LIPSCHITZ LECTURE 2 ACCOMPANYING EXERCISES

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Abstract. Feel free to come by office hours Wednesday and Thursday 3 - 5 p.m. in room 3-040.

(1) Prove that for any anti-symmetric polynomial $P(x) = P(x_1, x_2, \dots, x_N)$ (i.e. $\forall \sigma \in S_N$, $P(\sigma x) = \mathrm{sgn}(\sigma) P(x)$), the ratio $P(x)/a_\delta(x)$ is a symmetric polynomial (here a_δ is the Vandermonde determinant).

(2) Prove Cauchy's determinant identity

$$\det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^{N} = \frac{\prod_{1\le i<j\le N}(x_i - x_j)(y_i - y_j)}{\prod_{1\le i,j\le N}(1 - x_i y_j)}.$$

(3) Finish the proof of the Cauchy identity for Schur functions by proving that

$$\sum_{\lambda:\ell(\lambda)\le N} a_{\delta+\lambda}(x_1, \dots, x_N) a_{\delta+\lambda}(y_1, \dots, y_N) = \det\left(\frac{1}{1 - x_i y_j}\right)_{i,j=1}^{N}.$$

(Hint: Expand both sides into series in x's and y's and perform one of the summations over the symmetric group on the left-hand side to match with the right-hand side.)

(4) Restrict attention to Λ_2 (symmetric polynomials in x_1 and x_2. For $\lambda = (4, 1)$ and $\mu = (2)$ compute s_λ and $s_{\lambda/\mu}$.

(5) Let us prove the Jacobi Trudi formula which states that

$$s_\lambda(x) = \det\left(h_{\lambda_i - i + j}(x)\right)_{i,j=1}^{\ell(\lambda)}$$

with the convention $h_m(x) = 0$ for $m < 0$.

(a) Prove the following general fact: If $f(u) = \sum_{m=0}^{\infty} f_m u^m$ then

$$f(x_1) \cdots f(x_N) = \sum_{\lambda:\ell(\lambda)\le N} \det\left(f_{\lambda_i - i + j}\right)_{i,j=1}^{\ell(\lambda)} s_\lambda(x_1, \dots, x_N).$$

(Hint: To see this multiply by sides by $a_\delta(x)$ and match coefficients.)

(b) Now apply this fact with $f_m = h_m(y_1, \dots, y_N)$ and use the known generating function for the h_m to conclude that

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda \det\left(h_{\lambda_i - i + j}(y_1, \dots, y_N)\right)_{i,j=1}^{N} s_\lambda(x_1, \dots, x_N).$$

(c) Use orthogonality of the Schur functions to deduce the Jacobi Trudi formula.

(6) (Determinantal point processes) Consider $\mathcal{X}$ either $\mathbb{R}$ or $\mathbb{Z}$. A point configuration X in $\mathcal{X}$ is a locally finite (i.e. no accumulation points) collection of points in $\mathcal{X}$. We assume the points in X are always pair-wise distinct. The set of all such point configurations is denoted $\mathrm{Conf}(\mathcal{X})$. For a compact subset $A \subset \mathcal{X}$ and $X \in \mathrm{Conf}(\mathcal{X})$ set $N_A(X) = |A \cap X|$ (the number of points of X in A). Thus, N_A is a function $\mathrm{Conf}(\mathcal{X})$. We equip $\mathrm{Conf}(\mathcal{X})$ with the Borel structure (i.e. σ-algebra) generated by functions N_A for all compact A. A random point process on $\mathcal{X}$ is a probability measure on $\mathrm{Conf}(\mathcal{X})$.

(a) Assume that $\mathcal{X}$ is finite (e.g. $\mathbb{Z}$), then the correlation function of a finite subset $A \subset \mathcal{X}$ is $\rho(A) = \mathbb{P}(A \subset X)$. If $A = (x_1, \dots, x_n)$ then we also write $\rho(A) = \rho_n(x_1, \dots, x_n)$.

