s_\lambda(x_1, \ldots, x_N). \]

(Hint: To see this multiply by sides by $a_{\delta}(x)$ and match coefficients.)

(b) Now apply this fact with $f_m = h_m(y_1, \ldots, y_N)$ and use the known generating function for the $h_m$ to conclude that
\[ \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} \det (h_{\lambda_i-i+j}(y_1, \ldots, y_N))_{i,j=1}^{N} s_\lambda(x_1, \ldots, x_N). \]

(c) Use orthogonality of the Schur functions to deduce the Jacobi Trudi formula.

(6) (Determinantal point processes) Consider $\mathcal{X}$ either $\mathbb{R}$ or $\mathbb{Z}$. A point configuration $X$ in $\mathcal{X}$ is a locally finite (i.e. no accumulation points) collection of points in $\mathcal{X}$. We assume the points in $X$ are always pair-wise distinct. The set of all such point configurations is denoted $\text{Conf}(\mathcal{X})$. For a compact subset $A \subset \mathcal{X}$ and $X \in \text{Conf}(\mathcal{X})$ set $N_A(X) = |A \cap X|$ (the number of points of $X$ in $A$). Thus, $N_A$ is a function $\text{Conf}(\mathcal{X})$. We equip $\text{Conf}(\mathcal{X})$ with the Borel structure (i.e. $\sigma$-algebra) generated by functions $N_A$ for all compact $A$. A random point process on $\mathcal{X}$ is a probability measure on $\text{Conf}(\mathcal{X})$.

(a) Assume that $\mathcal{X}$ is finite (e.g. $\mathbb{Z}$), then the correlation function of a finite subset $A \subset \mathcal{X}$ is $\rho(A) = \mathbb{P}(A \subset X)$. If $A = (x_1, \ldots, x_n)$ then we also write $\rho(A) = \rho_n(x_1, \ldots, x_n)$.
Show that the random point process $X$ is then uniquely determined by its correlation functions.

(b) For $n \geq 1$ consider any compactly supported bounded Borel function $f : \mathcal{X}^n \to \mathbb{R}$ and show that

$$
\sum_{x_1, \ldots, x_n \in \mathcal{X}} f(x_1, \ldots, x_n) \rho_n(x_1, \ldots, x_n) = \mathbb{E} \left[ \sum_{x_1, \ldots, x_n \in X} f(x_1, \ldots, x_n) \right].
$$

(c) If $\mathcal{X} = \mathbb{R}$ we can still define $\rho_n(x_1, \ldots, x_n)$ as the density relative to Lebesgue that

$\{x_1, \ldots, x_n\} \subset X$. Show that a Poisson point process with intensity 1 has $\rho_n = 1$ for all $n$.

(d) Recall that a random point process is determinantal if $\rho_n(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{i,j=1}^n$ for all $n$ (with $K$ fixed). Compute the correlation kernel $K$ for the Poisson point process.

(7) Prove the finite dimensional Von-Koch formula: For an $n \times n$ matrix $K(x, y) : \{1, \ldots, n\}^2 \to \mathbb{R}$, and $\lambda \in \mathbb{C}$,

$$
\det(I + \lambda K) = 1 + \sum_{k=1}^{n} \frac{(\lambda)^k}{k!} \prod_{i=1}^{n} \prod_{j=1}^{n} \det(K(x_i, x_j))_{i,j=1}^k.
$$

This formula is generalized as the Fredholm determinant as $n \to \infty$.

(8) The Airy function $\text{Ai}(s)$ is the solution to $y''(s) = s y(s)$ with $y : \mathbb{R} \to \mathbb{R}$ and $y(s) \to 0$ exponentially fast that $s \to \infty$. Prove that

$$
\text{Ai}(s) = \frac{1}{2\pi i} \int e^{z^3/3 - sz} dz
$$

satisfies these conditions if the contour for $z$ comes from $\infty e^{-\pi i/3}$ to the origin, and then turns to $\infty e^{\pi i/3}$.

(9) Prove that the following versions of the Airy kernel are equivalent: (Hint: It may be useful to know that for $z$ with positive real part $1/z = \int_{-\infty}^{0} e^{z(t)} dt$)

(a) $K_1(x, y) = \frac{1}{(2\pi i)^2} \int \int e^{w^3/3 - yw} \frac{1}{w - v} dv dw$

where the contour of integration for $w$ comes from $\infty e^{-\pi i/3}$ to the origin, and then turns to $\infty e^{\pi i/3}$, and the contour of integration for $v$ comes from $\infty e^{-2\pi i/3}$ to a point just to the left of the origin, and then turns to $\infty e^{2\pi i/3}$.

(b) $K_2(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y}$

with the extension of $x = y$ via L’Hopital’s rule.

(c) $K_3(x, y) = \int_{-\infty}^{0} \text{Ai}(x - r)\text{Ai}(y - r) dr$.

(10) The ascending Schur process $\emptyset \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)}$ with specializations $\rho_0^+, \ldots, \rho_{N-1}^+$ and $\rho_N$ can be written as

$$
S_{\rho_0^+, \ldots, \rho_N^+} \left( \lambda^{(N)} \right) p_{\lambda^{(N)} \to \lambda^{(N-1)}} \left( \rho_0^+, \ldots, \rho_{N-2}^+; \rho_N^+ \right) \cdots \rho_{\lambda^{(2)}} \rho_{\lambda^{(1)}} \left( \rho_0^+ \right).
$$
Figure out, using $S$ and $p^\uparrow$ and $p^\downarrow$ how to write a general (not necessarily ascending) Schur process in a similar form.