LIPSCHITZ LECTURE 4 ACCOMPANYING EXERCISES

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Abstract. Feel free to come by office hours Wednesday and Thursday 3 - 5 p.m. in room 3-040.

(1) Prove that

$$\sum_{\sigma \in S_k} \prod_{1 \leq B < A < \leq k} \frac{z_{\sigma(A)} - q z_{\sigma(B)}}{z_{\sigma(A)} - z_{\sigma(B)}} = k_q! := \frac{(1-q)(1-q^2)\cdots(1-q^k)}{(1-q)(1-q)\cdots(1-q)}.$$

(2) Prove that $k_q! \to k!$ as $q \to 1$.

(3) Hahn (1949) introduced two q-deformed exponential functions

$$e_q(x) = \frac{1}{\big((1-q)x;q\big)_\infty}, \qquad E_q(x) = \big(-(1-q)x;q\big)_\infty.$$

(Recall $(a;q)_\infty = \prod_{i \geq 0}(1-q^i a)$) Prove that for any fixed x, as $q \to 1$, $e_q(x), E_q(x) \to e^x$.

(4) The q-Laplace transform is define for any function $f \in \ell^1(\mathbb{Z}_{\geq 0})$ as

$$\hat{f}^q(z) := \sum_{n \geq 0} \frac{f(n)}{(zq^n;q)_\infty},$$

where $z \in \mathbb{C} \setminus \{q^- M\}_{M \geq 0}$. Prove the following inversion formula:

$$f(n) = -q^n \frac{1}{2\pi i} \int_{C_n} (q^{n+1}z;q)_\infty \hat{f}^q(z) dz$$

where C_n is any positively oriented complex contour which encircles only the poles $z = q^{-M}$ for $0 \leq M \leq n$. (Hint: Start by computing the residues of the integrand above at these pole in terms of the values of $f(n)$.)

(5) Prove the duality statement between q-TASEP and q-TAZRP. Recall that

$$(L^{q-TASEP} f)(\vec{x}) = \sum_{i=1}^{N} (1 - q^{x_{i-1}-x_i-1}) \left[f(\vec{x}_i^+) - f(\vec{x}) \right]$$

$$(L^{q-TAZRP} h)(\vec{y}) = \sum_{i=1}^{N} (1 - q^{y_i}) \left[h(\vec{y}^{i,i-1}) - h(\vec{y}) \right]$$

$$H(\vec{x};\vec{y}) = \prod_{i=0}^{N} q^{(x_i+i)y_i}.$$

Then prove the identity

$$L^{q-TASEP} H(\vec{x};\vec{y}) = L^{q-TAZRP} H(\vec{x};\vec{y})$$

where on the left the operator acts on $\vec{x}$ and on the right it acts on $\vec{y}$.