The topic of the course is “Taming Moduli Problems in Algebraic Geometry.” The goal of the course is to give a working knowledge of stacks, which are usually abstract, and to give a survey of some nice results.

**Remark 1.1.** There are two types of exercises: those that can be solved easily using tools from the course or prerequisite tools and those that are more involved and for which a solution consists of reading a solution in some book or reference.

**1.1 Moduli problems**

There is a guiding meta-problem in mathematics: the classification of mathematical objects. A famous example is simple Lie algebras over \( \mathbb{C} \), whose classification is known in terms of Dynkin diagrams. There is a known classification which is discrete: there are \( A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \) types.

Classifying objects in algebraic geometry often involves finitely many continuous (or non-discrete) parameters. For example, consider the moduli of Riemann surfaces (compact, complex manifolds of complex dimension 1). Riemann originally suggested that it takes \( 3g - 3 \) complex parameters to specify a complex structure on a smooth surface of genus \( g \). Ahlfors and Bers confirmed that the first-order deformations of a complex structure are classified by \( H^1(X, TX) \) for a general complex manifold \( X \), which can be identified using Serre duality with \( H^0(C, K^{\otimes 2})^* \), when \( X \) is a curve \( C \). By Riemann-Roch, the dimension of this vector space is \( \text{deg}(2K) + 1 - g = 3(g - 1) \).

**Remark 1.2.** There are several ways to see the classification of first-order deformations given above. One way is from the so-called Kodaira-Spencer map. Another way is to give a direct identification between first-order deformations of integral almost complex structures and Dolbeault cohomology cycles in \( H^1(X, TX) \).

There is also a differential geometric perspective. If \( S \) is a smooth surface of genus \( g > 1 \), define the Teichmüller space \( T(S) \) to be the quotient space \( H(S)/\text{Diff}_0 \), where \( H(S) \) is the space of Riemannian
metrics of constant curvature $-1$ and $\text{Diff}_0$ is the group of diffeomorphisms isotopic to the identity map. This quotient space can be identified with $\mathbb{R}^{6(g-1)}$, which has a canonical complex structure. Moreover, if we let $\text{MCG} = \text{Diff}/\text{Diff}_0$ be the mapping class group, then $T(S)/\text{MCG}$ can be identified with $\mathcal{M}_g$, where $\mathcal{M}_g$ is the set of Riemann surface structures on $S$. One can show that $\mathcal{M}_g$ inherits a topology, and is homeomorphic to a quasi-projective variety over $\mathbb{C}$. This understanding of $\mathcal{M}_g$ is very useful because questions about metrics on $S$ can become questions about a quasi-projective surface, whose properties we understand.

Our goal is to have a general framework for studying moduli problems in algebraic geometry and for finding and constructing “moduli spaces.”

### 1.2 Equivariant geometry

This approach will be our most concrete method of constructing moduli spaces.

Let $G$ be a reductive group over $\mathbb{C}$. This means that $G$ is the complexification of a compact (real) Lie group. Suppose we have a linear action of $G$ on projective space $\mathbb{P}^n$. (In fact, every algebraic action is linear, because the automorphism group of $\mathbb{P}^n$ can be identified with $\text{PGL}_{n+1}$, which can be seen from a functor-of-points definition of $\mathbb{P}^n$.) Let $X \hookrightarrow \mathbb{P}^n$ be a locally closed quasi-projective variety, equivariant for the action of $G$.

In this course, we wish to discuss equivariant cohomology, equivariant $K$-theory, equivariant coherent sheaves. The guiding principle is that any equivariant construction should not depend on the quotient construction $X/G$. This is because you want equivariant geometry to be an extension of usual geometry. For example, if $G$ acts freely in a suitable sense, then there should be a space $X/G$ parameterizing $G$-orbits, and for example, we want $H^*_G(X) \simeq H^*(X/G)$. However, the existence of such a space $X/G$ is not always possible in general, as the following exercise demonstrates.

**Exercise 1.3.** Consider the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ by scaling. Then any $\mathbb{C}^*$-invariant map to a scheme $\varphi : \mathbb{C}^n \to X$ factors through $\mathbb{C}^n \to \text{pt}$. (Hint: This follows from the non-existence of invariant functions on $\mathbb{C}^n$.) It follows that there can be no orbit space. The issue is the origin, because once it is removed, there is a space parametrizing orbits.

What we will do is think of $X/G$ as a geometric object in its own right, namely as a quotient stack (as a “functor of points”).