Show that the random point process X is then uniquely determined by its correlation functions.

(b) For $n \geq 1$ consider any compactly supported bounded Borel function $f : \mathcal{X}^n \to \mathbb{R}$ and show that

$$\sum_{x_1,\ldots,x_n \in \mathcal{X}^n} f(x_1,\ldots,x_n)\rho_n(x_1,\ldots,x_n) = \mathbb{E}\left[\sum_{x_1,\ldots,x_n \in X} f(x_1,\ldots,x_n)\right].$$

(c) If $\mathcal{X} = \mathbb{R}$ we can still define $\rho_n(x_1,\ldots,x_n)$ as the density relative to Lebesgue that $\{x_1,\ldots,x_n\} \subset X$. Show that a Poisson point process with intensity 1 has $\rho_n \equiv 1$ for all n.

(d) Recall that a random point process is determinantal if $\rho_n(x_1,\ldots,x_n) = \det(K(x_i,x_j))_{i,j=1}^n$ for all n (with K fixed). Compute the correlation kernel K for the Poisson point process.

(7) Prove the finite dimensional Von-Koch formula: For an $n\times n$ matrix $K(x,y) : \{1,\ldots,n\}^2 \to \mathbb{R}$, and $\lambda \in \mathbb{C}$,

$$\det(I+\lambda K) = 1 + \sum_{k=1}^{n} \frac{(\lambda)^k}{k!} \sum_{x_1=1}^{n} \cdots \sum_{x_k=1}^{n} \det\left(K(x_i,x_j)\right)_{i,j=1}^k.$$

This formula is generalized as the Fredholm determinant as $n \to \infty$.

(8) The Airy function $\mathrm{Ai}(s)$ is the solution to $y''(s) = sy(s)$ with $y : \mathbb{R} \to \mathbb{R}$ and $y(s) \to 0$ exponentially fast that $s \to \infty$. Prove that

$$\mathrm{Ai}(s) = \frac{1}{2\pi i}\int e^{z^3/3 - sz}dz$$

satisfies these conditions if the contour for z comes from $\infty e^{-\pi i/3}$ to the origin, and then turns to $\infty e^{\pi i/3}$.

(9) Prove that the following versions of the Airy kernel are equivalent: (Hint: It may be useful to know that for z with positive real part $1/z = \int_{-\infty}^0 e^{zt}dt$)

(a)

$$K_1(x,y) = \frac{1}{(2\pi i)^2}\int\int \frac{e^{w^3/3 - yw}}{e^{v^3/3 - xv}} \frac{1}{w-v} dv dw$$

where the contour of integration for w comes from $\infty e^{-\pi i/3}$ to the origin, and then turns to $\infty e^{\pi i/3}$, and the contour of integration for v comes from $\infty e^{-2\pi i/3}$ to a point just to the left of the origin, and then turns to $\infty e^{2\pi i/3}$.

(b)

$$K_2(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}$$

with the extension of $x = y$ via L'Hopital's rule.

(c)

$$K_3(x,y) = \int_{-\infty}^{0} \mathrm{Ai}(x-r)\mathrm{Ai}(y-r)dr.$$

(10) The ascending Schur process $\emptyset \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(N)}$ with specializations $\rho_0^+,\ldots,\rho_{N-1}^+$ and ρ_N^- can be written as

$$\mathbb{S}_{\rho_0^+,\ldots,\rho_{N-1}^+;\rho_N^-}(\lambda^{(N)}) p^{\downarrow}_{\lambda^{(N)}\to\lambda^{(N-1)}}(\rho_0^+,\ldots,\rho_{N-2}^+;\rho_{N-1}^+)\cdots \rho^{\downarrow}_{\lambda^{(2)}\to\lambda^{(1)}}(\rho_0^+;\rho_1^+).$$

Figure out, using $\mathbb{S}$ and $p^{\uparrow}$ and $p^{\downarrow}$ how to write a general (not necessarily ascending) Schur process in a similar form.