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1.1 Geometric invariant theory

There is a classic book by Mumford-Fogarty-Kirwan, which we will not follow too closely. We will follow more closely a work by Alper, in which it is noticed that many properties of GIT quotients are consequences of simple axioms involving \( \text{QCoh}(\mathcal{X}) \), which we discuss now.

**Lemma 1.1.** Let \( \mathcal{X} \) be a geometric stack. Then every \( F \in \text{QCoh}(\mathcal{X}) \) is a union of its coherent subsheaves.

This lemma is a consequence of the fact that a geometric stack has a presentation \( X_\bullet \), where \( X_0 \) and \( X_1 \) are both affine. This comes from the fact that we may chose an affine scheme \( X_0 = \text{Spec}(R) \) and an fppf affine map \( X_0 \to X \), and taking the fiber product \( X_1 = X_0 \times_X X_0 \) is also affine.

Moreover, for a geometric stack \( \mathcal{X} \) we have \( \text{QCoh}(\mathcal{X}) = \text{Ind(}\text{Coh}(\mathcal{X})\text{)} \). This means that

(i) coherent sheaves are finitely presented objects and \( \text{Hom}(S,-) \) commutes with filtered colimits for a coherent sheaf \( S \).

(ii) \( \text{QCoh}(\mathcal{X}) \to \text{Fun(}\text{Coh}(\mathcal{X})^{\text{op}},\text{Ab}) \) is an equivalence of categories.

For any algebraic stack \( \mathcal{X} \), one can show that \( \text{QCoh}(\mathcal{X}) \) is a “Grothendieck abelian category.”

**Definition 1.2.** Say that a category is a **Grothendieck abelian category** if it has arbitrary direct sums and filtered colimits, filtered colimits are exact, and there is a generating object \( U \), meaning that for all \( M \subset N \), there is a map \( U \to N \) which doesn’t factor through \( M \).

**Theorem 1.3.** In a Grothendieck abelian category there is enough injective objects, and has enough K-injective complexes. (A K-complex is a special type of complex which plays the role of an injective resolution when forming the unbounded derived category.)

**Remark 1.4.** There are other definitions of bounded and unbounded derived categories, but they all agree for geometric stacks.

Any map of stacks \( f : \mathcal{X} \to \mathcal{Y} \) can be modeled as a map of groupoids

\[
\begin{array}{ccc}
\mathcal{X} & \to & \mathcal{Y} \\
V_0 & \to & U_0 \\
V_1 & \to & U_1 
\end{array}
\]
This implies that there is a pullback functor $f^* : \text{QCoh}(\mathfrak{Y}) \simeq \text{QCoh}(V \cdot) \to \text{QCoh}(U \cdot) \simeq \text{QCoh}(\mathfrak{X})$, which is independent of the choices. One can define a pushforward functor $f_* : \text{QCoh}(\mathfrak{X}) \to \text{QCoh}(\mathfrak{Y})$ as the right adjoint of $f^*$. We will sometimes think about the derived functor of pushforward $Rf_*$ using injective resolutions or $K$-injective resolutions.

**Example 1.5.** Suppose that $f : X/G \to Y$. Let $\tilde{f} : X \to Y$ be a corresponding lift. Given $E \in \text{QCoh}(X/G)$, then $f_*(E|_X) \in \text{QCoh}(Y)$ canonically belongs to $\text{QCoh}(Y \times (-/G))$. The map $f$ factors

$$
\begin{array}{ccc}
X/G & \xrightarrow{p} & Y \times (-/G) \\
\downarrow & & \downarrow q \\
X & \rightarrow & Y
\end{array}
$$

as $q \circ p$. Then $p_*(E)$ arises from $p'_*(E|_X)$ via smooth descent. Moreover $q_*$ is taking invariants under $G$.

**Theorem 1.6.** If $G$ is linearly reductive, then $R\Gamma^i(X/G, E) \simeq R\Gamma^i(X, E|_X)$.

### 1.2 Good moduli spaces

**Definition 1.7.** Let $q : \mathfrak{X} \to Y$ be a map from an algebraic stack $\mathfrak{X}$ to an algebraic space $Y$. We say that $q$ is a **good moduli space (GMS)** if

(i) $q_* : \text{QCoh}(\mathfrak{X}) \to \text{QCoh}(Y)$ is exact

(ii) $\mathcal{O}_Y \to q_* \mathcal{O}_\mathfrak{X}$ is an isomorphism.

**Example 1.8.** Let $G$ be linearly reductive. Let $X = \text{Spec}(R)$. Then the map

$$\text{Spec}(R)/G \to \text{Spec}(R^G)$$

is a GMS.

We study now the main properties of a GMS $q : \mathfrak{X} \to Y$.

(i) $q$ is surjective, universally closed, universally submersive

(ii) If $k$ is algebraically closed and $x_1, x_2 \in \mathfrak{X}(\hat{k})$, then $q(x_1) = q(x_2)$ if and only if $\overline{\{x_1\}} \cap \overline{\{x_2\}} \neq \emptyset$ in $\mathfrak{X} \times_{\text{Spec}(k)} \text{Spec}(k)$.

(iii) The property of being a GMS is stable under base change along $Y' \to Y$ and fppc local on $Y$.

(iv) If $\mathfrak{X}$ is locally Noetherian, then $Y$ is locally Noetherian. If $\mathfrak{X}$ is finite type over $k$, then $Y$ is finite type over $k$. 


Example 1.9. Let $\mathbb{C}^2 = \mathbb{C}(1) \oplus \mathbb{C}(-1)$ with the $\mathbb{C}^*$ action indicated by the 1 and $-1$. We can instead consider the scheme associated to the ring $R = \mathbb{C}[x, y]$ where $x$ has weight one and $y$ has weight $-1$. The ring of invariants is $R^G = \mathbb{C}[xy]$. There are three types of orbits.

(i) hyperbolas $xy = c \neq 0$ where $\mathbb{C}^*$ acts freely.

(ii) the axes

(iii) the origin

The origin is the intersection of the closures of the axes.

Example 1.10. This is a nonexample. Blow up $\mathbb{C}^2/\mathbb{C}^*$ at the origin. This is isomorphic to the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^1$ with $\mathbb{C}^*$ acting with weight 2 on $\mathbb{P}^1$. We have a map to $\text{Spec}(\mathbb{C}[xy])$, but the corresponding pushforward map will not be exact.