**Exercise 1.4.** There is a quasi-projective scheme $X_{g,d,n}$, constructed using Hilbert schemes parametrizing $C \hookrightarrow \mathbb{P}^n$ such that the action of $\text{PGL}_{n+1}$ extends to $X$ and $\mathcal{M}_g$ is an orbit space for the action of $\text{PGL}_{n+1}$ on $X_{g,d,n}$. (This done in Mumford’s book on GIT.)
1.3 Moduli of vector bundles on a curve

One can also consider the moduli of vector bundles over a curve, which will be an integral example for us. In fact, it is of fundamental interest in the geometric Langlands program. However, we will mostly study it because it is a beautiful example exhibiting much pathology yet much structure. In particular, it is highly non-separated, and there are too many vector bundles to be parametrized by a single scheme.

One can do a similar calculation as above to see that the first order deformations of a bundle $\xi$ are classified by $\text{Ext}^1(\xi, \xi)$ which has dimension $(n^2 - 1)(g - 1)$ for generic $\xi$.

There is an algebraic stack $\mathcal{M}_{n,d}(C)$ parametrizing rank $n$ degree $d$ vector bundles over $C$. There are several descriptions

(i) a functor of points description

(ii) a local quotient description

(iii) a global quotient description using infinite Grassmannians

(iv) a global quotient description due to Atiyah-Bott in mathematical gauge theory.

Understanding the equivalence of these construction is a fruitful way to understand this moduli problem.

The stack $\mathcal{M}_{n,d}$ has pathologies, but has special stratification (due to Harder-Narasimhan-Shatz), given by

$$\mathcal{M}_{n,d} = \mathcal{M}_{n,d}^{ss} \cup \bigcup_{\alpha} S_\alpha$$

where $\alpha$ ranges over 2-by-$k$ matrices

$$\alpha = \begin{bmatrix} d_1 & \cdots & d_k \\ n_1 & \cdots & n_k \end{bmatrix}$$

of integers such that $n_i > 0$, $d_1 + \cdots + d_k = d$, $n_1 + \cdots + n_k = n$, and $d_1/n_1 < \cdots < d_k/n_k$. The stratification has the properties that

- $\mathcal{M}_{n,d}^{ss}$ has a projective “good moduli space” $\underline{\mathcal{M}}_{n,d}^{ss}$, whose points parametrize “semistable bundles” up to “$S$-equivalence”

- The strata $S_\alpha$ deformation retracts onto $\mathcal{M}_{n_1,d_1}^{ss} \times \cdots \times \mathcal{M}_{n_k,d_k}^{ss}$ in a suitable sense, and the latter has a projective good moduli space as well.

Classically, the good moduli space $\underline{\mathcal{M}}_{n,d}^{ss}$ was studied because as a projective scheme, it is a bit more tractable. But thinking about $\mathcal{M}_{n,d}$ and the HNS stratification is the key to many results.

1.4 Striking results

If we restrict our attention to $\mathcal{M}_{2,d}$, then
(i) We have the Atiyah-Bott formula: If

\[ P_t(-) = \sum_{i \geq 0} t^i \dim H^i(-, \mathbb{Q}) \]

is the Poincaré polynomial, then

\[
P_t(M_{ss}^{\text{ss}}, d) = P_t(M_{2,d}^{ss}) - \sum_{k > d/2} t^\#_k P_t(M_{1,k}^{ss}) P_t(M_{1,d-k}^{ss})
\]

\[
= \frac{(1 + t)^{2g} (1 + t^3)^{2g}}{(1 - t^2)^2 (1 - t^4)} - \sum_{k > d/2} t^\#_k \left( \frac{(1 + t)^{2g}}{1 - t^2} \right)^2
\]

where \#_k = 2k - d + g - 1. The amazing fact is that when \( d \) is odd, we have that \( P_t(M_{ss}^{ss})(1 - t^2) \) is a polynomial.

(ii) Verlinde formula: There is a "unique" positive generator \( L \) of Pic(\( M_{ss}^{ss} \)), and the Verlinde formula says that \( H^i(M_{2,0}^{ss}, L^\otimes k) = 0 \) for \( i > 0 \) and

\[
\dim H^0(M_{2,0}^{ss}, L^\otimes k) = \left( \frac{k + 2}{2} \right)^{g-1} \sum_{j=1}^{k+1} \left( \sin(\pi j/(k+2)) \right)^{2-2g}
\]