Some facts about quotients

Thm (Chevalley)
Homogeneous spaces $G/H$ are always $G$-quasi-projective


$$\dim \text{ dim'} \quad V \subset I_H \subset k[G]$$

$$\quad \implies H = \text{Stab}(I_H) = \text{Stab}(V)$$

2) Find finite dim $V' > \sqrt{V}$ s.t. $G$ acts faithfully.

$$\quad \text{then still have } H = \text{Stab}(V \subset V')$$

3) Plücker embedding Grassmanian $\mathbb{G}(V) \hookrightarrow \mathbb{P}(\Lambda^{\dim(V)}(V'))$

$$\quad H = \text{Stab}(\Lambda^{\dim(V)}(V') \subset \mathbb{P}(\Lambda^{\dim(V)}(V'))$$

In fact this scheme structure is the "correct" one.
Existence of quotients: We don’t quite have the technology yet, but there is a very general
\[ G(\mathbb{R}) \text{ acts freely on } X(\mathbb{R}) \text{ for all } \mathbb{R} \]

**Thm:** if \( G \times X \rightarrow X \times X \) is a monomorphism, then the sheaf \( \mathcal{F}/G \) is an algebraic space

Can take this as motivation for thy of alg. spaces

What is an algebraic space?

Things to discuss: representable morphisms, properties of repr. morphisms, e.g. open immersions

**Def:** \( F : \text{Ring} \rightarrow \text{Set} \) such that

1) \( F \xrightarrow{\Delta} F \times F \) is representable by schemes

2) A surjective étale map from a scheme

**Equivalence relations in schemes:** \( R \rightarrow X \times X \),

*ex. relation associated to map \( X \rightarrow Y \)*

*NB, this is necessary*
Discuss construction $U/R$, involving sheafification (also note fiber products are sheaves).

L: example of surjective map $X \to Y$, then $Y$ is a quotient.

Main thm on alg. spaces: (Stacks Tag 0455)

TFAE for a sheaf $F : \text{Ring} \to \text{Set}$

1) $F$ alg. space (i.e. $F \cong \text{res}^\text{etale} \text{ surj } U \to X$) [easy]
2) $F$ is sheaf, $F$ reps. surjection, etale $U \to X$ [scheme]
3) $F$ equiv. relation $R \to U \times U$ in schemes s.t. $R \to U$ etale, and $F \cong U/R$

Also these are equiv. to the same conditions, but with "flat and finitely presented" instead of etale. [harder]

The point is that $G \times X \to X \times X$ is an equivalence relation and $G \times X \to X$ is Fppf (even smooth).

Another way to state the main theorem is that an Fppf equivalence relation is "equivalent" to an etale equivalence relation. Simply means $(U/R)(A)$.
Rem on separation axioms

The diagonal \( \Delta \) of a map of alg. spaces \( X \rightarrow Y \) will be a representable, locally \( F_1 \), monomorphism, separated, locally quasi-finite.

**Def.** \( f : X \rightarrow Y \) is separated if \( \Delta_{XY} \) is closed.

**Def.** \( f : X \rightarrow Y \) is quasi-separated if \( \Delta_{XY} \) is quasi-compact.

**Terrible example:** \( \mathbb{A}^1_{/\mathbb{Z}} \), \( \mathbb{A}^1_{/\mathbb{Z}}(\mathbb{R}) = \mathbb{R} \), \( \mathbb{Z}(\mathbb{R}) = \text{Map}(\mathbb{R}, \text{Spec}\mathbb{R}) \).

This is not quasi-separated, this is a nice class of things to work with.

**Thm.** If \( X \) is a locally finite type quasi-separated alg. space/k, then \( \exists \) a dense open subscheme \( X' \subset X \).

of \( G \) on a l.f.t. scheme
For free actions, you can at least form a quotient for some open subscheme

\[ x'/G \cong Y' \]
\[ \text{if } \emptyset \text{ open} \]
\[ x/G \cong Y \]

Discuss:

1) Definition \( G \times G \times X \xrightarrow{\Delta} X \times X \)
2) Representability: always an algebraic space

Cor.: \( G \) acts on a lift. scheme freely, then \( \exists \) dense open subscheme covered by \( G \)-equiv. open affines

\[ \text{quotient space is a scheme iff. it admits a } G \text{-equiv. open affine cover} \]