LIPSCHITZ LECTURE 3 ACCOMPANYING EXERCISES

IVAN CORWIN

Abstract. Feel free to come by office hours Wednesday and Thursday 3 - 5 p.m. in room 3-040.

(1) Let us prove the following asymptotics of the Airy function, via steepest descent analysis. Prove that as $s \to \infty$

$$\mathrm{Ai}(s) = \frac{e^{-\frac{2}{3}s^{3/2}}}{2\sqrt{\pi}s^{1/4}}(1+o(1)).$$

The idea splits into a few steps. First find an integral expression for the function and rewrite it integrand in the form $e^{Nf(z)}$ where N is the large parameter and $f(z)$ is independent of N. Then find the critical points of $f(z)$ and determine which one has the property that the contours of integration can be moved so as to cross the critical point and always have real part less than the real part of $f(z)$ at the critical point. Then rescale around the critical point and Taylor expand. Show that the value of the integral away from the critical point is negligible as compared to the contribution in a suitably scaled neighborhood of the critical point.

(a) Recall that

$$\mathrm{Ai}(s) = \frac{1}{2\pi i}\int e^{z^3/3 - sz}dz.$$

There is freedom in the choice of contours. The typical choice is that z comes from $\infty e^{-\pi i/3}$ and goes to 0 and then departs towards $\infty e^{-\pi i/3}$. However, using the ample decay of the integrand and Cauchy's theorem these contours can be varied. Now make the change of variables $z \mapsto s^{1/2}z$ so that

$$\mathrm{Ai}(s) = \frac{1}{2\pi i}\int e^{s^{3/2}f(z)}s^{1/2}dz,$$

with $f(z) = \frac{z^3}{3} - z$.

(b) The function $f(z)$ has a critical point at $z=1$. Show it is justified in choosing the contour for z to be $1+iy$ for $y\in\mathbb{R}$. Make a change of variables $z = 1 + is^{-3/4}y$ and observe that by Taylor approximation $f(z) = -\frac{2}{3} - s^{-3/2}y^2 + O(s^{-9/4})$. This means that

$$\mathrm{Ai}(s) = \frac{1}{2\pi i}e^{-\frac{3}{2}s^{3/2}}s^{1/2}is^{-3/4}\int_{-\infty}^{\infty} e^{-y^2+O(s^{-3/4})}dy.$$

Show that the $O(s^{-3/4})$ term can be neglected so as to prove the claimed result.

(2) Consider a point process X defined with respect to a single particle state space $\mathcal{X} = \mathbb{Z}$. Consider some finite set $I \subset \mathcal{X}$ and let $\rho_n(x_1,\ldots,x_n)$ be the correlation functions and for convention, assume that if $x_i = x_j$ for $i \neq j$, then $\rho_n(x_1,\ldots,x_n) = 0$. Prove using inclusion-exclusion that

$$\mathbb{P}(X\cap I = \emptyset) = 1 + \sum_{k\geq 1}\frac{(-1)^k}{k!}\sum_{x_1\in I}\cdots\sum_{x_k\in I}\rho_k(x_1,\ldots,x_k).$$

Note that by the convention adopted above, this sum will terminate as long as I is finite. Now, using the von-Koch formula, prove that if X is determinantal with correlation kernel

$K : \mathcal{X} \times \mathcal{X} \to \mathbb{C}$, then

$$\mathbb{P}(X \cap I = \emptyset) = \det(I - K_I)_{L^2(\mathcal{X})}$$

where $K_I = \chi_I K \chi_I$ and χ_I is the projection operator on the subspace $L^2(I)$.

(3) Consider a Schur measure specified via $\rho_0^+, \rho_1^-, \rho_1^+, \ldots, \rho_{N-1}^+, \rho_N^-$.

(a) Prove that the normalization constant for the Schur process is given by

$$\prod_{i<j} H(\rho_i^+; \rho_j^-).$$

(b) Prove that marginally, $\lambda^{(i)}$ (and similarly $\mu^{(i)}$) is distributed as a Schur measure. Determine what are the specializations for that measure.

(4) In this exercise we will prove that boxed plane partitions distributed according to q^{volume} are equivalent to a certain Schur process. Fix a box width L. A boxed plane partition is a filling of the $L \times L$ square grid of boxes labeled by $\{(i,j) : 1 \le i, j \le L\}$ with non-negative integers $x_{i,j}$ such that $x_{i,j} \ge x_{i+1,j}$ and also $x_{i,j} \ge x_{i,j+1}$. This can be represented as a piling of boxes so that as one increases in i or j, the height of the pile weakly decreases. For instance on such filling when $L = 3$ is

1	0	0
2	2	2
4	3	3

Define a probability measure on all such matrices – called boxed plane partitions – via weighting a given matrix like $q^{\sum_{i,j} x_{i,j}}$ for $q \in (0,1)$. The sum $\sum_{i,j} x_{i,j}$ is called the volume of the boxed plane partition.

It is possible to read a sequence of partition off from this matrix in the following way. Consider the diagonal line $y = x + m$ for $m = -L+1, -L+2, \ldots, L-1$. For each m define $\lambda^{(m)}$ as the partition which lies along the corresponding line. For the above example,

$$\lambda^{(-2)} = (3), \quad \lambda^{(-1)} = (3,2), \quad \lambda^{(0)} = (4,2,0), \quad \lambda^{(1)} = (2,0), \quad \lambda^{(2)} = (1).$$

Notice that by definition, for $m < 0$, $\lambda^{(m)} \preceq \lambda^{(m+1)}$ (they interlace) and for $m \ge 0$, $\lambda^{(m+1)} \preceq \lambda^{(m)}$ (they interlace in the other order). If $\mu \preceq \lambda$ this is the same as saying that λ/μ is a horizontal strip.

Prove that the q^{vol} measure pushes forward to a Schur measure on

$$\emptyset \preceq \lambda^{(-L+1)} \preceq \cdots \preceq \lambda^{(0)} \succeq \lambda^{(1)} \succeq \cdots \succeq \lambda^{(L-1)} \succeq \emptyset$$

and determine the corresponding Schur positive specializations. (Hint: One only needs to consider specializations $\rho = (\alpha; \beta; \gamma)$ where $\alpha = (c, 0, \ldots)$ for wisely chosen c's and $\beta = \gamma = 0$. Recall that for such a specialization $s_{\lambda/\mu} = c^{|\lambda| - |\mu|}$ for λ/μ a horizontal strip, and otherwise 0.)